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The Serre Spectral Sequence

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Introduction

In algebraic topology, one studies topological spaces through algebraic invariants such as groups : for instance, the homotopy groups $\pi_n(X, x_0)$ of a pointed topological space (X, x_0). The homology theories (and cohomology theories) provide other algebraic invariants on topological spaces. The reader might be familiar with the notion of fibration where one of the main properties is the long exact sequence of homotopy groups. Whence, fibrations can relate homotopy groups of different topological spaces, but what about homology groups ? What is the relationship between the homology groups of the total space, the base space, and the fiber of a fibration ?

The aim of this paper is to present these relationships through the notion of spectral sequence : a powerful algebraic object from homological algebra. Every fibration (or more generally every Serre fibration) gives rise to a spectral sequence called the *Serre spectral sequence*. It will describe a very particular relation between the homology groups. It will induce many different general results in homology theory, and we will present some of them in this paper.

In the first chapter, we present the algebraic concept and construction of a spectral sequence. In the second chapter, we construct the Serre spectral sequence of a fibration following the article [4] by the mathematician ANDREAS DRESS. We will also present in details the notion of homology with local coefficients. In the third chapter, we introduce some applications of the Serre spectral sequence. In appendix A, we will present the singular homology with integer coefficients which is the homology theory that we use for our description of the Serre spectral sequence. We will sketch briefly some useful results.

Convention

- We write I for $[0, 1] \subseteq \mathbb{R}$.
- We will always omit the composition of maps, namely, we write fg instead of $f \circ g$.
- For any topological space X, we write $H_n(X) = H_n(X; \mathbb{Z})$ for the n-th singular homology group of X (with integer coefficients) defined in appendix A.
- We write $\mathrm{Map}(X, Y)$ for the function space Y^X together with the compact-open topology.
- We label the vertices of the n-standard simplex Δ^n by $(0, 1, \ldots, n)$, defined in appendix A.
- For each $n > 0$, and $0 \leq i \leq n-1$, we write $\varepsilon_i = \varepsilon_i^n : \Delta^{n-1} \to \Delta^n$ for the i-th face map, defined in appendix A.

Chapter 1

The General Notion of Spectral Sequences

In this chapter, we present the algebraic notion of spectral sequence. Frank Adams said that *«a spectral sequence is an algebraic object, like an exact sequence, but more complicated»*.
It was first invented by the French mathematician Jean Leray from 1940 to 1945 as prisoner of war, in a concentration camp, during World War II. He was an applied mathematician, but because he did not want the Nazis to exploit his expertise, he did abstract work in algebraic topology. He invented spectral sequences in order to compute the homology (or cohomology) of a chain complex, in his work of sheaf theory. They were made algebraic by Jean-Louis Koszul in 1945.
Most of the work in this chapter will be based on [11], and [14].

1.1 Definitions and Basic Properties

Bigraded Abelian Groups A *bigraded abelian group* is a doubly indexed family of abelian groups $A_{\bullet\bullet} = \{A_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$. Subsequently, we will often write $A_{\bullet\bullet}$ as A. Let A and B be bigraded abelian groups, and let $(a,b) \in \mathbb{Z}\times\mathbb{Z}$. A *bigraded map of bidegree* (a,b), denoted $f : A \to B$, is a family of (abelian) group homomorphisms $f = \{f_{p,q} : A_{p,q} \to B_{p+a,q+b}\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$. The *bidegree* of f is (a,b). The *kernel* is defined as : $\ker f = \{\ker f_{p,q}\} \subseteq \{A_{p,q}\}$, and the *image* is defined by : $\mathrm{im}\ f = \{\mathrm{im}\ f_{p-a,q-b}\} \subseteq \{B_{p,q}\}$.

Definition 1.1.1. A *differential bigraded abelian group* (E,d) is a bigraded abelian group $E_{\bullet\bullet}$, together with a bigraded map $d : E \to E$, called the *differential*, such that $dd = 0$.

Definition 1.1.2. The *homology* $H_{\bullet\bullet}(E,d)$ of a differential bigraded abelian group (E,d) is defined by :

$$H_{p,q}(E,d) = \frac{\ker d_{p,q}}{\mathrm{im}\ d_{p-a,q-b}},$$

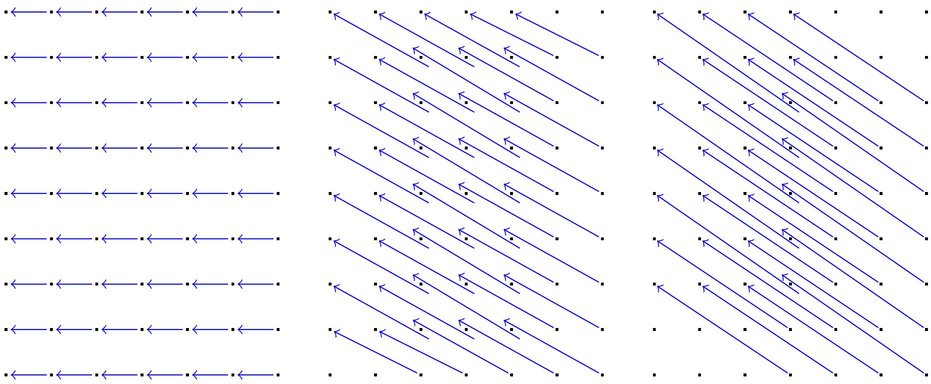
where (a,b) is the bidegree of the differential d.

Combining these notions, we can give the definition.

Definition 1.1.3. A *spectral sequence (of homological type)* is a collection of differential bigraded group $\{E^r_{\bullet\bullet}, d^r\}$, $r = 1,2,\ldots$, where the differentials are all of bidegree $(-r, r-1)$, such that $E^{r+1}_{p,q} \cong H_{p,q}(E^r, d^r)$. The bigraded abelian group E^r is called the E^r*-term*, or the E^r*-page*, of the spectral sequence.

The isomorphisms are fixed as part of the structure of the spectral sequence, so henceforth we will fudge the distinction between «$\cong$» and «$=$» in the above context.

One way to look at a spectral sequence is to imagine an infinite book, where each page is a Cartesian plane with the integral lattice points (p,q) which are the abelian groups $E_{p,q}$. There are homomorphisms (the differentials) between the groups forming «chain complexes». The homology groups of these chain complexes are precisely the groups which appear on the next page. One the first page, the differentials go one unit to the left, on the second page two units to the left and one unit up, on the third page three units to the left and two units up, and so on. The customary picture is shown below, where only parts of the pages are represented. Each dot represents a group.



E^1 E^2 E^3

It often happens that we do not look at the first page and we start at page 2 (it will be the case of the Serre spectral sequence).

The Limit Page It is instructive to describe a spectral sequence $\{E^r, d^r\}$ in terms of subgroups of E^1. Denote $Z^1_{p,q} = \ker d^1_{p,q}$, and $B^1_{p,q} = \mathrm{im}\ d^1_{p+1,q}$. We get $B^1 \subseteq Z^1 \subseteq E^1$ (from $d^1 d^1 = 0$). By definition : $E^2 \cong Z^1/B^1$. Write $\overline{Z^2} := \ker d^2$, it as a subgroup of E^2, whence, by the correspondence theorem, it can be written as Z^2/B^1, where Z^2 is a subgroup of Z^1. Similarly, write $\overline{B^2} := \mathrm{im}\ d^2$, which is isomorphic to B^2/B^1, where B^2 is a subgroup of Z^2, and so :

$$E^3 \cong \overline{Z^2}/\overline{B^2} \cong \frac{Z^2/B^1}{B^2/B^1} \cong Z^2/B^2,$$

by the third isomorphism theorem. This data can be presented as : $B^1 \subseteq B^2 \subseteq Z^2 \subseteq Z^1 \subseteq E^1$. Iterating this process, one can present the spectral sequence as an infinite sequence of subgroups of E^1 :

$$B^1 \subseteq B^2 \subseteq \ldots \subseteq B^n \subseteq \ldots \subseteq Z^n \subseteq \ldots \subseteq Z^2 \subseteq Z^1 \subseteq E^1,$$

with the property that : $E^{n+1} \cong Z^n/B^n$. Denote $Z^\infty_{p,q} := \bigcap_{r=1}^\infty Z^r_{p,q}$, and $B^\infty_{p,q} := \bigcup_{r=1}^\infty B^r_{p,q}$, subgroups of E^1. Clearly : $B^\infty \subseteq Z^\infty$. Thus, we get the following definition.

Definition 1.1.4. A spectral sequence determines a bigraded abelian group :

$$E^\infty_{p,q} := \frac{Z^\infty_{p,q}}{B^\infty_{p,q}},\ E^\infty_{\bullet\bullet} := \{E^\infty_{p,q}\},$$

called the *limit page* of the spectral sequence.

One can regard the E^r-pages as successive approximations of E^∞. It is usually the E^∞-page that is the general goal of a computation.

Lemma 1.1.5. *Let $\{E^r, d^r\}$ be a spectral sequence.*

(i) $E^{r+1} = E^r$ *if and only if* $Z^{r+1} = Z^r$ *and* $B^{r+1} = B^r$.

(ii) *If* $E^{r+1} = E^r$ *for all* $r \geq s$, *then* $E^s = E^\infty$.

Proof. Let us prove **(i)**. Suppose $E^{r+1} = E^r$. We have : $B^r \subseteq B^{r+1} \subseteq Z^{r+1} \subseteq Z^r$, and $E^{r+1} = Z^{r+1}/B^{r+1} = E^r = Z^r/B^r$. So $B^{r+1} = \{0\}$ in $E^r = Z^r/B^r$, that is $B^{r+1} = B^r$. Hence, we get $Z^{r+1}/B^r = Z^r/B^r$, so that $Z^{r+1} = Z^r$. The converse follows directly.
For **(ii)**, we get $Z^s = Z^r$ for all $r \geq s$, hence $Z^s = \bigcap_{r \geq s} Z^r = Z^\infty$. Similarly, we obtain the following : $B^s = \bigcup_{r \geq s} B^r = B^\infty$. Thus $E^s = Z^s/B^s = Z^\infty/B^\infty = E^\infty$. □

Definition 1.1.6. A *first quadrant spectral sequence* $\{E^r, d^r\}$ is one with $E^r_{p,q} = 0$ when $p < 0$ or $q < 0$. This condition for $r = 1$ implies the same condition for higher r. We have $E^{r+1}_{p,q} = E^r_{p,q}$, for $\max(p, q+1) < r < \infty$. In words, for fixed degrees p and q, $E^r_{p,q}$ is ultimately constant in r.

Definition 1.1.7. A *filtration $F_\bullet$ on an abelian group A*, is a family of subgroups $\{F_pA\}_{p\in\mathbb{Z}}$ of A, so that :

$$\ldots \subseteq F_{p-1}A \subseteq F_pA \subseteq F_{p+1}A \subseteq \ldots$$

Each filtration F of A determines an associated graded abelian group $E^0_\bullet(A)$ given by :

$$\{E^0_p(A) = F_pA/F_{p-1}A\}.$$

If A itself is graded, then the filtration is assumed to *preserve the grading*, which means : $F_pA_n \cap A_n \subseteq F_{p+1}A_n \cap A_n$, for all p and n. Write $n = p + q$, and define the associated graded abelian group as :

$$E^0_{p,q}(A_\bullet, F_\bullet) = \frac{F_pA_{p+q}}{F_{p-1}A_{p+q}}.$$

We can now define the notion of convergence.

Definition 1.1.8. A spectral sequence $\{E^r, d^r\}$ is said to *converge* to $A_\bullet$, a graded abelian group, if there is a given filtration $F_\bullet$ together with isomorphisms $E^\infty_{p,q} \cong E^0_{p,q}(A_\bullet, F_\bullet)$, and we denote it :

$$E^1_{p,q} \Rightarrow A_{p+q}.$$

We will now try to answer the following question : how does a spectral sequence arise ?

1.2 Spectral Sequence of a Filtered Complex

Now that we can describe a spectral sequence, how do we build one ? Most spectral sequences arise from a filtered complex.

Definition 1.2.1. A *filtered (chain) complex* is a chain complex $(K_\bullet, \partial)$ (see appendix A for definition), together with a filtration $\{F_pK_n\}_{p\in\mathbb{Z}}$ of each K_n such that the boundary operator *preserves the filtration* : $\partial(F_pK_n) \subseteq F_pK_{n-1}$, for all p and n. The family $\{F_pK_n\}_{n\in\mathbb{Z}}$ is itself a complex with induced boundary opeartors $\partial : F_pK_n \to F_pK_{n-1}$. The filtration induces a filtration on the homology of $K_\bullet$, with $F_pH_n(K_\bullet)$ defined as the image of $H_n(F_pK)$ under the (graded) map induced by the inclusion $F_pK_\bullet \hookrightarrow K_\bullet$.

With fundamental definitions in place, one can give the main theorem.

Theorem 1.2.2. *A filtration $F_\bullet$ of a chain complex $(K_\bullet, \partial)$ determines a spectral sequence $\{E^r_{\bullet\bullet}, d^r\}$, $r = 1, 2, \ldots$, where :*

$$E^1_{p,q} = H_{p+q}\left(\frac{F_pK_\bullet}{F_{p-1}K_\bullet}\right),$$

and the map d^1 :

$$\begin{array}{rcl} d^1 : H_{p+q}\left(\dfrac{F_pK_\bullet}{F_{p-1}K_\bullet}\right) & \longrightarrow & H_{p+q-1}\left(\dfrac{F_{p-1}K_\bullet}{F_{p-2}K_\bullet}\right) \\ {[z + F_{p-1}K_\bullet]} & \longmapsto & [\partial(z) + F_{p-2}K_\bullet] \end{array}$$

where z is in F_pK such that $\partial(z) \in F_{p-1}K$. Suppose further that the filtration is bounded, that is, for each n, there are values $s = s(n)$, $t = t(n)$, so that :

$$\{0\} = F_sK_n \subseteq F_{s+1}K_n \subseteq \ldots \subseteq F_{t-1}K_n \subseteq F_tK_n = K_n,$$

then the spectral sequence converges to $H_\bullet(K)$: $E^1_{p,q} \Rightarrow H_{p+q}(K_\bullet)$; more explicitly :

$$E^\infty_{p,q} \cong \frac{F_p(H_{p+q}(K_\bullet))}{F_{p-1}(H_{p+q}(K_\bullet))}.$$

Proof. We construct «by hand» the spectral sequence. Let's recall that we have the filtration :

$$\ldots \subseteq F_{p-1}K_{p+q} \subseteq F_pK_{p+q} \subseteq F_{p+1}K_{p+q} \subseteq \ldots,$$

and the fact that $\partial(F_pK_{p+q}) \subseteq F_pK_{p+q-1}$. For all integers $r \geq 0$, we introduce :

$$\begin{array}{rcl} Z^r_{p,q} & := & \text{elements in } F_pK_{p+q} \text{ that have boundaries in } F_{p-r}K_{p+q-1} \\ & = & \{a \in F_pK_{p+q} \mid \partial(a) \in F_{p-r}K_{p+q-1}\} \\ & = & F_pK_{p+q} \cap \partial^{-1}(F_{p-r}K_{p+q-1}), \\ B^r_{p,q} & := & \text{elements in } F_pK_{p+q} \text{ that form the image of } \partial \text{ from } F_{p+r}K_{p+q+1} \\ & = & F_pK_{p+q} \cap \partial(F_{p+r}K_{p+q+1}), \\ Z^\infty_{p,q} & := & \ker\partial \cap F_pK_{p+q}, \\ B^\infty_{p,q} & := & \operatorname{im}\partial \cap F_pK_{p+q}. \end{array}$$

However, one must be careful here, since $\bigcup_s F_sK_\bullet$ is not necessary equal to $K_\bullet$, the equalities $B^\infty_{p,q} = \bigcup_{r\geq 1} B^r_{p,q}$ and $Z^\infty_{p,q} = \bigcap_{r\geq 1} Z^r_{p,q}$ do *not* hold. So the definitions of Z^r and B^r are *not* as before. We obtain :

$$B^0_{p,q} \subseteq B^1_{p,q} \subseteq \ldots \subseteq B^\infty_{p,q} \subseteq Z^\infty_{p,q} \subseteq \ldots \subseteq Z^1_{p,q} \subseteq Z^0_{p,q},$$

and :

$$\begin{array}{rcl} \partial(Z^r_{p+r,q-r+1}) & = & \partial\left(F_{p+r}K_{p+q+1} \cap \partial^{-1}(F_pK_{p+q})\right) \\ & = & F_pK_{p+q} \cap \partial(F_{p+r}K_{p+q+1}) \\ & = & B^r_{p,q}. \end{array}$$

Define for all $0 < r \leq \infty$:

$$E^r_{p,q} := \frac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}}.$$

Since $Z^{r-1}_{p-1,q+1} \subseteq Z^r_{p,q}$ (use only the definitions), the quotient is well defined. Write the canonical projection $\eta^r_{p,q} : Z^r_{p,q} \to E^r_{p,q}$. By our previous work : $\partial(Z^r_{p,q}) = B^r_{p-r,q+r-1} \subseteq Z^r_{p-r,q+r-1}$, thus we get :

$$\begin{array}{rcl} \partial(Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}) & \subseteq & \partial(Z^{r-1}_{p-1,q+1}) + \partial(B^{r-1}_{p,q}) \\ & \subseteq & B^{r-1}_{p-r,q+r-1} + 0, \text{ because } \partial\partial = 0, \\ & \subseteq & Z^{r-1}_{p-1-r,q+r} + B^{r-1}_{p-r,q+r-1}. \end{array}$$

Thus the boundary operator, as a mapping $\partial : Z^r_{p,q} \to Z^r_{p-r,q+r-1}$, induces a homomorphism d^r, $r \geq 1$, such that the following diagram commutes :

$$\begin{tikzcd} Z^r_{p,q} \arrow[r, "\partial"] \arrow[d, "\eta"'] & Z^r_{p-r,q+r-1} \arrow[d, "\eta"] \\ E^r_{p,q} \arrow[r, "d^r"] & E^r_{p-r,q+r-1} \end{tikzcd}$$

It follows that $d^r d^r = 0$. Whence we have constructed $\{E^r, d^r\}$.

