Lemma 0.1 *Suppose $(F,<)$ is an ordered field and R_0 and R_1 are real closures of F such that $(R_i,<)$ is an ordered field extension of $(F,<)$. If $\alpha \in R_0 \setminus F$, there is an ordered field embedding of $F(\alpha)$ into R_1 fixing F.*

Proof Let $f \in F[X]$ be the minimal polynomial of α over F. Let $\alpha_1 < \ldots < \alpha_n$ be all zeros of f in R_0. By Corollary B.18, f has exactly n zeros $\beta_1 < \ldots < \beta_n \in R_1$. Let

$$\sigma : F(\alpha_1,\ldots,\alpha_n) \to F(\beta_1,\ldots,\beta_n)$$

be the map obtained by sending α_i to β_i. We claim that σ is an ordered field isomorphism.

For $i = 1,\ldots,n-1$, let $\gamma_i = \sqrt{\alpha_{i+1}-\alpha_i} \in R_0$. By the Primitive Element Theorem, there is $a \in F$ such that

$$F(a) = F(\alpha_1,\ldots,\alpha_n,\gamma_1,\ldots,\gamma_{n-1}).$$

Let $g \in F[X]$ be the minimal polynomial of a over F. By Corollary B.18, g has a zero $b \in R_1$ and there is a field isomorphism $\phi : F(a) \to F(b)$. Because $F(a)$ contains n zeros of F, so does $F(b)$. Thus $\beta_1,\ldots,\beta_n \in F(b)$ and for each i there is a j such that $\phi(\alpha_i) = \beta_j$. But

$$\phi(\gamma_i)^2 = \phi(\alpha_{i+1}) - \phi(\alpha_i).$$

Thus $\phi(\alpha_i) = \beta_i$ for $i = 1,\ldots,n$. We still must show that σ is order preserving. Suppose $c \in F(\alpha_1,\ldots,\alpha_n)$ and $c > 0$. There is $d \in R_0$ such that $d^2 = c$. Arguing as above, we can find a field embedding

$$\psi : F(\alpha_1,\ldots,\alpha_n,d) \subseteq R_1$$

fixing F. As above, $\psi(\alpha_i) = \beta_i$ and $\psi \supseteq \sigma$. Because

$$\psi(d)^2 = \psi(c) = \sigma(c),$$

we have $\sigma(c) > 0$. Thus σ is order preserving.

Proof of Theorem B.14 Let $\mathbb{P}$ be the set of all order preserving $\sigma : K \to R_1$ where $F \subseteq K \to R_0$ and $\sigma|F$ is the identity. By Zorn's Lemma, there is a maximal $\sigma : K \to R_1$ in $\mathbb{P}$. By identifying K and $\sigma(K)$ and applying the previous lemma, we see that $K = R_0$. A similar argument shows that $\sigma(K) = R_1$.

Uniqueness follows because the ith root of $f(X)$ in R_0 must be sent to the ith root of $f(X)$ in R_1.