K-Theory of Finite Fields

Recall that last week we wanted to prove the following theorem. If p is a prime number, and $q = p^d$, then :

Theorem 1 (Quillen).
$K_{2n-1}(\mathbb{F}_q) = \mathbb{Z}/(q^n - 1)$; $K_{2n}(\mathbb{F}_q) = 0$, where $n > 0$.

This follows from a more general theorem where we have the diagram :

$$\begin{array}{ccccc} B\mathrm{GL}(\mathbb{F}_q)^+ & \longrightarrow & BU & \xrightarrow{\psi^q - 1} & BU \\ {\scriptstyle \overline{\theta}}\big\downarrow{\scriptstyle \simeq} & \nearrow & & & \\ F\psi^q, & & & & \end{array} \qquad (\star)$$

where $F\psi^q$ is the homotopy fiber of $\psi^q - 1$, where ψ^q is the Adams operation. Today we will explain how precisely we get $\overline{\theta} : B\mathrm{GL}(\mathbb{F}_q)^+ \longrightarrow F\psi^q$ and outline the computation of the homology of $B\mathrm{GL}(\mathbb{F}_q)$.

1. Adams Operations ψ^k

Definition 2.
A commutative ring R is called a **λ-ring**, if we are given operations $\lambda^k : R \to R$ for $k \geq 0$ such that, for all $x, y \in R$:

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$,
- $\lambda^k(x + y) = \lambda^k(x) + \lambda^{k-1}(x)\lambda^1(y) + \cdots \lambda^k(y)$.

For instance, let X be a topological space and $E \to X$ a complex vector bundle. Let $\lambda^k(E)$ be the exterior power bundle $\bigwedge^k E$. This induces a λ-structure on the ring $KU^0(X) = [X, BU \times \mathbb{Z}]$. Another example is given by the complex representation ring $R(G)$ where G is a finite group : consider the exterior powers of a representation (think of the exterior power of a $\mathbb{C}G$-module).

Definition 3.
Let R be a λ-ring. The **Adams operations** $\psi^k : R \to R$, for $k > 0$ are defined as follows. The subring of symmetric polynomials in the polynomial algebra $\mathbb{Z}[x_1, \ldots, x_k]$ is the polynomial algebra $\mathbb{Z}[\sigma_1, \ldots, \sigma_k]$, where $\sigma_i = \sum_{1 \leq j_1 \leq \ldots \leq j_i \leq k} x_{j_1} \cdots x_{j_i}$ is the i-th elementary symmetric function. One may write the power sum $x_1^k + \ldots + x_n^k$ as a polynomial $Q_k(\sigma_1, \ldots, \sigma_k)$. Then one define :

$$\psi^k(x) := Q_k(\lambda^1(x), \ldots, \lambda^k(x)),$$

in other words:

$$\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) - \lambda^2(x)\psi^{k-2}(x) + \cdots + (-1)^k\lambda^{k-1}(x)\psi^1(x) + (-1)^{k-1}k\lambda^k(x).$$

For instance, since $x_1^2 + x_2^2 = \sigma_1^2 - 2\sigma_2$, we get $\psi^2(x) = x^2 - 2\lambda^2(x)$.

Notice that $\psi^k(\ell) = \ell^k$ for any line element in R (the definition of line element can makes sense for any augmented λ-ring), as $\lambda^1(\ell) = \ell$ and $\lambda^k(\ell) = 0$ for any $k \geq 2$. For a space X, the Adams operations $\psi^k : KU^0(X) \to KU^0(X)$ are represented by a map $\psi^k : BU \to BU$ (use Yoneda's lemma). So if for a line bundle ℓ we have $(\ell - 1)^{\otimes 2} = 0$, then :

$$\psi^k(\ell - 1) = \psi^k(\ell) - \psi^k(1) = \ell^k - 1 = (\ell - 1 + 1)^k - 1 = k(\ell - 1).$$

Now recall that in the proof of Bott periodicity, one shows that $\widetilde{KU}^0(S^2)$ is generated as a ring by $[L]$ subject to $([L] - 1)^2 = 0$, where L is the tautological line bundle over $\mathbb{C}P^1 \cong S^2$. Now since $S^{2n} = S^2 \wedge \ldots \wedge S^2$, $\widetilde{KU}^0(S^{2n})$ is generated by $([L] - 1) \otimes \ldots \otimes ([L] - 1)$. Therefore $\psi^k(x) = k^n x$ for any $x \in \widetilde{KU}^0(S^{2n})$.