In order to show that it is indeed a spectral sequence, we need to prove that there is an isomorphism $H_{p,q}(E^r, d^r) \cong E^{r+1}_{p,q}$. Consider the following diagram, we will prove that it is commutative :

$$\begin{tikzcd} Z^r_{p-1,q+1} + B^r_{p,q} \arrow[r, hook] & Z^{r+1}_{p,q} \arrow[r, hook] \arrow[d, "\eta^r_{p,q}|_{Z^{r+1}_{p,q}}"] \arrow[dd, dashed, bend right, "\gamma"'] & Z^r_{p,q} \arrow[r, "\partial"] \arrow[d, "\eta^r_{p,q}"] & Z^r_{p-r,q+r-1} \arrow[d, "\eta^r_{p-r,q+r-1}"] \\ & \ker d^r \arrow[r, hook] \arrow[d] & E^r_{p,q} \arrow[r, "d^r"] & E^r_{p-r,q+r-1} \\ & H_{p,q}(E^r, d^r) \arrow[d] & & \\ & 0 & & \end{tikzcd}$$

We first prove that $\eta^r_{p,q}(Z^{r+1}_{p,q}) = \ker d^r$. Consider $(\eta^r_{p,q})^{-1}(\ker d^r)$. Since $d^r \eta = \eta \partial$, it implies that $d^r(\eta(z)) = 0$ if and only if $\partial(z) \in Z^{r-1}_{p-r-1,q+r} + B^{r-1}_{p-r,q+r-1}$; which is the case if and only if $z \in Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}$ (use the definitions). Whence, $\eta^{-1}(\ker d^r) = Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}$. Finally, we obtain : $\ker d^r = \eta(Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}) = \eta(Z^{r+1}_{p,q})$, because $Z^{r-1}_{p-1,q+1} \subseteq \ker \eta^r_{p,q}$.
Now, we prove : $Z^r_{p-1,q+1} + B^r_{p,q} = Z^{r+1}_{p,q} \cap \Big((\eta^r_{p,q})^{-1}(\text{im } d^r)\Big)$. First, we know that we have : $\text{im } d^r = \eta^r_{p,q}(\partial(Z^r_{p+r,q-r+1})) = \eta^r_{p,q}(B^r_{p,q})$. And so :

$$\begin{array}{rcl} (\eta^r_{p,q})^{-1}(\text{im } d^r) & = & B^r_{p,q} + \ker \eta^r_{p,q} \\ & = & B^r_{p,q} + B^{r-1}_{p,q} + Z^{r-1}_{p-1,q+1} \\ & = & B^r_{p,q} + Z^{r-1}_{p-1,q+1}. \end{array}$$

Since by the definitions :

$$\begin{array}{rcl} Z^{r-1}_{p-1,q+1} \cap Z^{r+1}_{p,q} & = & \Big(F_{p-1}K_{p+q} \cap \partial^{-1}(F_{p-r}K_{p+q-1})\Big) \cap \Big(F_pK_{p+q} \cap \partial^{-1}(F_{p-r-1}K_{p+q-1})\Big) \\ & = & F_{p-1}K_{p+q} \cap \partial^{-1}(F_{p-r-1}K_{p+q-1}) \\ & = & Z^r_{p-1,q+1}, \end{array}$$

all together, we get : $Z^{r+1}_{p,q} \cap \left((\eta^r_{p,q})^{-1}(\text{im } d^r)\right) = Z^r_{p-1,q+1} + B^r_{p,q}$, because $Z^r_{p-1,q+1} \subseteq Z^{r-1}_{p-1,q+1}$. Now we can describe the isomorphism $H_{p,q}(E^r, d^r) \cong E^{r+1}_{p,q}$. Let $\gamma : Z^{r+1}_{p,q} \to H_{p,q}(E^r, d^r)$ be the dashed map in the diagram, where $\ker d^r \to H_{p,q}(E^r, d^r)$ is the canonical projection. The kernel of γ is : $Z^{r+1}_{p,q} \cap \left((\eta^r_{p,q})^{-1}(\text{im } d^r)\right) = Z^r_{p-1,q+1} + B^r_{p,q}$. Since γ is surjective, by the first isomorphism theorem :

$$E^{r+1}_{p,q} = \frac{Z^{r+1}_{p,q}}{Z^r_{p-1,q+1} + B^r_{p,q}} \cong H_{p,q}(E^r, d^r).$$

Now we must prove : $E^1_{p,q} \cong H_{p+q}\left(\dfrac{F_pK_\bullet}{F_{p-1}K_\bullet}\right)$. We will use our previous work. Define, as before, the case $r = 0$:

$$E^0_{p,q} = \frac{Z^0_{p,q}}{Z^{-1}_{p-1,q+1} + B^{-1}_{p,q}},$$

where $Z^{-1}_{p-1,q+1} := F_{p-1}K_{p+q}$ and $B^{-1}_{p,q} := \partial(F_{p-1}K_{p+q+1})$. Thus we get :

$$\begin{aligned} E^0_{p,q} &= \frac{F_pK_{p+q} \cap \partial^{-1}(F_pK_{p+q-1})}{F_{p-1}K_{p+q} + \partial(F_{p-1}K_{p+q+1})} \\ &= \frac{F_pK_{p+q}}{F_{p-1}K_{p+q}}, \text{because the boundary operator preserves the filtration.} \end{aligned}$$

Now define the differential $d^0 : E^0_{p,q} \to E^0_{p,q-1}$ as the induced map of $\partial : F_pK_{p+q} \to F_pK_{p+q-1}$. Our previous work still holds for $r = 0$, and so we get :

$$E^1_{p,q} \cong H_{p,q}(E^0, d^0) = H_{p+q}\left(\frac{F_pK_\bullet}{F_{p-1}K_\bullet}\right).$$

Using this isomorphism, one can determine the map $d^1 : H_{p+q}\left(\frac{F_pK_\bullet}{F_{p-1}K_\bullet}\right) \to H_{p+q-1}\left(\frac{F_{p-1}K_\bullet}{F_{p-2}K_\bullet}\right)$. A class in $H_{p+q}\left(\frac{F_pK_\bullet}{F_{p-1}K_\bullet}\right)$ can be written as $[z + F_{p-1}K_\bullet]$, where $z \in F_pK_\bullet$ and $\partial(z) \in F_{p-1}K_\bullet$. The map d^1 maps $[z + F_{p-1}K_\bullet]$ to $[\partial(z) + F_{p-2}K_\bullet]$.

Finally, we prove : $E^\infty_{p,q} \cong \dfrac{F_p(H_{p+q}(K_\bullet))}{F_{p-1}(H_{p+q}(K_\bullet))}$, when the filtration is bounded. The assumption that the filtration is bounded implies, for r large enough, that $Z^r_{p,q} = Z^\infty_{p,q}$ and $B^r_{p,q} = B^\infty_{p,q}$. Our previous definition of E^r still holds for $r = \infty$: more precisely, the limit term of the spectral sequence *is* really the case $r = \infty$ (use lemma 1.1.5). Consider the canonical projections $\eta^\infty_{p,q} : Z^\infty_{p,q} \to E^\infty_{p,q}$ and $\pi : \ker \partial \to H(K_\bullet, \partial)$. We get :

$$\begin{aligned} F_p(H_{p+q}(K_\bullet)) &= \text{im } (H_{p+q}(F_pK_\bullet) \hookrightarrow H_{p+q}(K_\bullet)) \\ &= \pi(F_pK_{p+q} \cap \ker \partial) \\ &= \pi(Z^\infty_{p,q}). \end{aligned}$$

Since $\pi(\ker \eta^\infty_{p,q}) = \pi(Z^\infty_{p-1,q+1} + B^\infty_{p,q}) = F_{p-1}(H_{p+q}(K_\bullet))$, the map π induces a mapping :

$$d^\infty : E^\infty_{p,q} \longrightarrow \frac{F_p(H_{p+q}(K_\bullet))}{F_{p-1}(H_{p+q}(K_\bullet))},$$

which is surjective. Finally, observe that :

$$\begin{aligned}\ker d^{\infty} &= \eta^{\infty}_{p,q}\left(\pi^{-1}(F_{p-1}H_{p+q}(K_\bullet)) \cap Z^{\infty}_{p,q}\right)\\ &= \eta^{\infty}_{p,q}\left((Z^{\infty}_{p-1,q+1} + \mathrm{im}\ \partial) \cap Z^{\infty}_{p,q}\right)\\ &= \eta^{\infty}_{p,q}(Z^{\infty}_{p-1,q+1} + B^{\infty}_{p,q}) = \{0\}.\end{aligned}$$

So d^{∞} is an isomorphism, which ends the proof. □

1.3 Spectral Sequences of a Double Complex

Definition 1.3.1. A *double (chain) complex* (or *bicomplex*) is an ordered triple $(K, \partial', \partial'')$, where K is a bigraded abelian group, and $\partial', \partial'' : K \to K$ are bigraded maps (called the *horizontal boundary* and the *vertical boundary*) of bidegree $(-1, 0)$ and $(0, -1)$ respectively, such that :

- **(i)** $\partial'\partial' = 0$ and $\partial''\partial'' = 0$,
- **(ii) (anticommutativity)** $\partial'_{p,q-1}\partial''_{p,q} + \partial''_{p-1,q}\partial'_{p,q} = 0$.

We associate to each double complex its *total complex*, denoted $\mathrm{tot}(K)_\bullet$, which is the complex with n-th term :

$$\mathrm{tot}(K)_n = \bigoplus_{p+q=n} K_{p,q},$$

where the boundary operator is given by $\partial := \partial' + \partial''$. One can easely check that $\partial\partial = 0$, whence $\mathrm{tot}(K)_\bullet$ is indeed a complex.

Definition 1.3.2. Let $(K, \partial', \partial'')$ be a double complex. The *transpose* of the double complex is $({}^tK, \delta', \delta'')$, where ${}^tK_{p,q} = K_{q,p}$, $\delta'_{p,q} = \partial''_{q,p}$ and $\delta''_{p,q} = \partial'_{q,p}$. Then $({}^tK, \delta', \delta'')$ is also a double complex and the total complexes are identical : $\mathrm{tot}({}^tK)_\bullet = \mathrm{tot}(K)_\bullet$ and $\delta = \delta' + \delta'' = \partial' + \partial'' = \partial$.

To each double complex, one can associate homology groups.

Definition 1.3.3. Let $(K, \partial', \partial'')$ be a double complex. One can see that (K, ∂') and (K, ∂'') are both complexes. The *homology of the rows* is the bigraded abelian group $H'_{\bullet\bullet}(K)$ defined as below :

$$H'_{p,q}(K) = \frac{\ker \partial'_{p,q}}{\mathrm{im}\ \partial'_{p+1,q}}.$$

Similarly, define the *homology of the columns* by $H''_{p,q}(K) = \dfrac{\ker \partial''_{p,q}}{\mathrm{im}\ \partial''_{p,q+1}}$. For each fixed q, the q-th row $H''_{\bullet,q}(K)$ of $H''(K)$ can be made into a complex if one defines the induced boundary operator :

$$\begin{aligned}\overline{\partial'} : H''_{p,q}(K) &\longrightarrow H''_{p-1,q}(K)\\ [z] = z + \mathrm{im}\ \partial''_{p,q+1} &\longmapsto \partial'(z) + \mathrm{im}\ \partial'_{p-1,q+1} = [\partial'(z)]\,,\end{aligned}$$

where $z \in \ker \partial''_{p,q}$. It is easy to check that $\overline{\partial'\partial'} = 0$. Thus one can again define a homology called the *first iterated homology* of the double complex, denoted $H'_\bullet H''_\bullet(K)$, which is defined as : $H'_pH''_q(K) := H_p(H''_{\bullet,q}(K), \overline{\partial'}) = \dfrac{\ker \overline{\partial'}_{p,q}}{\mathrm{im}\ \overline{\partial'}_{p+1,q}}$. Similarly, define the *second iterated homology* $H''_\bullet H'_\bullet(K)$ as : $H''_pH'_q(K) := H_p(H'_{q,\bullet}(K), \overline{\partial''})$, where $\overline{\partial''}$ is defined in a similar way.

Moreover, we can define two filtrations out of double complex :

$$^{\mathrm{I}}(K_{i,j})_p = \begin{cases} 0, & \text{if } i > p, \\ K_{i,j}, & \text{if } i \leq p, \end{cases}$$

$$^{\mathrm{II}}(K_{i,j})_p = \begin{cases} 0, & \text{if } j > p, \\ K_{i,j}, & \text{if } j \leq p. \end{cases}$$

These filtrations both define a filtration on the total complex.

Definition 1.3.4. Let $(K, \partial', \partial'')$ be a double complex. The *column-wise filtration* of $\mathrm{tot}(K)_\bullet$ is given by :

$$\begin{aligned} ^{\mathrm{I}}(F_p\mathrm{tot}(K))_n &= \bigoplus_{i \leq p} K_{i,n-i} \\ &= \ldots \oplus K_{p-2,q+2} \oplus K_{p-1,q+1} \oplus K_{p,q}, \end{aligned}$$

where $q = n - p$. For any fixed p, one can easily check that $^{\mathrm{I}}(F_p\mathrm{tot}(K))_\bullet$ is indeed a subcomplex of $\mathrm{tot}(K)_\bullet$, and a filtration. Similarly, the *row-wise filtration* of $\mathrm{tot}(K)_\bullet$ is given by :

$$\begin{aligned} ^{\mathrm{II}}(F_p\mathrm{tot}(K))_n &= \bigoplus_{j \leq p} K_{n-j,j} \\ &= \ldots \oplus K_{q+2,p-2} \oplus K_{q+1,p-1} \oplus K_{q,p}. \end{aligned}$$

With these two filtered complexes, there are induced spectral sequences, by theorem 1.2.2.

Theorem 1.3.5. *Given a double complex $(K, \partial', \partial'')$, there are two spectral sequences $\{^{\mathrm{I}}E^r, ^{\mathrm{I}}d^r\}$ and $\{^{\mathrm{II}}E^r, ^{\mathrm{II}}d^r\}$, with first pages :*

$$^{\mathrm{I}}E^1_{p,q} \cong H''_{p,q}(K), \textit{ and } ^{\mathrm{II}}E^1_{p,q} \cong H'_{p,q}(K),$$

and second pages :

$$^{\mathrm{I}}E^2_{p,q} \cong H'_pH''_q(K), \textit{ and } ^{\mathrm{II}}E^2_{p,q} \cong H''_pH'_q(K).$$

Moreover, if $K_{p,q} = \{0\}$ when $p < 0$ or $q < 0$, then both spectral sequences are first quadrant, and converge to the homology of the total complex $H_\bullet(\mathrm{tot}(K)_\bullet, \partial)$:

$$^{\mathrm{I}}E^1_{p,q} \Rightarrow H_{p+q}(\mathrm{tot}(K)_\bullet), \textit{ and } ^{\mathrm{II}}E^1_{p,q} \Rightarrow H_{p+q}(\mathrm{tot}(K)_\bullet).$$

Proof. The proof follows from theorem 1.2.2. Because $K_{p,q} = \{0\}$ when $p < 0$ or $q < 0$, the filtrations are bounded, thus we obtain two spectral sequences converging to $H_\bullet(\mathrm{tot}(K)_\bullet, \partial)$. In particular, there exists filtration $^{\mathrm{I}}F_\bullet$ and $^{\mathrm{II}}F_\bullet$ such that : $^{\mathrm{I}}E^\infty_{p,q} \cong \frac{^{\mathrm{I}}F_p(H_{p+q}(\mathrm{tot}(K)_\bullet))}{^{\mathrm{I}}F_{p-1}(H_{p+q}(\mathrm{tot}(K)_\bullet))}$ and $^{\mathrm{II}}E^\infty_{p,q} \cong \frac{^{\mathrm{II}}F_p(H_{p+q}(\mathrm{tot}(K)_\bullet))}{^{\mathrm{II}}F_{p-1}(H_{p+q}(\mathrm{tot}(K)_\bullet))}$, where for $p + q =: n$ fixed, $^{\mathrm{I}}F_t = 0$ and $^{\mathrm{II}}F_t = 0$, for all $t < 0$ and $^{\mathrm{I}}F_t = H_{p+q}(\mathrm{tot}(K)_\bullet))$ and $^{\mathrm{II}}F_t = H_{p+q}(\mathrm{tot}(K)_\bullet))$, for all $t \geq n$.
We first give a proof in the case of $\{^{\mathrm{I}}E^r, ^{\mathrm{I}}d^r\}$. Consider the column-wise filtration $^{\mathrm{I}}F_\bullet$. Let us prove $^{\mathrm{I}}E^1_{p,q} \cong H''_{p,q}(K)$. By theorem 1.2.2, we have :

$$^{\mathrm{I}}E^1_{p,q} = H_{p+q}\left(\left(\frac{^{\mathrm{I}}F_p\mathrm{tot}(K)}{^{\mathrm{I}}F_{p-1}\mathrm{tot}(K)}\right)_\bullet, \partial\right).$$

The $(p + q)$-th term of the quotient complex is given by :

$$\frac{^{\mathrm{I}}(F_p\mathrm{tot}(K))_{p+q}}{^{\mathrm{I}}(F_{p-1}\mathrm{tot}(K))_{p+q}} = \frac{\ldots \oplus K_{p-2,q+2} \oplus K_{p-1,q+1} \oplus K_{p,q}}{\ldots \oplus K_{p-2,q+2} \oplus K_{p-1,q+1}} \cong K_{p,q}.$$

The induced boundary operator $\partial : \frac{{}^{\mathrm{I}}(F_p\mathrm{tot}(K))_{p+q}}{{}^{\mathrm{I}}(F_{p-1}\mathrm{tot}(K))_{p+q}} \longrightarrow \frac{{}^{\mathrm{I}}(F_p\mathrm{tot}(K))_{p+q-1}}{{}^{\mathrm{I}}(F_{p-1}\mathrm{tot}(K))_{p+q-1}}$ maps the elements $a_{p+q} + {}^{\mathrm{I}}(F_{p-1}\mathrm{tot}(K))_{p+q}$ to $\partial(a_{p+q}) + {}^{\mathrm{I}}(F_{p-1}\mathrm{tot}(K))_{p+q-1}$, where $a_{p+q} \in {}^{\mathrm{I}}(F_p\mathrm{tot}(K))_{p+q}$, but we have just seen that we may assume $a_{p+q} \in K_{p,q}$. Now $\partial(a_{p+q}) = (\partial' + \partial'')(a_{p+q})$ which belongs to $K_{p-1,q} \oplus K_{p,q-1}$. But we have $K_{p-1,q} \subseteq {}^{\mathrm{I}}(F_{p-1}\mathrm{tot}(K))_{p+q-1}$, so only ∂'' survives. We have just proved :

$$H_{p+q}\left(\left(\frac{{}^{\mathrm{I}}F_p\mathrm{tot}(K)}{{}^{\mathrm{I}}F_{p-1}\mathrm{tot}(K)}\right)_{\bullet}, \partial\right) \cong H''_{p,q}(K).$$

Now, we prove ${}^{\mathrm{I}}E^2_{p,q} \cong H'_pH''_q(K)$. We will simplify the notation : ${}^{\mathrm{I}}F_p\mathrm{tot}(K)$ becomes F_p. We need to prove that the following diagram commutes :

$$\begin{tikzcd} H_{p+q}\left(\frac{F_p}{F_{p-1}}\right) \arrow[r, "d^1"] & H_{p+q-1}\left(\frac{F_{p-1}}{F_{p-2}}\right) \arrow[d, "\cong"] \\ H''_{p,q}(K) \arrow[u, "\cong"] \arrow[r, "\overline{\partial'}"] & H''_{p-1,q}(K) \end{tikzcd}$$

The elements in $H''_{p,q}(K)$ are the classes $[z]$, where z are in $\ker \partial''_{p,q}$. Now recall that d^1 sends $[x+F_{p-1}]$ to $[\partial(x)+F_{p-2}]$, where x in F_p and $\partial(x)$ in F_{p-1}. Since $\partial'' z = 0$, using the isomorphism $H''_{p,q}(K) \cong H_{p+q}\left(\frac{F_p}{F_{p-1}}\right)$, this determines $[\partial'(z)+F_{p-2}]$ as an element of $H_{p+q-1}\left(\frac{F_{p-1}}{F_{p-2}}\right)$. Finally, the isomorphism $H_{p+q-1}\left(\frac{F_{p-1}}{F_{p-2}}\right) \cong H''_{p-1,q}(K)$, sends $[\partial'(z) + F_{p-2}]$ to $[\partial'(z)]$ in $H''_{p-1,q}(K)$. We have just proved the commutativity of the diagram. And so : ${}^{\mathrm{I}}E^2_{p,q} \cong H'_pH''_q(K)$.

To get the second spectral sequence from the row-wise filtration ${}^{\mathrm{II}}F_{\bullet}$, reindex the double complex as its transpose (see definition 1.3.2), so that the total complex stays unchanged, but the row-wise filtration becomes the column-wise filtration. The same proof goes over to obtain the result. $\square$

Chapter 2

The Serre Spectral Sequence

2.1 The Notion of Serre Fibration

This section gathers results from algebraic topology that will be necessary subsequently, when we will introduce the Serre spectral sequence. We recall here the notion of Serre fibration and its properties.

Definition 2.1.1. A continuous map $p : E \to B$ is a *Serre fibration* (also called *weak fibration*) if it has *the homotopy lifting property* with respect to every cube I^n, $n \geq 0$; i.e. for any homotopy $H : I^n \times I \to B$, for any continuous map $\tilde{h}_0 : I^n \to E$ such that $p\tilde{h}_0 = Hi_0$ ($\tilde{h}_0$ is said to be a *lift* of Hi_0), where $i_0 : I^n \hookrightarrow I^n \times I$ is the inclusion $x \mapsto (x, 0)$, there exists a homotopy $\tilde{H} : I^n \times I \to E$ such that the diagram commutes :

$$\begin{tikzcd}
I^n \arrow[r, "\tilde{h}_0"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

whence $\tilde{H}$ is a *lift* of H, such that $\tilde{H}_{|_{I^n \times \{0\}}} = \tilde{h}_0$.

One of the fundamental properties of Serre fibration is the following theorem.

Theorem 2.1.2 (**Homotopy Sequence of a Serre Fibration**)**.** *Let $p : E \to B$ be a Serre fibration, equipped with basepoints $e_0 \in E$, $b_0 \in B$ such that $p(e_0) = b_0$, so that the* fiber *is defined by $F := p^{-1}(b_0)$. Then there is an exact sequence :*

$$\cdots \longrightarrow \pi_{n+1}(B) \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_1(B) \longrightarrow \pi_0(F) \longrightarrow \pi_0(E) \longrightarrow \pi_0(B)$$

Proof. See [3] for a proof. □

We will use throughout this chapter, the following results.

Proposition 2.1.3. *The pullback of a Serre fibration is a Serre fibration.*

Proof. An easy proof : use the fundamental property of a pullback. □

Proposition 2.1.4. *If $p : E \to B$ is a Serre fibration, then the induced continuous map $\widehat{p} : \mathrm{Map}(I^s, E) \to \mathrm{Map}(I^s, B)$ is also a Serre fibration, where* $\mathrm{Map}(X, Y)$ *denotes the function space Y^X together with the compact-open topology.*

Proof. Since I^s is locally compact and Hausdorff, any continuous map $I^n \to \mathrm{Map}(I^s, B)$ is equivalent to a continuous map $I^n \times I^s \to B$. Idem for the maps $I^{n-1} \to \mathrm{Map}(I^s, A)$. Now the assertion follows directly, using the fact that p is a Serre fibration. $\square$

Proposition 2.1.5. *For any topological space X, the continuous map $i : X \to \mathrm{Map}(I^n, X)$ which maps each element x to the constant map c_x, is a homotopy equivalence, with homotopy inverse $j : \mathrm{Map}(I^n, X) \to X$ which maps each map σ to $\sigma(0)$.*

Proof. Obviously, $ji = \mathrm{id}_X$, so we only need to prove : $ij \simeq \mathrm{id}_{\mathrm{Map}(I^n,X)}$, i.e. there is a homotopy $H : \mathrm{Map}(I^n, X) \times I \to \mathrm{Map}(I^n, X)$, such that $H(\sigma, 0) = c_{\sigma(0)}$ and $H(\sigma, 1) = \sigma$, for any σ in $\mathrm{Map}(I^n, X)$. Since I^n is locally compact and Hausdorff, it is equivalent to find a homotopy $\widehat{H} : \mathrm{Map}(I^n, X) \times I^n \times I \to X$, where $\widehat{H}(\sigma, s, 0) = \sigma(0)$ and $\widehat{H}(\sigma, s, 1) = \sigma(s)$, for any σ in $\mathrm{Map}(I^n, X)$ and any s in I^n. Since I^n is contractible, there is a homotopy : $H' : I^n \times I \to I^n$ where $H'(s, 0) = 0$ and $H'(s, 1) = s$, so define $\widehat{H}(\sigma, s, t) := \sigma(H'(s, t))$. The map $\widehat{H}$ is continuous because it is the composite of the following continuous maps :

$$\mathrm{Map}(I^n, X) \times I^n \times I \xrightarrow{\mathrm{id} \times H'} \mathrm{Map}(I^n, X) \times I^n \xrightarrow{\text{evaluation}} X,$$

whence $\widehat{H}$ is indeed the desired homotopy. $\square$

Proposition 2.1.6 (The 5-Lemma)**.** *Consider the category* **Ab** *of abelian groups. Suppose the following commutative diagram :*

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "f"] \arrow[d, "\ell"] & B \arrow[r, "g"] \arrow[d, "m"] & C \arrow[r, "h"] \arrow[d, "n"] & D \arrow[r, "j"] \arrow[d, "p"] & E \arrow[r] \arrow[d, "q"] & \cdots \\
\cdots \arrow[r] & A' \arrow[r, "r"] & B' \arrow[r, "s"] & C' \arrow[r, "t"] & D' \arrow[r, "u"] & E' \arrow[r] & \cdots
\end{tikzcd}$$

where rows are exact sequences, m and p isomorphims, l surjective, and q injective. Then n is an isomorphism.

Proof. It is a classical diagram chasing argument. Let us prove the surjectivity. Let $c' \in C'$. Since p is surjective, $\exists d \in D$ such that $p(d) = t(c')$. By commutativity, $u(p(d)) = q(j(d))$. Since $\mathrm{im}\ t = \ker u$, we have : $0 = u(t(c')) = u(p(d)) = q(j(d))$. Since q injective, $j(d) = 0$. So, $d \in \ker j = \mathrm{im}\ h$. Whence $\exists c \in C$ such that $h(c) = d$. By commutativity, $t(n(c)) = p(h(c)) = t(c')$. Since t is a group homomorphism, $t(n(c) - c') = 0$. Whence $n(c) - c' \in \ker t = \mathrm{im}\ s$. So $\exists b' \in B'$ such that $s(b') = c' - n(c)$. But m is surjective, hence $\exists b \in B$ such that $m(b) = b'$. By commutativity, $n(g(b)) = s(m(b)) = c' - n(c)$. However, n is a group homomorphism, whence $n(g(b) + c) = c'$. Thus n surjective.
Let us prove now the injectivity. Let $c \in C$ such that $n(c) = 0$. Since t is a group homomorphism, $t(n(c)) = 0$. By commutativity, $p(h(c)) = 0$. But p is injective, so $h(c) = 0$. Whence $c \in \ker h = \mathrm{im}\ g$. So $\exists b \in B$ such that $g(b) = c$. By commutativity, $s(m(b)) = n(g(b)) = n(c) = 0$. Hence $m(b) \in \ker s = \mathrm{im}\ r$. Whence $\exists a' \in A'$ such that $r(a') = m(b)$. But l surjective, so $\exists a \in A$ such that $l(a) = a'$. By commutativity, we have now : $m(f(a)) = r(l(a)) = r(a') = m(b)$. Since m is injective : $f(a) = b$. Whence $c = g(f(a))$. But $\mathrm{im}\ f = \ker g$, thus $c = 0$. $\square$

Scholia 2.1. *Obviously, the proposition still holds for non abelian groups. Moreover, one can notice that we only used the group structure for the groups C, C' and D', with homomorphisms n and t. All the other groups and homomorphisms can be replaced by pointed sets and pointed set maps respectively, where l and q are no longer isomorphisms, but bijections. The "kernel" of a pointed set map would then be defined as the (pointed) set of all points of the domain which are maped into the basepoint of the target.*

We will also need subsequently the following result from category theory.

Proposition 2.1.7. *Let $\mathscr{C}$ be a fixed category. Consider a pullback square :*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "d"] & B \arrow[d, "b"] \\ D \arrow[r, "c"] & C \end{tikzcd}$$

Consider then the following commutative diagram :

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "f"] & A \arrow[d, "d"] \\ F \arrow[r, "g"] & D \end{tikzcd}$$

The statement says that the latter diagram is a pullback, if and only if the following induced diagram is a pullback :

$$\begin{tikzcd} E \arrow[r, "ae"] \arrow[d, "f"] & B \arrow[d, "b"] \\ F \arrow[r, "cg"] & C \end{tikzcd}$$

Proof. It is a simple argument using the universal property of the pullback. □

2.2 The Singular Homology with Coefficients in a Local System

We present here a generalization of the singular homology on a topological space. We use the notations of appendix A. Our work is based on [9].

Definition 2.2.1. A *local system* $\underline{A} = \{A_x, \tau_\gamma\}$ on a topological space X consists of two functions : the first assigns to each point x (regarded as a singular 0-simplex) of X an abelian group A_x, the second assigns to each path $\gamma : \Delta^1 \to X$ (regarded as a singular 1-simplex) of X a group homomorphism $\tau_\gamma : A_{\gamma\varepsilon_1} \to A_{\gamma\varepsilon_0}$ such that :

(i) if γ is constant, then τ_γ is the identity map;

(ii) for each singular 2-simplex $h : \Delta^2 \to X$, there is an identity :

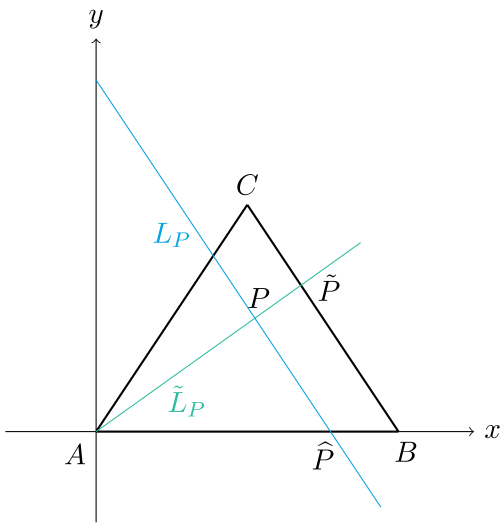
$$\tau_{h\varepsilon_0}\tau_{h\varepsilon_2} = \tau_{h\varepsilon_1}.$$

If $A_x = A$ for all x, where A is a fixed abelian group, and if all of the maps τ_γ are the identity map, then we say that $\underline{A}$ is a *constant local system with value A.*

Lemma 2.2.2. *Let $\underline{A} = \{A_x, \tau_\gamma\}$ be a local system on X. If two paths λ and λ' in X are homotopic : $\lambda \simeq_* \lambda'$, then $\tau_\lambda = \tau_{\lambda'}$.*

Proof. Consider the paths $\lambda, \lambda' : I \to X$ start at $x_0 = \lambda(0) = \lambda'(0)$ and end at $x_1 = \lambda(1) = \lambda'(1)$. There is a path homotopy $H : I \times I \to X$, such that $H(t,0) = \lambda(t)$, $H(t,1) = \lambda'(t)$, $H(0,s) = x_0$, $H(1,s) = x_1$, for all t and s in I. We need to define a singular 2-simplex $h : \Delta^2 \to X$, such that $h\varepsilon_2 = \lambda$, $h\varepsilon_1 = \lambda'$ and $h\varepsilon_0 = c_{x_1}$.
We will give an explicit formula, using geometry. Regard (via an affine map) Δ^2 as a subspace of $\mathbb{R}^2$: a triangle (with interior) with vertices $A = (0,0)$, $B = (1,0)$ and $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The equation of the line (BC) is : $y = -\sqrt{3}x + \sqrt{3}$. Now let $P = (p_1, p_2)$ be any point in Δ^2 except the point $A = (0,0)$. Let L_P be the line passing through P, parallel to the line (BC). Its equation is of the form : $y = -\sqrt{3}x + \alpha$. Since the point P belongs to L_P, we get $\alpha = \sqrt{3}p_1 + p_2$. Let $\hat{P} = (\hat{p}, 0)$ be the point of intersection of the lines L_P and (AB). The equation of the line L_P gives us that $\hat{p} = p_1 + \frac{\sqrt{3}}{3}p_2$. Now let $\tilde{L}_P$ be the line passing through A and P, and let $\tilde{P}$ be the point of intersection of $\tilde{L}_P$ with the line (BC). Denote m_P the value of $\|B\tilde{P}\|$. By Thales, we get $m_P = \frac{\|AB\| \cdot \|P\hat{P}\|}{\|A\hat{P}\|}$. Since $\|AB\| = 1$, and by computation $\|P\hat{P}\| = \frac{2\sqrt{3}}{3}p_2$, we get : $m_P = \frac{2p_2}{\sqrt{3}p_1 + p_2}$. Now define a map :

$$\begin{aligned} f : \Delta^2 \setminus \{(0,0)\} &\longrightarrow I^2 \setminus (\{0\} \times I) \\ P = (p_1, p_2) &\longmapsto (\|A\hat{P}\|, \|B\tilde{P}\|) = (\hat{p}, m_P) = (p_1 + \frac{\sqrt{3}}{3}p_2, \frac{2p_2}{\sqrt{3}p_1 + p_2}). \end{aligned}$$



It is well-defined, continuous and bijective. Now we can define the map h by :

$$\begin{aligned} h : \Delta^2 &\longrightarrow X \\ (p_1, p_2) &\longmapsto \begin{cases} Hf(p_1, p_2), & \text{if } (p_1, p_2) \neq (0,0), \\ x_0, & \text{if } (p_1, p_2) = (0,0). \end{cases} \end{aligned}$$

We must check that h is continuous. Since the maps H and f are both continuous, we only need to check the continuity at $(0,0)$. Let $(a_n), (b_n) \subset \mathbb{R}$ be sequences such that (a_n, b_n) are in Δ^2 for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Take U an open set in X containing x_0. Since H is continuous, the pre-image $H^{-1}(U)$ is an open set in I^2 containing $\{0\} \times I$. For any s in I, by continuity of H, there exists $V_s := [0, d_s[\times (]c_s, c'_s[\cap I) \subseteq H^{-1}(U)$ open set in I^2, such that $(0,s) \in V_s$ and : $H(0,s) = x_0 \in H(V_s) \subseteq U$. Whence $\{0\} \times I \subseteq \bigcup_{s \in I} V_s$. Since $\{0\} \times I$ is

compact in I^2, there exists $s_1, \dots, s_r$ such that $\{0\} \times I \subseteq \bigcup_{i=1}^r V_{s_i}$. Let m be the minimum of d_{s_i}, for all $i = 1, \dots, r$. Since $\lim_{n\to\infty} a_n + \frac{\sqrt{3}}{3} b_n = 0$, there exists $N \in \mathbb{N}$, such that $a_n + \frac{\sqrt{3}}{3} b_n \leq m$, for any $n \geq N$. Hence $f(a_n, b_n) \in \bigcup_{i=1}^r V_{s_i}$, for all $n \geq N$. Thus $h(a_n, b_n) \in U$, for all $n \geq N$. We get : $\lim\limits_{n\to\infty} Hf(a_n, b_n) = x_0$, so h is continuous.
Now one can see that $h\varepsilon_2 = \lambda$, $h\varepsilon_1 = \lambda'$ and $h\varepsilon_0 = c_{x_1}$. The equality $\tau_{h\varepsilon_0}\tau_{h\varepsilon_2} = \tau_{h\varepsilon_1}$ becomes $\tau_{c_{x_1}}\tau_\lambda = \tau_{\lambda'}$. Since c_{x_1} is a constant map, $\tau_{c_{x_1}}$ is the identity map, whence $\tau_\lambda = \tau_{\lambda'}$. □

Proposition 2.2.3. *For any local system $\underline{A} = \{A_x, \tau_\gamma\}$ on X, the maps τ_γ are isomorphism, with inverses $\tau_{\overline{\gamma}}$, where $\overline{\gamma}$ is the inverse path of γ.*

Proof. For any two paths γ and γ' in X, if their concatenation $\gamma \star \gamma'$ is well defined, then $\tau_{\gamma\star\gamma'} = \tau_{\gamma'}\tau_\gamma$. Now choose γ' to be $\overline{\gamma}$. Since $\gamma \star \overline{\gamma} \simeq_* c_{\gamma(0)}$, the constant map at $\gamma(0)$, we get $\tau_{\overline{\gamma}}\tau_\gamma = \mathrm{id}_{A_{\gamma(0)}}$, and whence τ_γ is an isomorphism. □

Local systems on a topological space X form a category in which a map $\underline{A} \to \underline{B}$ is defined to be a collection of maps $\{A_x \to B_x\}_{x\in X}$, such that for every singular 1-simplex γ, the following diagram commutes :

$$\begin{array}{ccc} A_{\gamma\varepsilon_1} & \longrightarrow & B_{\gamma\varepsilon_1} \\ \big\downarrow & & \big\downarrow \\ A_{\gamma\varepsilon_0} & \longrightarrow & B_{\gamma\varepsilon_0} \end{array}$$

Recall that the *fundamental groupoid*[1] of a topological space X, denoted $\Pi_1(X)$, is a category where the objects are the points of X, the maps are the homotopy classes of paths relative to the boundary, and composition comes from concatenation of paths. A local system in X is then equivalent to the data of a functor from $\Pi_1(X)$ to **Ab** : it follows directly from lemma 2.2.2.