Proof of Theorem 1 : Suppose $(\star)$ is true. Then the long exact sequence of the homotopy fiber, together with the fact that ψ^q acts on $\widetilde{KU}^0(S^{2n}) = \pi_{2n}(BU) \cong \mathbb{Z}$ as multiplication by q^n, and $\widetilde{KU}^0(S^{2n-1}) = \pi_{2n-1}(BU) = 0$ implies directly the Theorem. □

2. Brauer Lifting

Let G be a finite group. The **Borel construction** gives an additive functor :

$$\begin{array}{ccc} {}_{\mathbb{C}G}\mathbf{Mod} & \longrightarrow & \mathbf{Vect}_{\mathbb{C}}(BG) \\ V & \longmapsto & \left(\begin{array}{ccc} EG & \times_G & V \\ & \downarrow & \\ & BG & \end{array}\right), \end{array}$$

where $\mathbf{Vect}_{\mathbb{R}}(BG)$ denotes the category of complex vector bundles over BG. Since it commutes with tensor products and exterior powers, we also get a λ-ring homomorphism :

$$\Phi : R(G) \longrightarrow K^0(BG) = [BG, BU \times \mathbb{Z}].$$

We will consider the case $G = \mathrm{GL}_n(\mathbb{F}_q)$. We have the standard representations $\mathrm{GL}_n(\mathbb{F}_q)$ of $(\mathbb{F}_q)^n$. To obtain representations over $\mathbb{C}$, we use the following idea.

Definition 4.
Let $\overline{\mathbb{F}_q}$ be the algebraic closure of $\mathbb{F}_q$. Fix an embedding $\rho : \mathbb{F}_q^* \hookrightarrow \mathbb{C}^*$, via the roots of unity. Suppose V is a representation of G over $\mathbb{F}_q$. The **Brauer character** χ_V is defined as :

$$\chi_V(g) := \sum_{\alpha \in S_g} \rho(\alpha)$$

where $S_g \subseteq \mathbb{F}_q^*$ is the spectrum of $\varphi_V(g)$, where $\varphi_V : G \to \mathrm{Aut}(V)$, i.e. the set of eigenvalues, with multiplicity, of $\varphi_V(g)$. Then χ_V is the character of a unique virtual complex representation of G over $\mathbb{C}$

We obtain a ring homomorphism $R_{\mathbb{F}_q}(G) \longrightarrow R(G)$. Let us look how the Adams operation ψ^k behave under Brauer lifting.

Proposition 5.
Let χ be a virtual character of G. Then $(\psi^k\chi)(g) = \chi(g^k)$ for all $g \in G$.

Proof : Let us do by induction on the degree of the representation. If $\chi = \chi_\ell$ is the character of a one dimensional ℓ representation of G over $\mathbb{C}$, then we know:

$$(\psi^k\chi_\ell)(g) = \chi_{\ell^k}(g) = (\chi_\ell(g))^k = \chi_\ell(g^k),$$

since $\chi_\ell = \varphi_\ell$ as ℓ is one-dimensional, i.e., χ_ℓ is a group homomorphism. The same can be said for representation V of direct sum of one-dimensional representation. If G is cyclic, then any V has this form. The general case follows immediately by restricting to the cyclic subgroups $< g >$ for any g in G. □

Corollary 6.
If χ is the Brauer character of a representation V over $\mathbb{F}_q$, then $(\psi^q\chi) = \chi$.

Proof : Fix $g \in G$. Let us show that $S_g = S_{g^q}$. This follows from the fact that the eigeinvalues of any linear map lie in a finite Galois extension $K \geq \mathbb{F}_q$, and are permuted by the Galois group $\mathrm{Gal}(K/\mathbb{F}_q)$, which is cyclic and generated by the Frobenius morphism $a \mapsto a^q$. So $S_{g^q} = S_g$. □

So the Brauer lifting takes value in $R(G)^{\psi^q}$. So using our previous λ-ring homomorphism $\Phi : R(G) \longrightarrow K^0(BG)$, and the projection $KU^0(BG) \to \widetilde{KU}^0(BG)$, we have constructed a mapping :

$$R_{\mathbb{F}_q}(G) \longrightarrow \widetilde{KU}^0(BG)^{\psi^q} = [BG, BU]^{\psi^q}.$$