Proposition 2.2.4. *If X is a simply connected topological space, then any local system on X is isomorphic to a constant local system on X.*

Proof. Fix $x_0 \in X$. We will write $[\lambda]_*$ for the homotopy class of a path λ relative to the endpoints. Let $\underline{A} = \{A_x, \tau_\gamma\}$ be a local system on X. It is equivalent to a functor $F : \Pi_1(X) \to$ **Ab**. A constant local system on X can be regarded as a functor $G : \Pi_1(X) \to$ **Ab** which sends each object x in $\Pi_1(X)$ to A_{x_0}, and each morphism $[\lambda]_*$ to the identity map. To prove the theorem, we only need to show there is a natural isomorphism $\eta : F \Rightarrow G$, that is to say a family $\{\eta_x : A_x \to A_{x_0}\}_{x\in X}$ of abelian group isomorphisms, such that for each class $[\lambda]_*$ with endpoints x and y, we have the following commutative diagram :

$$\begin{array}{ccc} A_x & \xrightarrow{\tau_\lambda} & A_y \\ \big\downarrow{\scriptstyle \eta_x} & & \big\downarrow{\scriptstyle \eta_y} \\ A_{x_0} & \xrightarrow{\mathrm{id}} & A_{x_0} \end{array} \tag{2.1}$$

For each point x in X, since X is simply connected, define γ_x to be the unique path, up to relative homotopy with endpoints, from x to x_0, and define then $\eta_x := \tau_{\gamma_x}$. It is well defined by lemma 2.2.2 and it is an isomorphism by proposition 2.2.3. Now for any path λ in X from x to y, we have $[\gamma_y \star \lambda]_* = [\gamma_x]_*$, because X is simply connected. So we get $\eta_y\tau_\lambda = \eta_x$ by lemma 2.2.2, and so we have proved the commutativity of (2.1). □

[1] Also named the *Poincaré groupoid* of X.

We will often use the following notation subsequently.

Notation 2.2.5. Let X be topological space. For any singular n-simplex $\sigma : \Delta^n \to X$, for any $0 \leq i \leq n$, write $\sigma_i = \sigma(i)$. In addition, for any $0 \leq i \leq j \leq n$ define the singular 1-simplex $\sigma_{ij} : \Delta^1 \to X$ to be the composite :

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[rr, bend right, "\sigma_{ij}"'] & \Delta^n \arrow[r, "\sigma"] & X \end{tikzcd}$$

where the unlabeled map is defined as the affine map sending 0 and 1, to i and j respectively.

Construction of the Homology with Coefficients in a Local System Let X be a topological space, and $n \in \mathbb{N}$. Let $\underline{A} = \{A_x, \tau_\gamma\}$ be a local system on X. We want to construct a chain complex using $\underline{A}$. Define the abelian group $C_n(X; \underline{A})$ to be :

$$C_n(X; \underline{A}) := \bigoplus_{\sigma : \Delta^n \to X} A_{\sigma_0}.$$

Set $C_{-1}(X; \underline{A}) := 0$. We need to define the boundary operator $\partial : C_n(X; \underline{A}) \to C_{n-1}(X; \underline{A})$. Recall the universal property of direct sums of abelian groups : for any direct sum $\bigoplus_{j \in \mathscr{J}} A_j$ of abelian groups, for any abelian group B and group homomorphisms $\varphi_k : A_k \to B$, where k in $\mathscr{J}$, there is a unique group homomorphism $\tilde{\varphi} : \bigoplus_{j \in \mathscr{J}} A_j \to B$ such that the following diagram commutes for each k.

$$\begin{tikzcd} A_k \arrow[d, hook] \arrow[dr, "\varphi_k"] & \\ \bigoplus_{j \in \mathscr{J}} A_j \arrow[r, dashed, "\tilde{\varphi}"'] & B \end{tikzcd}$$

Now, as we did for the construction of the singular complex $S(X_\bullet)$ in appendix A, we will use the face maps ε_i. Since $C_n(X; \underline{A})$ is defined by using σ_0, we need to see how the face map acts on σ_0, where σ is a singular n-simplex. We have :

$$(\sigma\varepsilon_i)_0 = \begin{cases} \sigma_0, & \text{if } i > 0, \\ \sigma_1, & \text{if } i = 0. \end{cases}$$

So the case $i > 0$ will be easy. Define ∂_n^i to be the unique group homomorphism such that the following diagram commutes :

$$\begin{tikzcd} A_{\sigma_0} \arrow[d, hook] \arrow[dr] & \\ C_n(X; \underline{A}) \arrow[r, dashed, "\partial_n^i"'] & C_{n-1}(X; \underline{A}) \end{tikzcd}$$

where the unlabeled map is defined as the identity map which sends A_{σ_0} to $A_{(\sigma\varepsilon_i)_0} = A_{\sigma_0}$, followed by the inclusion to $C_{n-1}(X; \underline{A})$. We have only regarded the singular n-simplex σ as a singular $(n-1)$-simplex thanks to ε_i.
For the case $i = 0$, since $(\sigma\varepsilon_0)_0 = \sigma_1$, consider the singular 1-simplex σ_{01} : the path from σ_0 to σ_1, use the induced group isomorphim $\tau_{\sigma_{01}} : A_{\sigma_0} \to A_{\sigma_1}$ given by the structure of the

local system $\underline{A}$, and finally regard A_{σ_1} as $A_{(\sigma\varepsilon_0)_0}$. Namely, define ∂_n^0 to be the unique group homomorphism such that the following diagram commutes :

$$\begin{tikzcd}
A_{\sigma_0} \arrow[d, hook] \arrow[dr] & \\
C_n(X;\underline{A}) \arrow[r, dashed, "\partial_n^0"'] & C_{n-1}(X;\underline{A})
\end{tikzcd}$$

where the unlabeled map is defined to be the composite :

$$\begin{tikzcd}
A_{\sigma_0} \arrow[r, "\tau_{\sigma_{01}}"] \arrow[rr, bend right] & A_{(\sigma\varepsilon_0)_0} = A_{\sigma_1} \arrow[r, hook] & C_{n-1}(X;\underline{A})
\end{tikzcd}$$

Finally, we define the boundary operator for all $n > 0$ as :

$$\partial_n := \sum_{i=0}^{n}(-1)^i\partial_n^i,$$

and set $\partial_0 := 0$. To prove now that $C_\bullet(X;\underline{A}) := \{C_n(X;\underline{A}), \partial\}$ is a chain complex, we need to show that $\partial\partial = 0$. It suffices to show that $\partial_n\partial_{n+1}(a,\sigma) = 0$ for all elements $(a,\sigma) \in A_{\sigma_0}$, where σ is a singular $(n+1)$-simplex (we indexed the element a of the group by σ).

$$\begin{aligned}
\partial\partial(a,\sigma) &= \partial\left(\sum_i(-1)^i\partial_{n+1}^i(a,\sigma)\right) \\
&= \partial\left((\tau_{\sigma_{01}}(a), \sigma\varepsilon_0^{n+1}) + \sum_{i>0}(-1)^i(a,\sigma\varepsilon_i^{n+1})\right) \\
&= \left(\tau_{(\sigma\varepsilon_0^{n+1})_{01}}\tau_{\sigma_{01}}(a), \sigma\varepsilon_0^{n+1}\varepsilon_0^n\right) + \sum_{i>0}(-1)^i\left(\tau_{(\sigma\varepsilon_i^{n+1})_{01}}(a), \sigma\varepsilon_i^{n+1}\varepsilon_0^n\right) \\
&\quad + \sum_{j>0}(-1)^j\left(\tau_{\sigma_{01}}(a), \sigma\varepsilon_0^{n+1}\varepsilon_j^n\right) + \underbrace{\sum_{i,j>0}(-1)^{i+j}(a,\sigma\varepsilon_i^{n+1}\varepsilon_j^n)}_{=0,\text{ using (A.1) page 40}} \\
&= \underbrace{\left(\tau_{(\sigma\varepsilon_0^{n+1})_{01}}\tau_{\sigma_{01}}(a), \sigma\varepsilon_0^{n+1}\varepsilon_0^n\right) - \left(\tau_{(\sigma\varepsilon_1^{n+1})_{01}}(a), \underbrace{\sigma\varepsilon_1^{n+1}\varepsilon_0^n}_{=\sigma\varepsilon_0^{n+1}\varepsilon_0^n}\right)}_{=0,\text{ because : } \tau_{(\sigma\varepsilon_0^{n+1})_{01}}\tau_{\sigma_{01}}=\tau_{(\sigma\varepsilon_1^{n+1})_{01}} \text{ by } \textbf{(ii)} \text{ of 2.2.1}} \\
&\quad + \sum_{i>1}(-1)^i\left(\tau_{(\sigma\varepsilon_i^{n+1})_{01}}(a), \sigma\varepsilon_i^{n+1}\varepsilon_0^n\right) + \sum_{j>0}(-1)^j\left(\tau_{\sigma_{01}}(a), \sigma\varepsilon_0^{n+1}\varepsilon_j^n\right) \\
&= \sum_{q>0}(-1)^{q+1}\left(\tau_{(\sigma\varepsilon_{q+1}^{n+1})_{01}}(a), \sigma\varepsilon_0^{n+1}\varepsilon_q^n\right) + \sum_{j>0}(-1)^j\left(\tau_{\sigma_{01}}(a), \sigma\varepsilon_0^{n+1}\varepsilon_j^n\right), \text{ by (A.1) page 40,} \\
&= 0, \text{ because } (\sigma\varepsilon_{q+1}^{n+1})_{01} = \sigma_{01}, \ \forall q > 0.
\end{aligned}$$

Definition 2.2.6. The *n-th singular homology group $H_n(X;\underline{A})$ of a topological space X with coefficients in a local system $\underline{A}$* is the n-th homology group $H_n(C_\bullet(X;\underline{A}))$ of the complex $C_\bullet(X;\underline{A})$.

Remark 2.2.7. When $\underline{A}$ is the constant system with value in $\mathbb{Z}$, for any topological space X, we have $H_n(X;\mathbb{Z}) = H_n(X)$, which *is* the usual singular homology from appendix A (hence the fact that we often specify «with integer coefficients»).

We give a very useful result that will help us to compute homologies with constant coefficients.

Theorem 2.2.8. *Let X be a topological space and A any abelian group. Then X is path-connected if and only if $H_0(X;A) \cong A$.*

Proof. It is the same argument as theorem A.4.2 (we used only the abelian structure of $\mathbb{Z}$). $\square$

Theorem 2.2.9. *Let A be any abelian group. We have :*

$$\begin{cases} H_0(S^0;A) = A \oplus A, \\ H_k(S^0;A) = 0, \textit{if } k > 0, \\ H_0(S^n;A) = H_n(S^n;A) = A, \textit{if } n > 0, \\ H_k(S^n;A) = 0, \textit{if } k \neq 0, n. \end{cases}$$

Proof. Omitted. $\square$

Local System Induced by the Fibers of a Serre Fibration We want now to exhibit an example of local systems, using Serre fibrations, that will be useful subsequently. We need the following general result.

Lemma 2.2.10. *Suppose that :*

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

is a pullback square in **Top***, in which $p : E \to B$ is a Serre fibration. If $g : B' \to B$ is a weak equivalence, then so is $f : E' \to E$.*

Proof. By proposition 2.1.3, the map $p' : E' \to B'$ is a Serre fibration. Choose the basepoints $e_0' \in E'$, $e_0 := f(e_0') \in E$, $b_0 := p(e_0) \in B$, and $b_0' := p'(e_0') \in B'$. Set the fibers of the Serre fibrations $F := p^{-1}(b_0) \subseteq E$ and $F' := p^{-1}(b_0') \subseteq E'$. Name $h : F' \to F$ the restriction and corestriction of f. Let us prove that h is a homeomorphism. Note that we have the two following pullbacks :

$$\begin{tikzcd} F' \arrow[r, hook] \arrow[d, "p'"] & E' \arrow[d, "p'"] \\ b_0' \arrow[r, hook] & B' \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, hook] \arrow[d, "p"] & E \arrow[d, "p"] \\ b_0 \arrow[r, hook] & B \end{tikzcd}$$

Whence, the former diagram together with the diagram of the statement, induce the following pullback (use the proposition 2.1.7):

$$\begin{tikzcd} F' \arrow[r, hook] \arrow[d, "p'"] & E' \arrow[r, "f"] & E \arrow[d, "p"] \\ b_0' \arrow[r, hook] & B' \arrow[r, "g"] & B \end{tikzcd}$$

Thus, by the universal property of pullbacks, $F \cong F'$. In particular, h is a weak equivalence. The homotopy sequence of a Serre fibration enounced in the proposition 2.1.2 gives rise to two exact sequences : the rows of the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(B') \arrow[r] \arrow[d, "g_*"] & \pi_n(F') \arrow[r] \arrow[d, "h_*"] & \pi_n(E') \arrow[r] \arrow[d, "f_*"] & \pi_n(B') \arrow[r] \arrow[d, "g_*"] & \pi_{n-1}(F') \arrow[r] \arrow[d, "h_*"] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}(B) \arrow[r] & \pi_n(F) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(B) \arrow[r] & \pi_{n-1}(F) \arrow[r] & \cdots
\end{tikzcd}$$

The scholia 2.1 proves that f_* is a group isomorphism for $n \geq 1$. We need to show now that f_* is a bijection for the case $n = 0$. Let us prove first that $f_* : \pi_0(E') \to \pi_0(E)$ is surjective. Recall that $\pi_0(X)$ of a topological space X is the set of path components of X. We will write P_x for the path component containing $x \in X$. Recall that since E' is a pullback, it can be written as $B' \times_B E := \{(b', e) \in B' \times E \mid g(b') = p(e)\}$, and both p' and f are regarded as projection maps. Take $P_e \in \pi_0(E)$. To prove the surjectivity, one must find $e' \in E'$, such that there exists a path in E from e to $f(e')$. Define $b := p(e)$ and consider $P_b \in \pi_0(B)$. Since g is a weak equivalence, the induced map $g_* : \pi_0(B') \to \pi_0(B)$ is a bijection. Thus, there exists $b' \in B'$ such that $g_*(P_{b'}) = P_b$. Whence there exists a path $\lambda : I \to B$ where $\lambda(0) = b = p(e)$, and $\lambda(1) = g(b')$. Using the homotopy lifting property of the Serre fibration p :

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
I \arrow[r, "\lambda"] \arrow[ur, dashed, "\lambda'"] & B
\end{tikzcd}$$

where the unlabeled map maps 0 to e, there exists a path $\lambda' : I \to E$ where $\lambda'(0) = e$ and $p\lambda'(1) = \lambda(1) = g(b')$. Take $e' := (b', \lambda'(1)) \in E'$. We have now that $f(e') = \lambda'(1)$. Hence, λ' is a path in E from e to $f(e')$, which ends the proof of the surjectivity.

Now, we need to show that f_* is injective. It is rather simple, but one needs to be careful. The proof will be somehow very similar to the 5-lemma. The idea is to choose the basepoints such that one can always work with the fibers. Take e'_1 and e'_2 in E' such that there exists a path in E from $f(e'_1)$ to $f(e'_2)$. We need to show that there is a path in E' between e'_1 and e'_2. One can notice that the definition of $\pi_0(E')$ and $\pi_0(E)$ is independent of the choice of basepoints e'_0 and e_0, and hence this same fact holds for $f_* : \pi_0(E') \to \pi_0(E)$. Thus, one can set the basepoints of E' and E to be e'_1 and $f(e'_1)$ respectively ; and set $p'(e'_1)$ and $p(f(e'_1))$ for the basepoints of B' and B respectively. The fibers F, F' will although change, but the same consequences hold : one can define the weak equivalence h as before, and p and p' give rise to two exact sequences. In particular, with the inclusions $i' : F' \hookrightarrow E'$, $i : F \hookrightarrow E$, we have :

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(B') \arrow[r, "\delta'"] \arrow[d, "g_*"] & \pi_0(F') \arrow[r, "i'_*"] \arrow[d, "h_*"] & \pi_0(E') \arrow[r, "p'_*"] \arrow[d, "f_*"] & \pi_0(B') \arrow[d, "g_*"] \\
\cdots \arrow[r] & \pi_1(B) \arrow[r, "\delta"] & \pi_0(F) \arrow[r, "i_*"] & \pi_0(E) \arrow[r, "p_*"] & \pi_0(B)
\end{tikzcd} \tag{2.2}$$

Since there is a path in E from $f(e'_1)$ to $f(e'_2)$, there is an induced path in B from $pf(e'_1)$ to $pf(e'_2)$. Whence, by commutativity of the diagram (2.2), we get :

$$g_* p'_*(P_{e'_1}) = p_* f_*(P_{e'_1}) = p_* f_*(P_{e'_2}) = g_* p'_*(P_{e'_2}).$$

By bijectivity of g_*, we get : $p'_*(P_{e'_1}) = p'_*(P_{e'_2})$, so there is a path in B' from $p'(e'_1)$ to $p'(e'_2)$. Hence $P_{e'_1}$ and $P_{e'_2}$ are in the «kernel» of p'_* which is equal to the image of i'_* by exactness of the top row of (2.2) : there exist x'_1 and x'_2 in F' such that there are paths in E' from x'_1 to e'_1 and from x'_2 to e'_2. Since x'_1 and e'_1 are both in the fibers, we can make the assumption that $x'_1 = e'_1$. Now, in order to prove the injectivity, we only need to prove that the path component $P_{x'_2}$ lies in the «kernel» of i'_*, which is defined by : $\ker i'_* = \{P_x \in \pi_0(F') \mid i'_*(P_x) = P_{e'_1}\}$. By commutativity of (2.2) :

$$i_*(h_*(P_{x'_2})) = f_*(i'_*(P_{x'_2})) = f_*(P_{e'_2}) = f_*(P_{e'_1}).$$

Then $h_*(P_{x'_2}) \in \ker i_* = \{P_x \in \pi_0(F) \mid i_*(P_x) = P_{f(e'_1)} = f_*(P_{e'_1})\}$. By exactness, this kernel equals to im δ. Whence there exists b in B such that $\delta(P_b) = h_*(P_{x_2})$. Since the map g_* is surjective, there exists b' in B' such that $g_*(P_{b'}) = P_b$. So by commutativity, we get :

$$h_*(\delta'(P_{b'})) = \delta(g_*(P_{b'})) = \delta(P_b) = h_*(P_{x'_2}).$$

By bijectivity of h_*, we get $P_{x'_2} = \delta'(P_{b'})$. Whence we obtain : $i'_*(\delta(P_{b'})) = i'_*(P_{x'_2}) = P_{e'_2}$. Since we have the exactness : im $\delta' = \ker i'_*$, we get $P_{x'_2} \in \ker i'_*$. Thus : $P_{e'_2} = i'_*(P_{x'_2}) = P_{e'_1}$. □

Example 2.2.11. Suppose $p : E \to B$ is a Serre fibration, and $q \in \mathbb{Z}$ a fixed integer. For each x in B we use the notation $F_x := p^{-1}(x)$: the fiber of x over B. Define the *local system induced by the fibers of the Serre fibration* p : $\underline{A} = \{A_x, \tau_\gamma\}$ as follows. The abelian groups are defined by $A_x := H_q(F_x)$, for each x in B. Given a singular 1-simplex $\gamma : \Delta^1 \to B$, define $E_\gamma \to \Delta^1$ by the pullback square :

$$\begin{tikzcd} E_\gamma \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta^1 \arrow[r, "\gamma"] & B \end{tikzcd}$$

that is to say : $E_\gamma = \Delta^1 \times_B E$, and the unlabeled maps are the projections maps. We have then a diagram with pullbacks at fibers :

$$\begin{tikzcd} F_{\gamma(0)} \arrow[r] \arrow[d] & E_\gamma \arrow[d] & F_{\gamma(1)} \arrow[l] \arrow[d] \\ \{0\} \arrow[r, hook] & \Delta^1 & \{1\} \arrow[l, hook'] \end{tikzcd} \tag{2.3}$$

Clearly, the maps of the bottom row of (2.3) are weak equivalences. The map $E_\gamma \to \Delta^1$ is a Serre fibration (use proposition 2.1.3). Whence, by the lemma 2.2.10, the maps of the upper row of (2.3) are weak equivalences. Thus, we have the induced group isomorphisms $H_q(F_{\gamma(0)}) \to H_q(E_\gamma)$ and $H_q(F_{\gamma(1)}) \to H_q(E_\gamma)$. The latter isomorphism induces an isomorphism $H_q(E_\gamma) \to H_q(F_{\gamma(1)})$ which is its inverse. Define then τ_γ as the composite :

$$\begin{tikzcd} H_q(F_{\gamma(0)}) \arrow[r, "\cong"] \arrow[rr, bend right, "\tau_\gamma"'] & H_q(E_\gamma) \arrow[r, "\cong"] & H_q(F_{\gamma(1)}) \end{tikzcd} \tag{2.4}$$

Of course, if γ is constant, τ_γ is the identity map. To prove that $\underline{A}$ is indeed a local system on B, we must now show that for any singular 2-simplex $h : \Delta^2 \to B$, the following identity holds :

$$\tau_{h\varepsilon_0}\tau_{h\varepsilon_2} = \tau_{h\varepsilon_1}.$$

Define E_h as the following pullback :

$$\begin{tikzcd} E_h \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta^2 \arrow[r, "h"] & B \end{tikzcd}$$

The identity follows from the commutativity of the following diagram :

$$\begin{tikzcd} & & F_{h(2)} \arrow[dl] \arrow[dd] \arrow[dr] & & \\ & E_{h\varepsilon_1} \arrow[dr] & & E_{h\varepsilon_0} \arrow[dl] & \\ & & E_h & & \\ F_{h(0)} \arrow[uur] \arrow[urr] \arrow[rr] & & E_{h\varepsilon_2} \arrow[u] & & F_{h(1)} \arrow[ll] \arrow[uul] \arrow[ull] \end{tikzcd} \tag{2.5}$$

where the maps $F_{h(i)} \to E_h$ and $E_{h\varepsilon_i} \to E_h$, with $i = 0, 1, 2$, are defined as follows. Recall that $F_{h(0)}$ and $E_{h\varepsilon_2}$ are obtained as the pullbacks of the following diagrams respectively :

$$\begin{tikzcd} F_{h(0)} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \{h(0)\} \arrow[r, hook] & B \end{tikzcd} \qquad \begin{tikzcd} E_{h\varepsilon_2} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta^1 \arrow[r, "h\varepsilon_2"] & B \end{tikzcd}$$

But one can define $F_{h(0)}$ and $E_{h\varepsilon_2}$ equivalently as the pullbacks of the induced diagram (use repeatedly the proposition 2.1.7) :

$$\begin{tikzcd} F_{h(0)} \arrow[r] \arrow[d] \arrow[rr, dashed, bend left] & E_{h\varepsilon_2} \arrow[r, dashed] \arrow[d] & E_h \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \{0\} \arrow[r, hook] & \Delta^1 \arrow[r, "\varepsilon_2"] & \Delta^2 \arrow[r, "h"] & B \end{tikzcd}$$

and let $F_{h(0)} \to E_h$ and $E_{h\varepsilon_2} \to E_h$ be the dashed maps. The commutativity of the triangle diagram follows immediately by construction :

$$\begin{tikzcd} & E_h \\ F_{h(0)} \arrow[ur] \arrow[r] & E_{h\varepsilon_2} \arrow[u] \end{tikzcd}$$

Now, we wish do define $F_{h(0)} \to E_{h\varepsilon_1}$. By the same argument, one can take $E_{h\varepsilon_1}$ as the pullback square of the diagram :

$$\begin{tikzcd} E_{h\varepsilon_1} \arrow[r] \arrow[d] & E_h \arrow[d] \\ \Delta^1 \arrow[r, "\varepsilon_1"] & \Delta^2 \end{tikzcd}$$

Whence, by the universal property of the pullbacks and the proposition 2.1.7 :

$$\begin{tikzcd}
F_{h(0)} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, bend right] & & & \\
& E_{h\varepsilon_1} \arrow[r] \arrow[d] & E_h \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\{0\} \arrow[r, hook] & \Delta^1 \arrow[r, "\varepsilon_1"] & \Delta^2 \arrow[r, "h"] & B
\end{tikzcd}$$

there is a unique map (dashed in the diagram) $F_{h(0)} \to E_{h\varepsilon_1}$ such that the diagram commutes. Whence, by construction, we have the commutativity of the following diagram which corresponds to left bottom corner of (2.5).

$$\begin{tikzcd}
E_{h\varepsilon_1} \arrow[r] & E_h \\
F_{h(0)} \arrow[u] \arrow[ur] \arrow[r] & E_{h\varepsilon_2} \arrow[u]
\end{tikzcd}$$

One can get the whole diagram (2.5) inductively. Thus we have showed that the diagram (2.5) commutes. All the maps $F_{h(i)} \to E_h$ and $E_{h\varepsilon_i} \to E_h$, with $i = 0, 1, 2$, are weak equivalences (use the lemma 2.2.10). Using directly the definition (2.4) of τ_γ, the identity follows : $\tau_{h\varepsilon_0}\tau_{h\varepsilon_2} = \tau_{h\varepsilon_1}$.

2.3 Dress' Construction

We now finally introduce the *Serre spectral sequence*. It is sometimes called the *Leray-Serre spectral sequence* to acknowledge earlier work of Jean Leray in the Leray spectral sequence. The result is due to the great mathematician Jean-Pierre Serre in his doctoral dissertation [15], in 1951. We will give an elegant construction due to Andreas Dress, from his article [4], in 1967. Our work is based on [10]. Basically, the idea is to construct a double complex from any Serre fibration in order to apply theorem 1.3.5. We will get two spectral sequences. One will be the Serre spectral sequence of the Serre fibration, and the other will help us to determine the convergence of the Serre spectral sequence.

Theorem 2.3.1 (Leray-Serre). *Let $f : E \to B$ be a Serre fibration. Then, there is a first quadrant spectral sequence $\{E^r, d^r\}_{r\geq 2}$, called the* Serre spectral sequence*, with second page $E^2_{p,q} = H_p(B; \underline{H_q}(F))$: the singular homology with coefficients in the local system induced by the fibers of f. The spectral sequence converges to the singular homology of the topological space E :*

$$E^2_{p,q} = H_p(B; \underline{H_q}(F)) \Rightarrow H_{p+q}(E).$$

Proof. We will apply the theorem 1.3.5, so we will construct a double complex. For all natural numbers p, q, consider the set of continuous maps :

$$\mathscr{S}_{p,q} = \{(\sigma_{p,q}, \tau_p) \mid \sigma_{p,q} : \Delta^p \times \Delta^q \to E, \tau_p : \Delta^p \to B \text{ continuous with } f\sigma_{p,q} = \tau_p pr_1\},$$

where pr_1 is the projection $\Delta^p \times \Delta^q \to \Delta^p$. Thus $(\sigma_{p,q}, \tau_p)$ belongs to $\mathscr{S}_{p,q}$ if the following diagram commutes :

$$\begin{tikzcd}
\Delta^p \times \Delta^q \arrow[r, "\sigma_{p,q}"] \arrow[d, "pr_1"'] & E \arrow[d, "f"] \\
\Delta^p \arrow[r, "\tau_p"'] & B
\end{tikzcd} \tag{2.6}$$

Apply the free abelian group functor $\mathscr{F}_{Ab}$ (defined in appendix A) : define for all $p, q \geq 0$:

$$K_{p,q} = \mathscr{F}_{Ab}(\mathscr{S}_{p,q}),$$

and set $K_{p,q} = 0$ whenever $p < 0$ or $q < 0$. We must now define the horizontal and vertical boundaries : for $p, q > 0$,

$$\begin{array}{rcl} \partial'_{p,q} : K_{p,q} & \longrightarrow & K_{p-1,q} \\ (\sigma_{p,q}, \tau_p) & \longmapsto & \displaystyle\sum_{i=0}^{p} (-1)^i \left(\sigma_{p,q}(\varepsilon_i^p \times \mathrm{id}_{\Delta^q}), \tau_p \varepsilon_i^p\right) \\ \partial''_{p,q} : K_{p,q} & \longrightarrow & K_{p,q-1} \\ (\sigma_{p,q}, \tau_p) & \longmapsto & \displaystyle\sum_{j=0}^{q} (-1)^{j+p} (\sigma_{p,q}(\mathrm{id}_{\Delta^p} \times \varepsilon_i^q), \tau_p), \end{array}$$

and set $\partial'_{p,q} = \partial''_{p,q} = 0$ whenever $p \leq 0$ or $q \leq 0$. We get : $\partial'\partial' = \partial''\partial'' = \partial''\partial' + \partial'\partial'' = 0$. Thus $(K, \partial', \partial'')$ is a double complex. Now by theorem 1.3.5, we get two first quadrant spectral sequences $\{{}^{\mathrm{I}}E^r, {}^{\mathrm{I}}d^r\}$ and $\{{}^{\mathrm{II}}E^r, {}^{\mathrm{II}}d^r\}$. The former will be the *Serre spectral sequence*. The latter will help us to compute the homology of the total complex associated to the double complex.

Let us work first with the second spectral sequence $\{{}^{\mathrm{II}}E^r, {}^{\mathrm{II}}d^r\}$. We will prove that the homology of the total complex is the singular homology (with integer coefficients) of the topological space E : $H_{p+q}(\mathrm{tot}(K)_\bullet) = H_{p+q}(E)$. We drop the exponent II in our discussion. We start by computing the E^1-page of the spectral sequence. By theorem 1.3.5, it is given by the homology of the rows : $E^1_{p,q} = H'_{p,q}(K)$.
We reinterpret our construction of the double complex. Fix q, and think p as varying. Recall the diagram (2.6). Since we have the homeomorphism $\Delta^q \cong I^q$, the standard q-simplex is locally compact and Hausdorff, so there is a bijection between continuous maps $\sigma_{p,q} : \Delta^p \times \Delta^q \to E$ and continuous maps $\widehat{\sigma}_{p,q} : \Delta^p \to \mathrm{Map}(\Delta^q, E)$, where $\widehat{\sigma}_{p,q}(t)(s) = \sigma_{p,q}(t, s)$, for all $t \in \Delta^p$ and $s \in \Delta^q$. So one can regard the top row of the diagram (2.6) as a map $\Delta^p \to \mathrm{Map}(\Delta^q, E)$.
Now consider the topological space E' defined by the pullback square :

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^q, E) \arrow[d, "\widehat{f}"] \\ B \arrow[r, "i"'] & \mathrm{Map}(\Delta^q, B) \end{tikzcd}$$

where the map $i : B \to \mathrm{Map}(\Delta^q, B)$ assigns to every point b in B, the constant map $c_b : \Delta^q \to B$ in b. Define the set $\mathscr{P}_{p,q}$ to be all continuous maps $\Delta^p \to E'$. We claim that there is a bijection between $\mathscr{P}_{p,q}$ and $\mathscr{S}_{p,q}$.
For all $(\sigma_{p,q}, \tau_p) \in \mathscr{S}_{p,q}$, since E' is defined as a pullback, there is a (unique) continuous map $\ell : \Delta^p \to E'$ such that the following diagram commutes :

$$\begin{tikzcd} \Delta^p \arrow[dr, dashed, "\ell"] \arrow[drr, bend left, "\widehat{\sigma}_{p,q}"] \arrow[ddr, bend right, "\tau_p"'] & & \\ & E' \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^q, E) \arrow[d, "\widehat{f}"] \\ & B \arrow[r, "i"'] & \mathrm{Map}(\Delta^q, B) \end{tikzcd}$$

So define $F : \mathscr{S}_{p,q} \to \mathscr{P}_{p,q}$ as $(\widehat{\sigma}_{p,q}, \tau_p) \mapsto \ell$. Now, if we name $p_1 : E' \to \mathrm{Map}(\Delta^q, E)$ and $p_2 : E' \to B$ the projections associated to the pullback, define $G : \mathscr{P}_{p,q} \to \mathscr{S}_{p,q}$ as $\ell \mapsto (p_1\ell, p_2\ell)$. The maps F and G are mutual inverses, and so we have proved that there is a bijection between $\mathscr{P}_{p,q}$ and $\mathscr{S}_{p,q}$.
Applying the bijection to the horizontal boundary, we get :

$$\begin{array}{rcl} \partial'_{p,q} : K_{p,q} & \longrightarrow & K_{p-1,q} \\ \ell & \longmapsto & \sum_{i=0}^{p} (-1)^i \ell \varepsilon_i^p, \end{array}$$

Thus every q-th row of the double complex is exactly the singular chain complex of E'. Now since $B \to \mathrm{Map}(\Delta^q, B)$ is a homotopy equivalence by proposition 2.1.5, hence a weak equivalence, the top row map $E' \to \mathrm{Map}(\Delta^q, E)$ is a weak equivalence (by lemma 2.2.10). Consider now $E \to \mathrm{Map}(\Delta^q, E)$ defined the same way as $B \to \mathrm{Map}(\Delta^q, B)$. It is also a weak equivalence by proposition 2.1.5. Whence, we obtain the isomorphisms :

$$H_p(E') \xrightarrow{\cong} H_p(\mathrm{Map}(\Delta^q, E)) \xleftarrow{\cong} H_p(E).$$

So $H_p(E') \cong H_p(E)$, for every p. We have just proved that the homology of the rows $H'_{p,q}(K)$, that is to say the first page $E^1_{p,q}$ of the spectral sequence, is the singular homology $H_p(E)$.
Now let us compute the second page $E^2_{p,q}$. Recall that $E^2_{p,q} = H''_p H'_q(K) := H_p(H'_{q,\bullet}(K), \overline{\partial''})$, by theorem 1.3.5. Applying our previous bijection to the vertical map ∂'', we get :

$$\begin{array}{rcl} \partial''_{p,q} : K_{p,q} & \longrightarrow & K_{p,q-1} \\ \ell & \longmapsto & \sum_{j=0}^{q} (-1)^{j+p} \ell, \end{array}$$

whence the map alternates between the identity map and the zero map. Thus, for each column p, we get chain complexes :

$$\cdots = H_p(E) \xrightarrow{0} H_p(E) = H_p(E) \xrightarrow{0} H_p(E) \longrightarrow 0$$

Therefore, we get (pay attention to the swap between p and q) :

$$\begin{array}{rcl} E^2_{p,q} & = & H_p(H'_{q,\bullet}(K), \overline{\partial''}) \\ & = & \begin{cases} \dfrac{\ker H_q(E) \xrightarrow{0} H_q(E)}{\mathrm{im}\ H_q(E) = H_q(E)}, & p > 0 \text{ and odd}, \\ \dfrac{\ker H_q(E) = H_q(E)}{\mathrm{im}\ H_q(E) \xrightarrow{0} H_q(E)}, & p > 0 \text{ and even}, \\ \dfrac{\ker H_q(E) \longrightarrow 0}{\mathrm{im}\ H_q(E) \xrightarrow{0} H_q(E)}, & p = 0. \end{cases} \\ & = & \begin{cases} 0, & p > 0, \\ H_q(E), & p = 0. \end{cases} \end{array}$$

Hence we have computed the second page. It is easy to see now that $E^r_{p,q} = E^2_{p,q}$, for each $r \geq 2$, because $E^{r+1}_{p,q} = H_{p,q}(E^r, d^r)$. Hence $E^\infty_{p,q} = E^2_{p,q}$.
Now we can finally determine the homology of the total complex. Recall that there exists a filtration $F_\bullet$ of the homology of the total complex $H_{p+q}(\mathrm{tot}(K)_\bullet)$:

$$0 = F_{-1}H_{p+q}(\mathrm{tot}(K)_\bullet) \subseteq \ldots \subseteq F_{p+q-1}H_{p+q}(\mathrm{tot}(K)_\bullet) \subseteq F_{p+q}H_{p+q}(\mathrm{tot}(K)_\bullet) = H_{p+q}(\mathrm{tot}(K)_\bullet),$$

such that : $E^2_{p,q} = E^\infty_{p,q} \cong \dfrac{F_p(H_{p+q}(\mathrm{tot}(K)_\bullet))}{F_{p-1}(H_{p+q}(\mathrm{tot}(K)_\bullet))}$, according to theorem 1.3.5. Now fix the integer $n := p+q$. We will write F_p for $F_p(H_{p+q}(\mathrm{tot}(K)_\bullet))$. We get :

$$H_n(E) = E^2_{0,n} = \frac{F_0}{F_{-1}} = F_0 \Rightarrow F_0 = H_n(E),$$

$$0 = E^2_{1,n-1} = \frac{F_1}{F_0} = \frac{F_1}{H_n(E)} \Rightarrow F_1 = H_n(E),$$

$$\vdots$$

$$0 = E^2_{n,0} = \frac{F_n}{F_{n-1}} = \frac{H_n(\mathrm{tot}(K)_\bullet)}{H_n(E)} \Rightarrow H_n(E) = H_n(\mathrm{tot}(K)_\bullet).$$

So we have proved that the homology of the total complex is the singular homology of the topological space E : $H_{p+q}(\mathrm{tot}(K)_\bullet) = H_{p+q}(E)$.