3. The Homotopy Equivalence $B\mathrm{GL}(\mathbb{F}_q)^+ \longrightarrow F\psi^q$

We can now construct our map $\overline{\theta} : B\mathrm{GL}(\mathbb{F}_q)^+ \longrightarrow F\psi^q$. The standard action of $\mathrm{GL}_n(\mathbb{F}_q)$ over $(\mathbb{F}_q)^n$ will give maps $\theta_n : B\mathrm{GL}_n(\mathbb{F}_q) \to BU$ by the Brauer lifting. It is easy to see that these maps are compatible under restriction : $\theta_{n|B\mathrm{GL}_{n-1}(\mathbb{F}_q)} = \theta_{n-1}$. Using colimit, we obtain a map $\theta : B\mathrm{GL}(\mathbb{F}_q) \longrightarrow BU$. Now Corollary 6 shows that $\psi^q \circ \theta_n \simeq \theta_n$, so $(\psi^q - 1) \circ \theta_n$ is nullhomotopic. Now, as BU has abelian fundamental group, the universal property of the $+$-construction gives a unique map up to homotopy $\theta : B\mathrm{GL}(\mathbb{F}_q)^+ \longrightarrow BU$, denoted again θ, such that $(\psi^q - 1) \circ \theta$ is nullhomotopic. Property of the homotopy fiber says that the sequence :

$$[Z, F\psi^q] \longrightarrow [Z, BU] \overset{(\psi^q-1)^*}{\longrightarrow} [Z, BU]$$

is exact for any space Z. Hence we get a map $\overline{\theta} : B\mathrm{GL}(\mathbb{F}_q) \to F\psi^q$, which turns out to be unique up to homotopy with some work.

To prove that $\overline{\theta}$ is a homotopy equivalence, we only need to show it is a weak homotopy equivalence, since the spaces are CW-complexes. In fact, the spaces are simple.

Definition 7.
A **simple space** X is a space homotopy equivalent to a CW-complex, connected, such that $\pi_1(X)$ is abelian and acts trivially on the higher homotopy groups.

Theorem 8.
Let $f : X \to Y$ be a map between simple spaces. If f induces an isomorphism on integer homology $H_*(-;\mathbb{Z})$, then f is a homotopy equivalence.

Sketch of the Proof : For simplicity, let's omit $\mathbb{Z}$ in the coefficients of the homology. Consider Ff the homotopy fiber of f. Then Ff is connected, so $H_0(Ff) = \mathbb{Z}$. The Serre spectral sequence : $E^2_{pq} = H_p(Y, H_q(Ff)) \Rightarrow H_{p+q}(X)$ has no twisted coeficients and depicts as follows :

$$\begin{array}{cccc}
H_0(Y, H_1(Ff)) & H_1(Y, H_1(Ff)) & H_2(Y, H_1(Ff)) & \cdots \\
 & \quad \overset{d_2}{\longleftarrow\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-} & & \\
H_0(Y) & H_1(Y) & H_2(Y) & \cdots \\
\cong \uparrow & \cong \uparrow & \cong \uparrow & \\
H_0(X) & H_1(X) & H_2(X) & \cdots ,
\end{array}$$

where the bottom horizontal maps represent the edge isomorphisms which are exactly the maps f_*. Hence we get :

$$0 = H_0(Y, H_1(Ff)) = (H_1(Ff))^{\pi_0(Y)} = H_1(Ff),$$

and so $H_1(Ff) = \pi_1(Ff) = 0$. It is simply connected, and inductively, one can show that $H_*(Ff)$. Therefore f is a weak homotopy equivalence. $\square$

Therefore, it suffices to prove that $\overline{\theta} : B\mathrm{GL}(\mathbb{F}_q)^+ \longrightarrow F\psi^q$ induces a homology equivalence over $\mathbb{Z}$. A standard universal coefficient argument shows that it suffices to prove the isomorphism for homology with coefficients over $\mathbb{Q}$, and the finite cyclic groups. Since the homology of $F\psi^q$ was computed last week, it only remains to look for $B\mathrm{GL}(\mathbb{F}_q)$.

4. $\widetilde{H}_*(B\mathrm{GL}(\mathbb{F}_q);\mathbb{Q}) = 0$

This follows from a more general result.