We consider now the other spectral sequence $\{{}^{\mathrm{I}}E^r_{p,q}, d^r\}$. We drop the exponent I for our discussion. Recall from theorem 1.3.5 that $E^1_{p,q} = H''_{p,q}(K)$, the homology of the columns. Our goal is to show that $H''_{p,q}(K) \cong C_p(B; \underline{H_q}(F))$, the abelian group of the singular chain complex on B with coefficients in the local system induced by the fibers of the Serre fibration f. We reinterpret again our construction of the double complex. For all continuous maps $\tau_p : \Delta^p \to B$, denote $\mathscr{S}_{p,q}(\tau_p) = \{\sigma_{p,q} \mid \sigma_{p,q} : \Delta^p \times \Delta^q \to E,$ continuous with $f\sigma_{p,q} = \tau_p pr_1\}$. One sees that the set $\mathscr{S}_{p,q}$ equals $\displaystyle\coprod_{\tau_p:\Delta^p\to B} \mathscr{S}_{p,q}(\tau_p)$. Fix p and the map $\tau_p : \Delta^p \to B$. Now consider the fiber F_{τ_p} of the induced Serre fibration $\hat{f} : \mathrm{Map}(\Delta^p, E) \to \mathrm{Map}(\Delta^p, B)$, defined by the following pullback square :

$$\begin{tikzcd}
F_{\tau_p} \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^p, E) \arrow[d, "\hat{f}"] \\
\{*\} \arrow[r, "j"] & \mathrm{Map}(\Delta^p, B)
\end{tikzcd}$$

where j is defined by $j(*) = \tau_p$. Define the set $\mathscr{L}_{p,q}(\tau_p)$ to be all continuous maps $\Delta^q \to F_{\tau_p}$. As before, one can argue that there is a bijection between $\mathscr{L}_{p,q}(\tau_p)$ and $\mathscr{S}_{p,q}(\tau_p)$, for all continuous maps $\tau_p : \Delta^p \to B$, using the universal property of the pullback.
Now we prove that the singular homology of F_{τ_p} is isomorphic to the homology of the fiber $F_{\tau_p(0)} := f^{-1}(\tau_p(0))$. For that consider the diagram :

$$\begin{tikzcd}
F_{\tau_p} \arrow[r, hook] \arrow[d] & \mathrm{Map}(\Delta^p, E) \arrow[r, "\hat{f}"] \arrow[d] & \mathrm{Map}(\Delta^p, B) \arrow[d] \\
F_{\tau_p(0)} \arrow[r, hook] & E \arrow[r, "f"] & B
\end{tikzcd}$$

where the column maps are given by restricting a map from Δ^p to its zero vertex. The two right vertical maps are weak equivalences by proposition 2.1.5. Now using a similar argument as for lemma 2.2.10, since the rows induce exact sequences of homotopy groups, using the 5-lemma and scholia 2.1, one can show that the map $F_{\tau_p} \to F_{\tau_p(0)}$ is a weak equivalence : one just has to be careful for the case π_0 where sets appear. Whence $H_q(F_{\tau_p}) \cong H_q(F_{\tau_p(0)})$, for all q.

We can compute the homology of the columns. We have :

$$\begin{aligned} K_{p,q} &= \mathscr{F}_{Ab}(\mathscr{S}_{p,q}) \\ &= \mathscr{F}_{Ab}(\coprod_{\tau_p:\Delta^p\to B} \mathscr{S}_{p,q}(\tau_p)) \\ &= \bigoplus_{\tau_p:\Delta^p\to B} \mathscr{F}_{Ab}(\mathscr{S}_{p,q}(\tau_p)), \text{it follows from the definition of } \mathscr{F}_{Ab} \\ &= \bigoplus_{\tau_p:\Delta^p\to B} \mathscr{F}_{Ab}(\mathscr{L}_{p,q}(\tau_p)), \text{using the bijection.} \end{aligned}$$

Using again the bijection, one sees that the induced boundary map on $\mathscr{F}_{Ab}(\mathscr{L}_{p,q}(\tau_p))$, for a fixed τ_p, is given by, for $\ell : \Delta^q \to F_{\tau_p}$:

$$\begin{aligned} \partial''_{\tau_p} : \mathscr{F}_{Ab}(\mathscr{L}_{p,q}(\tau_p)) &\longrightarrow \mathscr{F}_{Ab}(\mathscr{L}_{p,q-1}(\tau_p)) \\ \ell &\longmapsto \sum_{j=0}^{q}(-1)^{j+p}\ell\varepsilon_i^q \end{aligned}$$

which is, up to a sign, the usual boundary operator of the singular chain complex (defined in page 40). Whence we get (using proposition A.1.8) :

$$\begin{aligned} H''_{p,q}(K) &= \bigoplus_{\tau_p:\Delta^p\to B} H_q(\mathscr{F}_{Ab}(\mathscr{L}_{p,q}(\tau_p)), \partial''_{\tau_p}) \\ &= \bigoplus_{\tau_p:\Delta^p\to B} H_q(F_{\tau_p}) \\ &\cong \bigoplus_{\tau_p:\Delta^p\to B} H_q(F_{\tau_p(0)}) \\ &= C_p(B; \underline{H_q}(F)). \end{aligned}$$

Now the second page is given by $E^2_{p,q} = H'_p H''_q(K) = H_p(C_\bullet(B; \underline{H_q}(F)), \overline{\partial'})$. So we only have to determine how the map $\overline{\partial'}$ works with $C_\bullet(B; \underline{H_q}(F)$. Using the bijection we get :

$$\begin{aligned} \partial'_{p,q} : K_{p,q} &\longrightarrow K_{p-1,q} \\ \ell : \Delta^q \to F_{\tau_p} &\longmapsto \sum_{i=0}^{p}(-1)^i\ell_i \end{aligned}$$

where $\ell_i : \Delta^q \to F_{\tau_p\varepsilon_i^p}$, and $\ell_i(x) = \ell(x)\varepsilon_i^p$, for all x in Δ^q. Now we want to prove the commutativity of the following diagram :

$$\begin{array}{ccc} \bigoplus_{\tau_p:\Delta^p\to B} H_q(F_{\tau_p}) & \xrightarrow{\cong} & \bigoplus_{\tau_p:\Delta^p\to B} H_q(F_{\tau_p(0)}) = C_p(B; \underline{H_q}(F)) \\ \Big\downarrow \overline{\partial'} & & \Big\downarrow \delta \\ \bigoplus_{\tau_{p-1}:\Delta^{p-1}\to B} H_q(F_{\tau_{p-1}}) & \xrightarrow{\cong} & \bigoplus_{\tau_{p-1}:\Delta^p\to B} H_q(F_{\tau_{p-1}(0)}) = C_{p-1}(B; \underline{H_q}(F)) \end{array} \tag{2.7}$$

where δ is the usual boundary map of the chain complex $C_\bullet(B; \underline{H_q}(F))$, defined on page 16. Fix a map $\tau_p : \Delta^p \to B$. To prove the commutativity of (2.7), it suffices to consider a class $[\ell]$ in

$H_q(F_{\tau_p})$, where $\ell : \Delta^q \to F_{\tau_p}$ is continuous such that $\partial''_{\tau_p}(\ell) = 0$. The top row isomorphism of (2.7) sends $[\ell]$ to $[\tilde{\ell}]$ where $\tilde{\ell} : \Delta^q \to F_{\tau_p(0)}$ is the continuous map such that $\tilde{\ell}(x) = \ell(x)(0)$, for all x in Δ^q. The bottom row isomorphism is defined in a similar way. Now take $[\ell]$ in $H_q(F_{\tau_p})$. The induced map $\overline{\partial'}$ maps $[\ell]$ to $[\partial'(\ell)] = \sum_{i=0}^{p}(-1)^i[\ell_i]$. Note that $\tilde{\ell_i} = \tilde{\ell}$ for all $i \geq 1$, and $\tilde{\ell_0}(x) = \ell(x)(1)$, for all x in Δ^q. Hence the composite :

$$H_q(F_{\tau_p}) \overset{\overline{\partial'}}{\to} \bigoplus_{\tau_{p-1}:\Delta^{p-1}\to B} H_q(F_{\tau_{p-1}}) \cong \bigoplus_{\tau_{p-1}:\Delta^{p}\to B} H_q(F_{\tau_{p-1}(0)}),$$

maps $[\ell]$ to $[\tilde{\ell_0}] + \sum_{i=1}^{p}(-1)^i[\tilde{\ell}]$. Now let us determine the other composite in the diagram (2.7). Again, take $[\ell]$ in $H_q(F_{\tau_p})$. The top row isomorphism sends $[\ell]$ to $[\tilde{\ell}]$. Now apply the boundary map δ on $[\tilde{\ell}]$. Recall that $\delta_p = \sum_{i=0}^{p}(-1)^i\delta_p^i$, where $\delta_p^i([\tilde{\ell}]) = [\tilde{\ell}]$, for all $i \geq 1$, and $\delta_p^0([\tilde{\ell}]) = [\tilde{\ell_0}]$. To see this, let γ be the path from $\tau_p(0)$ to $\tau_p(1)$ given by the composite : $\Delta^1 \hookrightarrow \Delta^p \overset{\tau_p}{\to} B$. Write τ_γ for the isomorphism induced by γ given by the structure of the local system $\underline{H_q}(F)$. Recall how we have defined τ_γ, we wrote $E_\gamma \subseteq \Delta^1 \times E$ the following pullback :

$$\begin{tikzcd}
E_\gamma \arrow[r] \arrow[d] & E \arrow[d, "f"] \\
\Delta^1 \arrow[r, "\gamma"] & B
\end{tikzcd}$$

which induced pullbacks at fibers :

$$\begin{tikzcd}
F_{\tau_p(0)} \arrow[r, "\varphi"] \arrow[d] & E_\gamma \arrow[d] & F_{\tau_p(1)} \arrow[l, "\psi"'] \arrow[d] \\
\{0\} \arrow[r, hook] & \Delta^1 & \{1\} \arrow[l, hook']
\end{tikzcd}$$

where we proved that φ and ψ are weak equivalences. They induced isomorphisms on homology groups, and so τ_γ is given by :

$$\begin{tikzcd}
H_q(F_{\tau_p(0)}) \arrow[r, "\cong"] \arrow[rr, bend right, "\tau_\gamma"'] & H_q(E_\gamma) \arrow[r, "\cong"] & H_q(F_{\tau_p(1)})
\end{tikzcd}$$

The map δ_0^p is given by τ_γ followed by the inclusion to $C_{p-1}(B; \underline{H_q}(F))$. Thus, to show that $\delta_p^0([\tilde{\ell}]) = [\tilde{\ell_0}]$, we need to prove that the following diagram commutes :

$$\begin{tikzcd}
H_q(F_{\tau_p}) \arrow[r, "\cong"] \arrow[dd] & H_q(F_{\tau_p(0)}) \arrow[dr, "\widehat{\varphi}", "\cong"'] & \\
& & H_q(E_\gamma) \\
H_q(F_{\tau_p\varepsilon_0^p}) \arrow[r, "\cong"] & H_q(F_{\tau_p(1)}) \arrow[ur, "\widehat{\psi}", "\cong"'] &
\end{tikzcd} \tag{2.8}$$

where the vertical unlabeled map sends $[\ell]$ to $[\ell_0]$. Now recall that homotopic maps induce the same map in homology (this stems from the homotopy invariance axiom of Eilenberg-Steenrod).

So we only need to show that the maps : $F_{\tau_p} \to F_{\tau_p(0)} \to E_\gamma$ and $F_{\tau_p} \to F_{\tau_p \varepsilon_0^p} \to F_{\tau_p(1)} \to E_\gamma$ are homotopic. Let us name the two composites $G, G' : F_{\tau_p} \to E_\gamma$ respectively. We have, for any g in F_{τ_p}, $G(g) = (0, g(0))$ and $G'(g) = (1, g(1))$. Name g_{01} the composite $\Delta^1 \hookrightarrow \Delta^p \xrightarrow{g} E$. Define the map H by :

$$\begin{array}{rcl} H : F_{\tau_p} \times I & \longrightarrow & E_\gamma \\ (g,t) & \longmapsto & (t, g_{01}(t)) \end{array}$$

By definition of F_{τ_p}, we have $fg_{01}(t) = \gamma(t)$ for any t in $I \cong \Delta^1$. So H is well-defined. It is also continuous since it is an evaluation on I. Notice that $H(g,0) = G(g)$ and $H(g,1) = G'(g)$, for any g. Thus H is a homotopy from G to G'. Thus, the diagram (2.8) commutes, and so we proved that $\delta_p^0([\tilde{\ell}]) = [\tilde{\ell_0}]$.
Whence the composite :

$$H_q(F_{\tau_p}) \cong H_q(F_{\tau_p(0)}) \xrightarrow{\delta} \bigoplus_{\tau_{p-1}:\Delta^p \to B} H_q(F_{\tau_{p-1}(0)}),$$

maps $[\ell]$ to $[\tilde{\ell_0}] + \sum_{i=1}^p (-1)^i [\tilde{\ell}]$. We have just proved the commutativity of (2.7). Thus, the induced map $\overline{\partial'}$ is exactly the boundary map of the chain complex $C_\bullet(B; \underline{H_q}(F))$, and so we get $E^2_{p,q} = H_p(B; \underline{H_q}(F))$, which ends the proof. ☐

Corollary 2.3.2. *Let $f : E \to B$ be a Serre fibration with fiber F, where B is simply connected. Then the second page of the Serre spectral sequence is : $E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E)$.*

Proof. Apply proposition 2.2.4. ☐

We give now an immediate easy application of the Serre spectral sequence.

Proposition 2.3.3. *Let $f : E \to B$ be a Serre fibration with fiber F. If B is simply connected and E is path-connected, then F is path-connected.*

Proof. Consider the Serre spectral sequence. We have : $E^2_{0,0} = E^\infty_{0,0} = H_0(E)$, because it is a first quadrant sequence. Since E is path-connected, we get $H_0(E) = \mathbb{Z}$. But we also have : $E^2_{0,0} = H_0(B; H_0(F)) = H_0(F)$ by theorem 2.2.8, and so $H_0(F) = \mathbb{Z}$. Conclude with theorem A.4.2. ☐

The following result will be very useful subsequently.

Proposition 2.3.4. *If $f : E \to B$ is a Serre fibration, where E is simply connected, then the limit page of the Serre spectral sequence induced by f is given by : $E^\infty_{p,q} = 0$ for all p and q, except $E^\infty_{0,0} = \mathbb{Z}$.*

Proof. Since E is contractible, we get $H_n(E) = 0$ for all $n > 0$, and $H_0(E) = \mathbb{Z}$. The proposition follows from the convergence of the Serre spectral sequence. ☐

Chapter 3

Applications of the Serre Spectral Sequence

3.1 The Path Fibration

We recall the definition of a fibration.

Definition 3.1.1. A map $p : E \to B$ is a *fibration* if it is a continuous map which has the homoptopy lifting property with respect to any topological space (see definition 2.1.1). If (B, b_0) is a pointed space, the subspace $F := p^{-1}(b_0)$ is called the fiber of the fibration p. We write the fibration : $F \hookrightarrow E \xrightarrow{p} B$.

Of course, any fibration is a Serre fibration. From any space B, it gives rise to a fibration $\Omega B \hookrightarrow PB \to B$ that will be very useful subsequently.

Definition 3.1.2. Let (B, b_0) be a pointed space. The *path space* PB of (B, b_0) is the topological space : $PB = \{\lambda \in \mathrm{Map}(I, B) \mid \lambda(0) = b_0\} \subseteq \mathrm{Map}(I, B)$.

Proposition 3.1.3. *Let (B, b_0) be a pointed space. The map $p : PB \to B$ defined by $p(\lambda) = \lambda(1)$ for any λ in PB is a fibration, with fiber ΩB : the loop space of B.*

Proof. Since the map p can be seen as an evaluation : $\mathrm{Map}(I, B) \times \{1\} \to B$, it is continuous. Let X be any topological space. Let $H : X \times I \to B$ be a homotopy, and $\tilde{h}_0 : X \to PB$ a continuous map, such that the following diagram commutes :

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & PB \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd} \tag{3.1}$$

where $\tilde{H}$ is still to be defined. For all $0 < t \leq 1$, for all $x \in X$, let $H_{[0,t]}(x) : I \to B$ be the path defined by, for all s in I : $H_{[0,t]}(x)(s) = H(x, st)$ Define the map :

$$\begin{aligned} \tilde{H} : X \times I &\longrightarrow PB \\ (x, t) &\longmapsto \tilde{h}_0(x) \star H_{[0,t]}(x), \end{aligned}$$

where $\star$ represents the concatenation of paths. We have : $\tilde{h}_0(x)(1) = H(x, 0)$, because (3.1) commutes, so the map $\tilde{H}$ is well defined. The continuity follows from the fact that $\tilde{H}$ is equivalent to a continuous map $G : X \times I \times I \to B$, where $G(x, t, s) = \tilde{H}(x, t)(s)$. It is straightforward to see that $p^{-1}(b_0) = \Omega B$. $\square$

Definition 3.1.4. The *path fibration* of a pointed space (B, b_0) is the fibration of the previous proposition : $\Omega B \hookrightarrow PB \to B$.

Proposition 3.1.5. *For any pointed space (B, b_0), the path space PB is contractible.*

Proof. Define the map $f : PB \to \{b_0\}$ by $f(\lambda) = \lambda(0)$, for all λ in PB. Let $g : \{b_0\} \to PB$ be the constant path in B at b_0. Of course $fg = \mathrm{id}_{\{b_0\}}$. For any λ in PB, the map $gf(\lambda)$ is the constant path in B at $\lambda(0)$. We must show that $gf \simeq \mathrm{id}_{PB}$. Define the map $H : PB \times I \to PB$ by $H(\lambda, t)(s) = \lambda(s(1-t))$, for any λ in PB, and any t, s in I. We get $H(\lambda, 0) = \lambda$ and $H(\lambda, 1) = gf(\lambda)$. The continuity follows from the fact that H is equivalent to the map $\widehat{H}$, which is the composition of the following continuous maps :

$$PB \times I \times I \xrightarrow{\mathrm{id}_{PB} \times h} PB \times I \xrightarrow{\text{evaluation}} B,$$

where $h(t, s) = s(1-t)$, for all t and s in I. So H is the desired homotopy. $\square$

Corollary 3.1.6. *For a pointed space (B, b_0), we get isomorphisms : $\pi_{n+1}(B, b_0) \cong \pi_n(\Omega B, b_0)$, for all $n \geq 1$. For the case $n = 0$, there is a bijection between $\pi_1(B)$ and $\pi_0(\Omega B)$.*

Proof. Consider the path fibration $\Omega B \hookrightarrow PB \to B$. Apply theorem 2.1.2. By the previous proposition, we get $\pi_n(PB, b_0) = 0$, for all n, which gives the isomorphisms.
For the case $n = 0$, define the map $F : \pi_1(B) \to \pi_0(\Omega B)$ by $F([\lambda]_*) = P_\lambda$, the path component of the loop λ. It is a well defined bijective set map : two path components P_λ and $P_{\lambda'}$ are equal if and only if there exists a path $\widehat{H} : I \to \Omega B$, from λ to λ'. But since I is locally compact and Hausdorff, the map $\widehat{H}$ is equivalent to a map $H : I \times I \to B$: a path homotopy from λ to λ', which is the case if and only if $[\lambda]_* = [\lambda']_*$. $\square$

Remark 3.1.7. The preceeding result could have been proved by using the suspension functor Σ and the fact that $[\Sigma X, Y] = [X, \Omega Y]$.

We have constructed a fibration for any topological space. In the following sections, the path fibration will be extremely useful when applied to the Serre spectral sequence.

3.2 The Hurewicz Isomorphism Theorem

Theorem 3.2.1. *Let X be a path-connected space. The the first group of homology $H_1(X)$ is the Abelianization of the fundamental group $\pi_1(X)$ of X. In other words, we have the isomorphism :*

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

Proof. Omitted, a proof can be found for instance in [13]. $\square$

Theorem 3.2.2 (Hurewicz Isomorphism Theorem). *Let X be a simply connected space. Then the firsts nontrivial homotopy and homology groups occur in the same dimension and are equal, i.e., given a positive integer $n \geq 2$, if $\pi_q(X) = 0$, for $1 \leq q < n$, then $H_q(X) = 0$, for $1 \leq q < n$, and $H_n(X) = \pi_n(X)$.*

Proof. We will prove the statement by induction. Start with $n = 2$. Consider the Serre spectral sequence induced by the path fibration $\Omega X \hookrightarrow PX \to X$. The second page is given by $E^2_{p,q} = H_p(X; H_q(\Omega X))$. When $q = 0$, since ΩX is path-connected (because $\pi_0(\Omega X) = \pi_1(X) = 0$ by corollary 3.1.6), we get $H_0(\Omega X) = \mathbb{Z}$. So the 0-th row of the second page is given by the singular homologies of X : $E^2_{p,0} = H_p(X)$. We have $H_0(X) = \mathbb{Z}$ because X is path connected, and $H_1(X) = 0$ because $\pi_1(X) = 0$ (use theorem 3.2.1). When $p = 0$, we have $H_0(X; H_q(\Omega X)) = H_q(\Omega X)$ by theorem 2.2.8. So the 0-th column is given by the homologies of ΩX : $E^2_{0,q} = H_q(\Omega X)$. So a part of the second page can be represented as :

2	$H_2(\Omega X)$			
1	$H_1(\Omega X)$			
0	$\mathbb{Z}$	0	$H_2(X)$	$H_3(X)$
	0	1	2	3

The differential $d^2_{2,0} : H_2(X) \to H_1(\Omega X)$ must be an isomorphism. If not, some elements in $H_2(X)$ or in $H_1(\Omega X)$ would survive in the third page, meaning that $E^3_{2,0}$ or $E^3_{0,1}$ would not be zero, because $E^3_{2,0} = \frac{\ker d^2_{2,0}}{\mathrm{im}\ 0 \to H_2(X)}$ and $E^3_{0,1} = \frac{\ker H_1(\Omega X) \to 0}{\mathrm{im}\ d^2_{2,0}}$. And because of the structure of a first quadrant spectral sequence, we have $E^\infty_{2,0} = E^3_{2,0}$ and $E^\infty_{0,1} = E^3_{0,1}$, meaning these elements will survive all the way to the limit page. However the only non trivial group of the limit page is $E^\infty_{0,0} = \mathbb{Z}$, by proposition 2.3.4 because PX is contractible. Whence $d^2_{2,0}$ is really an isomorphism.
By theorem 3.2.1, we have $H_1(\Omega X) \cong \pi_1(\Omega X)/[\pi_1(\Omega X), \pi_1(\Omega X)]$, but we have the isomorphism $\pi_1(\Omega X) \cong \pi_2(X)$, which is an abelian group. Whence : $H_1(\Omega X) \cong \pi_1(\Omega X) \cong \pi_2(X)$. Thus we obtain : $H_2(X) \cong \pi_2(X)$.

Now let $n > 2$ be any fixed positive integer. By the induction hypothesis applied to ΩX, $H_q(\Omega X) \cong \pi_q(\Omega X) \cong \pi_{q+1}(X) = 0$, for $q < n-1$, and $H_{n-1}(\Omega X) \cong \pi_{n-1}(\Omega X) \cong \pi_n(X)$. By the same argument as before, the second page E^2 of the Serre spectral sequence of the path fibration is :

$n-1$	$H_{n-1}\Omega X$				
$\vdots$	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	$H_n(X)$
	0	1	2	$\cdots$	n

As before, because PX is contractible, $d^n_{n,0} : H_n(X) \to H_{n-1}(\Omega X)$ is also an isomorphism. Thus : $H_n(X) \cong H_{n-1}(\Omega X) \cong \pi_{n-1}(\Omega X) \cong \pi_n(X)$. $\square$

Corollary 3.2.3. *For any $n \geq 2$, we have : $\pi_q(S^n) = 0$, for $q < n$, and $\pi_n(S^n) = \mathbb{Z}$.*

Proof. Apply theorems A.4.6 and 3.2.2. $\square$

3.3 The Gysin and Wang sequences

The Serre spectral sequence of a fibration induces exact sequences of homology groups. First we recall some basic results in homological algebra. We define the *cokernel* of an abelian group homomorphism $f : G \to H$ by the quotient $\mathrm{coker}(f) = H/(\mathrm{im} f)$.

Proposition 3.3.1 (Splicing exact sequences)**.** *Let $A \to B \xrightarrow{f} C$ and $D \to E \xrightarrow{g} F$ be exact sequences of abelian groups. Suppose there is an isomorphism $\varphi : \mathrm{coker}(f) \xrightarrow{\cong} \ker g$. Then there is an exact sequence :*

$$\begin{array}{ccccccccccc} A & \longrightarrow & B & \xrightarrow{f} & C & \xrightarrow{\psi} & D & \xrightarrow{g} & E & \longrightarrow & F \\ & & & & c & \longmapsto & \varphi(\overline{c}) & & & & \end{array}$$

where $\overline{c}$ is the class of c in $\mathrm{coker}(f)$*.*

Proof. Let c be in $\ker \psi$. It is equivalent to say that $\psi(c) = \varphi(\overline{c}) = 0$, which is equivalent to say that $\overline{c} = 0$, because φ is an isomorphism. But this means that c is in im f. We have just proved that $\ker \psi = \mathrm{im}\ f$. Let d be in $\ker g$, this means that there exists c in C such that $\varphi(\overline{c}) = d$, because φ is an isomorphism. This is equivalent to say that $\psi(c) = d$ and so d is in im ψ. We have just proved that $\ker g = \mathrm{im}\ \psi$. $\square$

Lemma 3.3.2. *Given the following diagram of abelian groups :*

$$\begin{tikzcd} & A \arrow[d, "f"] & & \\ & B \arrow[d, "g"] \arrow[dr, dashed, "hg"] & & \\ 0 \arrow[r] & C \arrow[r, "h"] \arrow[d] & D \arrow[r, "k"] & E \\ & 0 & & \end{tikzcd}$$

where the row and the column are both exact sequences, the following induced sequence is exact :

$$A \xrightarrow{f} B \xrightarrow{hg} D \xrightarrow{k} E$$

Proof. This is an easy proof. $\square$

Theorem 3.3.3 (The Gysin Sequence). *Let $S^n \hookrightarrow E \to B$ be a Serre fibration, where B is simply connected and $n \geq 1$. Then there exists an exact sequence :*

$$\cdots \longrightarrow H_r(E) \longrightarrow H_r(B) \longrightarrow H_{r-n-1}(B) \longrightarrow H_{r-1}(E) \longrightarrow \cdots$$

In particular, for $0 \leq r \leq n-1$, we have isomorphisms : $H_r(E) \cong H_r(B)$.

Proof. The second page of the Serre spectral sequence of the fibration is given by :

$$E^2_{p,q} = H_p(B; H_q(S^n)) = \begin{cases} H_p(B;\mathbb{Z}), & \text{if } q = 0, n, \\ 0, & \text{otherwise,} \end{cases}$$

using theorem A.4.6. Whence, the only non-zero differentials are $d^{n+1}_{p,0} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$ and $E^{n+1}_{p,q} = E^2_{p,q}$. It follows that :

$$H_{p,q}(E^{n+1}, d^{n+1}) = E^{n+2}_{p,q} = \ldots = E^\infty_{p,q} = \begin{cases} 0, & \text{if } q \neq 0, n, \\ \ker d^{n+1}_{p,0}, & \text{if } q = 0, \\ \operatorname{coker} d^{n+1}_{p+n+1,0}, & \text{if } q = n. \end{cases}$$

And so we get exact sequences :

$$0 \longrightarrow \underbrace{\ker d^{n+1}_{p,0}}_{=E^\infty_{p,0}} \hookrightarrow E^{n+1}_{p,0} = E^2_{p,0}$$

$$E^2_{p-n-1,n} = E^\infty_{p-n-1,n} \twoheadrightarrow \underbrace{\operatorname{coker} d^{n+1}_{p,0}}_{=E^\infty_{p-n-1,n}} \longrightarrow 0$$

Since by first isomorphism theorem we have : $\frac{E^{n+1}}{\ker d^{n+1}_{p,0}} \cong \operatorname{im} d^{n+1}_{p,0}$, we get by splicing the above exact sequences (proposition 3.3.1) the following exact sequence :

$$0 \longrightarrow E^\infty_{p,0} \longrightarrow E^2_{p,0} \xrightarrow{d^{n+1}} E^2_{p-n-1,n} \longrightarrow E^\infty_{p-n-1,n} \longrightarrow 0 \tag{3.2}$$

Now, by the structure of the limit page E^∞, the fitration of $H_r(E)$ is given by, for all r :

$$\{0\} = F_0 = \cdots = F_{r-n-1} \subseteq \underbrace{F_{r-n}}_{=E^\infty_{r-n,n}} = \cdots = F_{r-1} \subseteq F_r = H_r(E)$$

And so we get exact sequences for all r :

$$0 \longrightarrow E^\infty_{r-n,n} \longrightarrow H_r(E) \twoheadrightarrow \frac{H_r(E)}{E^\infty_{r-n,n}} = E^\infty_{r,0} \longrightarrow 0 \tag{3.3}$$

Putting together the exact sequences (3.2) and (3.3) :

$$\begin{tikzcd}
{} \arrow[dr, dashed] & \vdots \arrow[d] \\
& H_r(E) \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] \\
0 \arrow[r] & E^\infty_{r,0} \arrow[r] \arrow[d] & \underbrace{E^2_{r,0}}_{=H_r(B)} \arrow[r, "d^{n+1}"] & \underbrace{E^2_{r-n-1,n}}_{=H_{r-n-1}(B)} \arrow[r] \arrow[dr, dashed] & E^\infty_{r-n-1,n} \arrow[r] \arrow[d] & 0 \\
& 0 & & & H_{r-1}(E) \arrow[d] \arrow[dr, dashed] \\
& & & 0 \arrow[r] & E^\infty_{r-1,0} \arrow[r] \arrow[d] & \cdots \\
& & & & 0
\end{tikzcd}$$

Using repeatedly lemma 3.3.2, we obtained the desired exact sequence (using the dashed map of the previous diagram) :

$$\cdots \longrightarrow H_r(E) \longrightarrow H_r(B) \xrightarrow{d^{n+1}} H_{r-n-1}(B) \longrightarrow H_{r-1}(E) \longrightarrow \cdots$$

In particular, when $0 \leq r \leq n-1$, we have $H_{r-n-1}(B) = 0$ and so :

$$\cdots \longrightarrow 0 \longrightarrow H_r(E) \longrightarrow H_r(B) \longrightarrow 0 \longrightarrow \cdots$$

Whence : $H_r(E) \cong H_r(B)$. $\square$

Example 3.3.4. Recall there is a fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$, for all $n \geq 1$, and $\mathbb{C}P^n$ is simply connected. Knowing that the homology $H_p(\mathbb{C}P^n) = 0$ for all $p > 2n$ (this stems from cellular homology), we can use the Gysin sequence. We will show that, for $p \leq 2n$:

$$H_p(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & p \text{ even}, \\ 0, & p \text{ odd}. \end{cases}$$

From the Gysin sequence, we have in particular :

$$\underbrace{H_{2n+2}(\mathbb{C}P^n)}_{=0} \longrightarrow H_{2n}(\mathbb{C}P^n) \longrightarrow \underbrace{H_{2n+1}(S^{2n+1})}_{=\mathbb{Z}} \longrightarrow \underbrace{H_{2n+1}(\mathbb{C}P^n)}_{=0}$$

and so $H_{2n}(\mathbb{C}P^n) = \mathbb{Z}$. Now consider the exact sequence (from the Gysin sequence) :

$$\underbrace{H_{2n}(S^{2n+1})}_{=0} \longrightarrow \underbrace{H_{2n}(\mathbb{C}P^n)}_{=\mathbb{Z}} \longrightarrow H_{2n-2}(\mathbb{C}P^n) \longrightarrow \underbrace{H_{2n-1}(S^{2n+1})}_{=0}$$

and so $H_{2n-2}(\mathbb{C}P^n) = \mathbb{Z}$. Iterating this argument, we get : $H_p(\mathbb{C}P^n) = \mathbb{Z}$, for $p \leq 2n$ and p even. Now notice that :

$$\underbrace{H_{2n+1}(\mathbb{C}P^n)}_{=0} \longrightarrow H_{2n-1}(\mathbb{C}P^n) \longrightarrow \underbrace{H_{2n}(S^{2n+1})}_{=0}$$

So : $H_{2n-1}(\mathbb{C}P^n) = 0$. Now from the exact sequence :

$$\underbrace{H_{2n-1}(S^{2n+1})}_{=0} \longrightarrow \underbrace{H_{2n-1}(\mathbb{C}P^n)}_{=0} \longrightarrow H_{2n-3}(\mathbb{C}P^n) \longrightarrow \underbrace{H_{2n-2}(S^{2n+1})}_{=0}$$

we get : $H_{2n-3}(\mathbb{C}P^n) = 0$. Iterating this argument, we get : $H_p(\mathbb{C}P^n) = 0$, for $p \leq 2n$ and p odd.

Example 3.3.5. Like previous exemple, knowing there is a fibration $S^3 \hookrightarrow S^{4n+3} \to \mathbb{H}P^n$, for all $n \geq 1$, and $H_p(\mathbb{H}P^n) = 0$, for all $p > 4n$, one can prove with the Gysin sequence :

$$H_p(\mathbb{H}P^n) = \begin{cases} \mathbb{Z}, & p = 0, 4, 8, \ldots, 4n, \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 3.3.6 (**The Wang Sequence**). *Let $F \hookrightarrow E \to S^n$ be a Serre fibration, with $n \geq 2$. Then there exists an exact sequence :*

$$\cdots \longrightarrow H_r(F) \longrightarrow H_r(E) \longrightarrow H_{r-n}(F) \longrightarrow H_{r-1}(F) \longrightarrow \cdots$$

In particular, for $0 \leq r \leq n-2$, we have isomorphisms : $H_r(E) \cong H_r(F)$.

Proof. The proof will be similar to the Gysin sequence. We will thus give less details. The second page of the Serre spectral sequence is given by :

$$E^2_{p,q} = H_p(S^n; H_q(F)) = \begin{cases} H_q(F), & \text{if } p = 0, n, \\ 0, & \text{otherwise,} \end{cases}$$

using theorem 2.2.9. Hence the only possible non-zero differentials are $d^n_{p,q}$. We get $E^2_{p,q} = \ldots = E^n_{p,q}$ and $E^{n+1}_{p,q} = \ldots = E^\infty_{p,q}$. Since $E^{n+1}_{p,q} = H_{p,q}(E^n, d^n)$, we get the exact sequences :

$$0 \longrightarrow E^\infty_{n,q} \longrightarrow E^2_{n,q} \xrightarrow{d^n} E^2_{0,q+n-1} \longrightarrow E^\infty_{0,q+n-1} \longrightarrow 0 \tag{3.4}$$

Now, by the structure of the limit page E^∞, the fitration of $H_r(E)$ is given by, for all r :

$$0 = F_{-1} \subseteq F_0 = \ldots = F_{n-1} \subseteq F_n = \ldots = H_r(E),$$

and we get $E^\infty_{n,r-n} = \dfrac{H_r(E)}{E^\infty_{0,r}}$, and whence the exact sequences :

$$0 \longrightarrow E^\infty_{0,r} \longrightarrow H_r(E) \longrightarrow E^\infty_{n,r-n} \longrightarrow 0 \tag{3.5}$$

Putting together the exact sequences (3.4) and (3.5) :

$$\begin{tikzcd}
{} \arrow[dr, dashed] & \vdots \arrow[d] & & & & \\
& H_r(E) \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E^\infty_{n,r-n} \arrow[r] \arrow[d] & \underbrace{E^2_{n,r-n}}_{=H_{r-n}(F)} \arrow[r, "d^n"] & \underbrace{E^2_{0,r-1}}_{=H_{r-1}(F)} \arrow[r] \arrow[dr, dashed] & E^\infty_{0,r-1} \arrow[r] \arrow[d] & 0 \\
& 0 & & & H_{r-1}(E) \arrow[d] \arrow[dr, dashed] & \\
& & & 0 \arrow[r] & E^\infty_{n,r-n-1} \arrow[r] \arrow[d] & \cdots \\
& & & & 0 &
\end{tikzcd}$$

Conclude with lemma 3.3.2. $\square$

Corollary 3.3.7. *The loop space ΩS^n of the n-sphere, where $n \geq 2$, has singular homology :*

$$H_r(\Omega S^n) = \begin{cases} \mathbb{Z}, & \text{if } r \text{ is a multiple of } n-1, \\ 0, & \text{otherwise,} \end{cases}$$

Proof. Apply the Wang sequence to the path fibration $\Omega S^n \hookrightarrow PS^n \to S^n$. Since PS^n is contractible, every third term $H_r(PS^n)$ in the Wang sequence is zero, except for the case $H_0(PS^n) = \mathbb{Z}$. Whence we get isomorphisms $H_{r-n}(\Omega S^n) \cong H_{r-1}(\Omega S^n)$. Knowing the initial value $H_0(\Omega S^n) = \mathbb{Z}$ (using proposition 2.3.3), one can conclude. □

Conclusion

We are able to solve our initial problem : for any (Serre) fibration $F \hookrightarrow E \to B$, is there a link between the homology groups of the spaces E, B and F ? The anwser is given by the Serre spectral sequence :

$$E^2_{p,q} = H_p(B; \underline{H_q}(F;\mathbb{Z})) \Rightarrow H_{p+q}(E;\mathbb{Z}).$$

This relationship is particularly strong and, with more knowledge in homology theory and homotopy theory, one can deduce many results through the Serre spectral sequence. For instance, one can prove that : $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}$. Even better, one can show that : if n is odd, then $\pi_m(S^n)$ is finite whenever $m \neq n$.

We worked only with homology groups, but the dual case exists. In words, there is a cohomological spectral sequence which gives the same kind of relations between the cohomology groups of the spaces E, B and F.
We also dealt only with the singular homology, but the same result holds for any ordinary homology theory (it is usually called the Leray-Serre-Atiyah-Hirzebruch spectral sequence).

Appendix A

Singular Homology

The singular homology is one of the most important homology theories in algebraic topology. The modern definition is due to Samuel Eilenberg in [5] in 1944. We will present here briefly the concept of singular homology. We follow [12] and [13].

A.1 Homology of Complexes

We introduce the fundamental notion in homological algebra used throughout this paper.