Proposition 9.
If G is a finite group, then $\widetilde{H}_*(BG;\mathbb{Q}) = 0$.

Proof : We have :

$$H_*(BG;\mathbb{Q}) = H_*(G;\mathbb{Q}) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Q}).$$

Since for any $\mathbb{Z}G$-module M, we clearly have a natural isomorphism :

$$\mathbb{Q} \otimes_{\mathbb{Z}G} M \cong \mathbb{Q} \otimes_{\mathbb{Q}G} (\mathbb{Q} \otimes_{\mathbb{Z}} M),$$

then since Tor is a left derived functor, we get :

$$\mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Q}) \cong \mathrm{Tor}_*^{\mathbb{Q}G}(\mathbb{Q},\mathbb{Q}).$$

Since the augmentation map $\varepsilon : \mathbb{Q}G \to \mathbb{Q}$, defined as $e_g \mapsto 1$, splits in :

$$0 \longrightarrow \ker(\varepsilon) \longrightarrow \mathbb{Q}G \overset{\varepsilon}{\longrightarrow} \mathbb{Q} \longrightarrow 0$$

via the mapping : $1 \mapsto \frac{1}{|G|}\sum_{g\in G} e_g$, we get that the trival module $\mathbb{Q}$ is a projective $\mathbb{Q}G$-module. Thus $H_0(G;\mathbb{Q}) = \mathbb{Q}$ and $H_n(G;\mathbb{Q}) = 0$ for $n \geq 1$. $\square$

To conclude, use the fact that $\widetilde{H}_*(B\mathrm{GL}(\mathbb{F}_q);M) \cong \mathrm{colim}_n \widetilde{H}_*(B\mathrm{GL}_n(\mathbb{F}_q);M)$, for any coefficient M.

5. $\widetilde{H}_*(B\mathrm{GL}\mathbb{F}_q;\mathbb{Z}/p)=0$

We will make use of the transfer in algebraic K-theory. Recall that in general, a ring homomorphism $R\to S$ would induce $K_nR\to K_nS$ using functoriality. Now if $R\subseteq S$ is a subring, then we may define in certain cases a homomorphism $\tau: K_*S\to K_*R$, called the **transfer**. In our case, it sufficient to think of finite field extensions $F\leq E$, of degree, say d. Then, recalling the definition of K_0, it is easy to define $\tau: K_0E\to K_0F$: let V be a finite dimensional E-vector space, set $\tau([V])=[V_F]$, where V_F is V regarded as a F-vector space. If $i:F\hookrightarrow E$, then $\tau\circ i_*: K_0F\to K_0F$ is multiplication by d. For higher cases, fix a basis of E over F. It induces maps :

$$\mathrm{GL}_nE\to\mathrm{GL}_{dn}F,$$

which are compatible, so that we get : $\mathrm{GL}E\to\mathrm{GL}F$. Using the universal property of the $+$-construction, we get the desired map :

$$\tau: B\mathrm{GL}(E)^+\longrightarrow B\mathrm{GL}(F)^+.$$

It turns out that the above construction is independant of the choice of the basis. And as the case for K_0, we have that the composite :

$$B\mathrm{GL}(F)^+\xrightarrow{i}B\mathrm{GL}(E)^+\xrightarrow{\tau}B\mathrm{GL}(F)^+,$$

is homotopic to the d-power map (as a H-space), so that $\pi_*(\tau\circ i)$ is multiplication by d. Suppose now we have proved the following.

Lemma 10.
Let $q=p^\nu$, then $\widetilde{H}_i(B\mathrm{GL}_n(\mathbb{F}_q);\mathbb{Z}/p)=0$ for $i<\nu(p-1)$, for all n.

Choose r prime to p and consider now :

$$B\mathrm{GL}(\mathbb{F}_q)^+\xrightarrow{i}B\mathrm{GL}(\mathbb{F}_{q^r})^+\xrightarrow{\tau}B\mathrm{GL}(\mathbb{F}_q)^+.$$

If we localize our spaces at p, then the map $\tau\circ i$ become an equivalence (as $\pi_*(\tau\circ)$ is multiplication by r and r prime to p). So $\tau\circ i$ induces an isomorphism on homology with coefficient $\mathbb{Z}_{(p)}$, and so with coefficient $\mathbb{Z}/p$. Now, by Lemma 10 applied on $\mathbb{F}_{q^r}$, (recall that $q=p^d$) the map $\widetilde{H}_n(\tau\circ i,\mathbb{Z}/p)$ is the trivial map for $n<dr(p-1)$, and so :

$$\widetilde{H}_n(B\mathrm{GL}(\mathbb{F}_q);\mathbb{Z}/p)=0,$$

for all $n<dr(p-1)$. But r was arbitrary, so $\widetilde{H}_*(B\mathrm{GL}\mathbb{F}_q;\mathbb{Z}/p)=0$.