Definition A.1.1. A *(chain) complex* $(K_\bullet, \partial)$ *of abelian groups*, is a family $\{K_n, \partial_n\}_{n\in\mathbb{Z}}$ of abelian groups K_n and (abelian) group homomorphisms $\partial_n : K_n \to K_{n-1}$ such that $\partial_n\partial_{n+1} = 0$ for each $n \in \mathbb{Z}$. The last condition is equivalent to im $\partial_{n+1} \subseteq \ker \partial_n$. We will usually write simply $K_\bullet$ for $(K_\bullet, \partial)$. The homomorphisms ∂_n are called the *boundary operators* or *differentials*. A complex $K_\bullet$ thus appears as a doubly infinite sequence :

$$K_\bullet : \qquad \cdots \longrightarrow K_{n+1} \xrightarrow{\partial_{n+1}} K_n \xrightarrow{\partial_n} K_{n-1} \longrightarrow \cdots$$

with each composite map zero. An *n-cycle* of $K_\bullet$ is an element of the subgroup $Z_n(K_\bullet) := \ker \partial_n$, an *n-boundary* of $K_\bullet$ is an element of the subgroup $B_n(K_\bullet) :=$ im ∂_{n+1}.[1]

Definition A.1.2. Let $K_\bullet$ be a complex. The *homology* $H(K_\bullet)$ of the complex $K_\bullet$ is the family of abelian groups $H_n(K_\bullet)$ called *n-th homology group* of the complex $K_\bullet$:

$$H_n(K_\bullet) := \frac{\ker \partial_n}{\text{im } \partial_{n+1}} = \frac{Z_n(K_\bullet)}{B_n(K_\bullet)} \quad \text{(cycles mod boundaries)}.$$

Thus, $H_n(K_\bullet) = 0$ means that the sequence $K_\bullet$ is exact at K_n. The coset of a cycle c in H_n is written cls $c := c + B_n$, and is called the *homology class* of c. Subsequently, we usually omit the subscript n on ∂_n.

Definition A.1.3. A complex $K_\bullet$ is *positive* if $K_n = 0$ for $n < 0$.

Definition A.1.4. If $K_\bullet$ and $K'_\bullet$ are complexes, a *chain transformation* $f : K_\bullet \to K'_\bullet$ is a family of (abelian) group homomorphisms $f_n : K_n \to K'_n$, one for each n, such that $\partial'_n f_n = f_{n-1}\partial_n$ for all n. In other words, we have the commutativity of the diagram :

$$\begin{tikzcd}
K_\bullet : & \cdots \arrow[r] & K_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & K_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & K_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
K'_\bullet : & \cdots \arrow[r] & K'_{n+1} \arrow[r, "\partial'_{n+1}"'] & K'_n \arrow[r, "\partial'_n"'] & K'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

[1]The symbol Z_n is from the German *Zykel*

The function $H_n(f) = f_*$ defined by $f_*(c+B_n) = f(c)+B'_n$ is an (abelian) group homomorphism $H_n(f) : H_n(K_\bullet) \to H_n(K'_\bullet)$. With this definition, each H_n is a (covariant) functor on the category **Comp** of chain complexes and chain transformations to the category **Ab** of abelian groups.

Definition A.1.5. A *subcomplex* $S_\bullet$ of $K_\bullet$ is a family of (abelian) subgroups $S_n \subseteq K_n$, for each n, such that $\partial S_n \subseteq S_{n-1}$. We will write then : $S_\bullet \subseteq K_\bullet$. Hence $S_\bullet$ is itself a complex (with boundary operator induced by $K_\bullet$).

Definition A.1.6. Consider the complexes $S_\bullet \subseteq K_\bullet$. The *quotient complex* $(K_\bullet/S_\bullet)$ is a complex with family K_n/S_n, together with boundary $\partial' : K_n/S_n \to K_{n-1}/S_{n-1}$ induced by the boundary ∂ of $K_\bullet$.

Definition A.1.7. If $\{(K^i_\bullet, \partial^i_\bullet)\}_{i\in\mathscr{I}}$ is a family of complexes, then their *direct sum* is the complex $\bigoplus_{i\in\mathscr{I}} K^i_\bullet$ with boundary maps :

$$\begin{aligned} \bigoplus_{i\in\mathscr{I}} \partial^i_n : \bigoplus_{i\in\mathscr{I}} K^i_n &\longrightarrow \bigoplus_{i\in\mathscr{I}} K^i_{n-1} \\ c^i_n &\longmapsto \partial^i_n(c^i_n). \end{aligned}$$

Proposition A.1.8. *Homology commutes with direct sums : for all n, there are group isomorphisms :* $H_n(\bigoplus_{i\in\mathscr{I}} K^i_\bullet) \cong \bigoplus_{i\in\mathscr{I}} H_n(K^i_\bullet)$.

Proof. Define the map :

$$\begin{aligned} H_n\left(\bigoplus_{i\in\mathscr{I}} K^i_\bullet\right) &\longrightarrow \bigoplus_{i\in\mathscr{I}} H_n(K^i_\bullet) \\ \text{cls}\left(\sum c_i\right) &\longmapsto \sum \text{cls } c_i \end{aligned}$$

It is straightfoward to see that this is a well defined bijective abelian group homomorphism. □

A.2 The Notion of Simplex

Before introducing the singular homology, we have to define a geometric notion, very useful in algebraic topology : the *simplices*.

Let E be an n-dimensional euclidean space. It is a metric space and hence a topological space. In particular, E may be the space $\mathbb{R}^n$. A subset A of E is called *affine* if, for every pair of distinct points $x, x' \in A$, the line determined by x, x' is contained in A. By convention, the empty set $\emptyset$ and the point-set $\{*\}$ are affine.
It is easy to see that any intersection of affine subsets of $\mathbb{R}^n$ is also an affine subset. Thus, one can define the *affine set in* $\mathbb{R}^n$ *spanned* by a subset X of $\mathbb{R}^n$, by the intersection of all affine subsets of $\mathbb{R}^n$ containing X. Similarly for convex sets, the *convex set spanned* by a subset X (also called the *convex hull* of X) is the intersection of all convex subsets containing X. We denote it by $\langle X\rangle$.
An *affine combination* of points $p_0, p_1, \dots, p_m$ in $\mathbb{R}^n$ is a point x with $x = \sum_{i=0}^m t_i p_i$, where $\sum_{i=0}^m t_i = 1$. If moreover $t_i \geq 0$ for all i, it is a *convex combination*. One can prove that the affine (respectively convex) set spanned by $\{p_0, p_1, \dots, p_m\} \subset \mathbb{R}^n$ consists of all affine (respectively convex) combinations of these points.
An ordered set of points $\{p_0, \dots, p_m\} \subset \mathbb{R}^n$ is *affine independent* if $\{p_1 - p_0, \dots, p_m - p_0\}$ is a linearly independent subset of $\mathbb{R}^n$. One can prove the following result.

Proposition A.2.1. *The following conditions on an ordered set of points $\{p_0, \ldots, p_m\}$ in $\mathbb{R}^n$ are equivalent.*

(i) $\{p_0, \ldots, p_m\}$ *is affine independent,*

(ii) *if* $\{s_0, \ldots, s_m\} \subset \mathbb{R}$ *satisfies* $\sum_{i=0}^m s_i p_i = 0$ *and* $\sum_{i=0}^m s_i = 0$, *then* $s_i = 0$ *for all* i,

(iii) *each* x *in the affine set spanned by* $\{p_0, \ldots, p_m\}$ *has unique expression as an affine combination* $x = \sum_{i=0}^m t_i p_i$, *where* $\sum_{i=0}^m t_i = 1$.

Thus affine independence is a property of the set $\{p_0, \ldots, p_m\}$ that is independent of the given ordering. The entries t_i mentioned are called the *barycentric coordinates* of x, relative to the set $\{p_0, \ldots, p_m\}$.
Whence we have the following definition.

Definition A.2.2. Let $\{p_0, \ldots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$. The convex set spanned by this set, denoted by $\langle p_0, \ldots, p_m \rangle$, is called the *(affine) m-simplex* with *vertices* $p_0, \ldots, p_m$.

Hence, each x in the m-simplex $\langle p_0, \ldots, p_m \rangle$ has a unique expression of the form :

$$x = \sum_{i=0}^m t_i p_i, \text{ where } \sum t_i = 1 \text{ and each } t_i \geq 0.$$

For instance, a 0-simplex is just a point, a 1-simplex is a line segment, a 2-simplex is a triangle with interior, a 3-simplex is a tetrahedron, and so on.

Definition A.2.3. Let $\langle p_0, \ldots, p_m \rangle$ be a m-simplex. A k-face of $\langle p_0, \ldots, p_m \rangle$ is a k-simplex spanned by $k+1$ of the vertices $\{p_0, \ldots, p_m\}$, where $0 \leq k \leq m-1$. The $(m-1)$ *face opposite* p_i is $\langle p_0, \ldots, \widehat{p_i}, \ldots p_m \rangle$.[2] The boundary of $\langle p_0, \ldots, p_m \rangle$ is the union of its faces.

Definition A.2.4. Let $(e_0, \ldots, e_n)$ be the usal canonical basis of $\mathbb{R}^{n+1}$. We define *the standard n-simplex* Δ^n to be $\langle e_0, \ldots, e_n \rangle$. Note that the barycentric coordinates and cartesian coordinates coincide. In order to simplify the notations, we label the vertices of Δ^n by $(0, 1, \ldots, n)$. Note that Δ^n is homeomorphic to the n-disk D^n.

Let $\{p_0, \ldots, p_m\}$ be affine independent and let A denote the affine set it spans. An *affine map* $T : A \to \mathbb{R}^k$, for some positive integer k, is a function satisfying $T(\sum t_i p_i) = \sum t_i T(p_i)$, whenever $\sum t_i = 1$. One can show that it is a continuous map and it is determined by its values on an affine independent subset. It is easy now to see that any two m-simplices are homeomorphic via an affine map, whence any m-simplex is homeomorphic to Δ^m. This is why we will usually work only with the standard simplex.

A.3 Construction of the Singular Complex

We will now construct a functor from the category of topological spaces **Top** to the category of complexes **Comp**. The key concept is the following.

Definition A.3.1. Let X be a topological space. A *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$.

Before introducing the chain complex associated to any topological space, we must do an algebraic detour.

[2]The circumflex notation means "delete".

Free Abelian Groups Let B be a subset of an abelian group F. Then F is *free abelian with basis* B if the cyclic subgroup $\langle b\rangle$ is infinite cyclic for each $b \in B$ and $F = \bigoplus_{b\in B}\langle b\rangle$.

A free abelian group is thus a direct sum of copies of $\mathbb{Z}$. Given a set T, there exists a free abelian group $\mathscr{F}_{Ab}(T)$ having T as a basis. Explicitely, there exists a functor $\mathscr{F}_{Ab} : \mathbf{Set} \to \mathbf{Ab}$, which is given by :

$$\mathscr{F}_{Ab}(T) = \bigoplus_{t\in T}\mathbb{Z}\cdot t = \left\{\sum_{t\in T} m_t \cdot t \mid m_t \in \mathbb{Z},\, \forall t \in T,\, |\{m_t \neq 0\}| < \infty\right\},$$

where the sum is defined as : $\left(\sum m_t \cdot t\right) + \left(\sum n_t \cdot t\right) := \sum (m_t + n_t)\cdot t$, and any set map $f : T \to Y$ induces :

$$\begin{array}{rccc} \mathscr{F}_{Ab}(f) : & \mathscr{F}_{Ab}(T) & \longrightarrow & \mathscr{F}_{Ab}(Y) \\ & \displaystyle\sum_{t\in T} m_t \cdot t & \longmapsto & \displaystyle\sum_{t\in T} m_t \cdot f(t). \end{array}$$

Free abelian groups satisfy the following universal property : for any free abelian group F with basis B, for any abelian group G and any set map $\varphi : B \to G$, there exists a unique group homomorphism $\tilde{\varphi} : F \to G$ such that $\tilde{\varphi}(b) = \varphi(b)$, for all b in B.

$$\begin{tikzcd} F \arrow[dr, dashed, "\tilde{\varphi}"] & \\ B \arrow[u, hook] \arrow[r, "\varphi"'] & G \end{tikzcd}$$

The Singular Chain Complex We are now ready to construct a complex defined for any topological space called the *singular complex*. Let X be a topological space. For each $n \in \mathbb{N}$, take $S_n(X)$ to be the free abelian group with basis all singular n-simplices (using the functor $\mathscr{F}_{Ab}$). Define $S_{-n}(X) := 0$ for each $n \geq 1$.
We must now specify the boundary operator ∂ of the complex. For each $n > 0$, and $0 \leq i \leq n-1$, define the *i-th face map* :

$$\begin{array}{rcl} \varepsilon_i = \varepsilon_i^n : \Delta^{n-1} & \longrightarrow & \Delta^n \\ (t_0, \ldots, t_{n-1}) & \longmapsto & (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}). \end{array}$$

One can see, by evaluating on each vertex, that if $j > k$, the face maps satisfy :

$$\varepsilon_j^{n+1}\varepsilon_k^n = \varepsilon_k^{n+1}\varepsilon_{j-1}^n : \Delta^{n-1} \to \Delta^{n+1}. \tag{A.1}$$

Then the i-th face maps define the boundary operators $\partial = \partial_n : S_n(X) \to S_{n-1}(X)$ as the following alterning sum : for any singular n-simplex σ :

$$\partial\sigma = \sigma\varepsilon_0 - \sigma\varepsilon_1 + \cdots + (-1)^n\sigma\varepsilon_n = \sum_{i=0}^{n}(-1)^i\sigma\varepsilon_i,$$

if $n > 0$, and $\partial_0\sigma := 0$. Using the universal property of free abelian groups, one can extend the formula of ∂_n for any element of $S_n(X)$ and the homomorphism ∂ obtained is unique. To prove now that $S_\bullet(X) := \{S_n(X), \partial\}$ is a chain complex, we need to show that $\partial\partial = 0$. It suffices to show that $\partial_n\partial_{n+1}\sigma = 0$ for all singular $(n+1)$-simplices σ, $n \geq 0$.

$$\begin{aligned}
\partial\partial\sigma &= \partial\left(\sum_j (-1)^j \sigma\varepsilon_j^{n+1}\right) \\
&= \sum_{j,k} (-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n \\
&= \sum_{j\leq k} (-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n + \sum_{j>k} (-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n \\
&= \sum_{j\leq k} (-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n + \sum_{j>k} (-1)^{j+k}\sigma\varepsilon_k^{n+1}\varepsilon_{j-1}^n, \text{ by (A.1)}, \\
&= \sum_{j\leq k} (-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n + \sum_{p\leq q} (-1)^{p+q+1}\sigma\varepsilon_p^{n+1}\varepsilon_q^n \\
&= 0.
\end{aligned}$$

Thus, we have constructed $S_\bullet(X)$ *the singular complex* of X.

Definition A.3.2. The *n-dimensional singular homology group* $H_n(X)$ of a topological space X is defined to be the n-th homology group $H_n(S_\bullet(X))$ of the singular complex $S_\bullet(X)$.

For any continuous map $f : X \to Y$, define $S_n(f) : S_n(X) \to S_n(Y)$ as follows :

$$S_n(f)\left(\sum m_\sigma \cdot \sigma\right) = \sum m_\sigma \cdot (f\sigma),$$

for all n, which induce all together a chain transformation $S(f) : S_\bullet(X) \to S_\bullet(Y)$. Thus, for all $n \geq 0$, H_n : **Top** $\to$ **Ab** is a (covariant) functor.

A.4 Some Properties of Singular Homology

We gather here some results that are useful.

Theorem A.4.1. *Let $\{X_\alpha\}_{\alpha\in\mathscr{I}}$ be all the path components of a topological space X. Then : $H_n(X) \cong \bigoplus_{\alpha\in\mathscr{I}} H_n(X_\alpha)$.*

Proof. A singular simplex $\sigma : \Delta^n \to X$ has a path-connected image, because Δ^n is path-connected. Whence the singular chain complex $S_n(X)$ equals $\bigoplus_{\alpha\in\mathscr{I}} S_n(X_\alpha)$. Now apply proposition A.1.8. □

Theorem A.4.2. *Let X be a non empty topological space. Then X is path-connected if and only if $H_0(X) \cong \mathbb{Z}$.*

Proof. Let us first show that if X is path-connected then $H_0(X) \cong \mathbb{Z}$. Recall that $\partial_0 = 0$, hence we get : $H_0(X) = S_0(X)/\text{im } \partial_1$. Define :

$$\begin{aligned}
\varphi : S_0(X) &\longrightarrow \mathbb{Z} \\
\sum m_i\sigma_i &\longmapsto \sum m_i
\end{aligned}$$

It is clearly a surjective group homomorphism. We only need to show that $\ker\varphi = \text{im } \partial_1$, and we can conclude that $H_0(X) \cong \mathbb{Z}$, by the first isomorphism theorem.
Let $\sigma : \Delta^1 \to X$ be a 1-singular simplex. We get : $\varphi(\partial_1(\sigma)) = \varphi(\sigma\varepsilon_0 - \sigma\varepsilon_1) = 1 - 1 = 0$. So im $\partial_1 \subseteq \ker\varphi$. Now take $\sum m_i\sigma_i$ in $S_0(X)$, such that $\sum m_i = 0$. For that, fix any point

x_0 in X, and regard this point as the 0-singular simplex σ_{x_0}. Denote $\lambda_i : \Delta^1 \to X$ a path from x_0 to $\sigma_i(0)$. We get $\partial_1(\lambda_i) = \sigma_i - \sigma_{x_0}$, for all i. Whence : $\partial_1(\sum m_i \lambda_i) = \sum m_i \sigma_i$, and so $\ker \varphi \subseteq \text{im } \partial_1$.
Let us prove the converse. By what we have just proved, and by the previous theorem, we get that $H_0(X)$ is isomorphic to $\bigoplus_p \mathbb{Z}$ where p is the number of path components of X. Here $H_0(X) = \mathbb{Z}$, so there is only one path component. □

Theorem A.4.3. *Consider $\{*\}$ the one-point set space. Then : $H_0(\{*\}) = \mathbb{Z}$ and $H_n(\{*\}) = 0$, for all $n > 0$.*

Proof. The case $n = 0$ is already proved by theorem A.4.2. Since there is only one continuous map $\sigma_n : \Delta^n \to \{*\}$, we get : $S_n(\{*\}) = \mathbb{Z}$, generated by σ_n. Hence the boundary map ∂_n alternates between the trivial map, if n is odd, and maps σ_n to σ_{n-1} if n is even. So we get a chain complex :

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So its homologies are all trivial, for $n > 0$. □

Theorem A.4.4. *If $f : X \to Y$ is a weak equivalence, then it induces a group isomorphism between $H_n(X)$ and $H_n(Y)$, for all n.*

Proof. Omitted, a proof can be found for instance in [13]. □

One consequence of this theorem is the following.

Theorem A.4.5. *If X is a contractible space, then $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$, for all $n > 0$.*

Proof. A consequence of the two previous theorems. □

Theorem A.4.6. *The singular homologies of the spheres are :*

$$\begin{cases} H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}, \\ H_k(S^0) = 0, \text{ if } k > 0, \\ H_0(S^n) = H_n(S^n) = \mathbb{Z}, \text{ if } n > 0, \\ H_k(S^n) = 0, \text{ if } k \neq 0, n. \end{cases}$$

Proof. Omitted, a proof can be found for instance in [13]. □

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