Sketch of the proof of Lemma 10 : Let $B_n\subseteq\mathrm{GL}_n(\mathbb{F}_q)$ denote the subgroup of upper triangular matrices. Since the inclusion induces an injection onto group cohomology, it turns out that it is sufficient to prove that in group homology :

$$\widetilde{H}_i(B_n;\mathbb{Z}/p)=0,$$

for $i<\nu(p-1)$. One proceeds by induction on n. For $n=1$, we have $B_1=\mathbb{F}_q^*$, so p does not divide the order of B_1, and so $\widetilde{H}_i(B_1;\mathbb{Z}/p)=0$ for all i. For the inductive step, we have the usual group extension :

$$A_n\to B_n\to B_{n-1},$$

where A_n is the top row subgroup. Using some Hochschild-Serre spectral sequence argument, it is sufficient to prove that the homology of A_n vanishes in dimensions i, for $i < d(p-1)$, and this is done by group cohomology, via the semidirect product structure of A_n :

$$V \to A_n \to \mathbb{F}_q^*,$$

where V is the additive group of a $(n-1)$-dimensional vector space over $\mathbb{F}_q$. $\square$

6. The isomorphism $\overline{\theta}_* : H_*(B\mathrm{GL}(\mathbb{F}_q); \mathbb{Z}/\ell) \longrightarrow H_*(F\psi^q; \mathbb{Z}/\ell)$

We only briefly outline the proof. Recall ℓ is a prime different than p. We will now omit the $\mathbb{Z}/\ell$ coefficients. Let $\mu = \mu_\ell$ denote a ℓ-th root of unity different than 1, and $G = \mathrm{Gal}(\mathbb{F}_q(\mu)/\mathbb{F}_q)$. Let r be the smallest integer such that ℓ divides $q^r - 1$. So G is a group of order r and generated by the Frobenius automorphism., which acts on $C = \mathbb{F}_q(\mu)^*$ as multiplication by q. We have an embedding :

$$C \hookrightarrow \mathrm{GL}_r(\mathbb{F}_q),$$

induced by the representation L of $\mathbb{F}_q(\mu)$ of C. We obtain an homomorphism :

$$H_*(BC)_G \longrightarrow H_*(B\mathrm{GL}_r(\mathbb{F}_q)),$$

where the subscript denote the invariants by the group action. The former has a basis consisting of elements ξ'_j of degree $2jr$ for each $j \geq 0$ and η'_j of degree $2jr-1$ for each $j \geq 1$. Their images under the homomorphism are denoted :

$$\xi_j \in H_{2jr}(B\mathrm{GL}_r(\mathbb{F}_q)),$$

$$\eta_j \in H_{2jr-1}(B\mathrm{GL}_r(\mathbb{F}_q)).$$

Let ε denote the distinguished generator of $H_0(\mathrm{GL}_1\mathbb{F}_q))$. Clearly, $\xi_0 = \varepsilon^r$. The key point is to realize that the representation of C lifts by Brauer to the representation W of last week :

$$W = \zeta \oplus \zeta^2 \oplus \cdots \oplus \zeta^{q^r-1},$$

where $\zeta : \mathbb{Z}/(q^r-1) \to \mathbb{C}^*$ is done via root of unity. So the ring homomorphism :

$$\bigoplus_n H_*(B\mathrm{GL}_n(\mathbb{F}_q)) \longrightarrow H_*(F\psi^q),$$

sends the ε to 1 and sends η_j and ξ_j to the same letters element we have defined last week. Now one can prove again that :

$$\bigoplus_n H_*(B\mathrm{GL}_n(\mathbb{F}_q)) \cong P[\varepsilon, \xi_1, \xi_2, \ldots] \otimes \Lambda[\eta_1, \eta_2, \ldots].$$

Thus $\overline{\theta}$ is an isomorphism on homology with coefficients $\mathbb{Z}/\ell$.