An Introduction to Borel Reducibility for Countable Structures

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CiE 2024 Tutorial, Part 2

Review from yesterday

Definition (H. Friedman, Stanley)

Suppose $\mathcal{C} \subseteq Mod(\mathcal{L})$ and $\mathcal{D} \subseteq Mod(\mathcal{L}')$ are closed under isomorphism. We say that $\mathcal{C}$ is Borel reducible to $\mathcal{D}$ if there is a Borel function $\Phi : \mathcal{C} \to \mathcal{D}$ such that for $\mathcal{A}, \mathcal{B} \in \mathcal{C}$,

$$\mathcal{A} \cong \mathcal{B} \Longleftrightarrow \Phi(\mathcal{A}) \cong \Phi(\mathcal{B}).$$

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A class of structures $\mathcal{C}$ is Borel complete if for every other class $\mathcal{D}$, $\mathcal{D} \leq_B \mathcal{C}$.

The following classes of structures are Borel complete:

- graphs,
- partial orders,
- rings,
- integral domains,
- 2-step nilpotent groups,
- fields.

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- fields.

Moreover, in each of these cases, we have something stronger.

The reduction is by a computable bi-interpretation, i.e., if Φ is the reduction and $\Phi(\mathcal{A}) = \mathcal{B}$, then a copy of $\mathcal{A}$ can be found inside of $\mathcal{B}$ and can be recovered computably.

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We called these classes universal or computably universal.

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Trees and linear orders are not universal; in particular, there are automorphism groups of structures which are not the automorphism group of a tree (or of a linear order).

A number of the hardest questions of computable structure theory are about whether, even though they are not universal, trees and linear orders still satisfy some of the consequences of universality.

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Is there a linear order whose degree spectrum is exactly the non-computable degrees?

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There is no Borel way to recover a copy of $\mathcal{A}$ from a copy of $\mathcal{T}(\mathcal{A})$.

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Theorem (Harrison-Trainor, Montalbán)

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Question

Is there a Borel reduction $\mathcal{T}^*$ from graphs to trees such that $\mathcal{A}$ can be recovered in a Borel way from $\mathcal{T}(\mathcal{A})$?

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Question

Is there a Borel reduction $\mathcal{T}^*$ from graphs to trees such that the image is Borel?

Structures which are Borel complete but not universal are some of the most interesting classes of structures in computable structure theory.

The Isomorphism Problem

For a class $\mathcal{C}$ let

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It is an analytic or Σ^1_1 subset of $Mod(\mathcal{L}) \times Mod(\mathcal{L})$:

$$\mathcal{A} \cong \mathcal{B} \iff \exists f \text{ isomorphism } f\colon \mathcal{A} \to \mathcal{B}.$$

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For graphs, it is Σ^1_1-complete and so not Borel. Thus if $\mathcal{C}$ is Borel complete, then $I(\mathcal{C})$ Σ^1_1-complete.

$\mathcal{C}$ is universal

$$\Downarrow$$

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Isomorphism for $\mathcal{C}$ is analytic complete

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Theorem (Laskowski, Rast, Ulrich)

Binary splitting, refining equivalence relations are not Borel complete, but the isomorphism problem is analytic-complete.

The graph isomorphism problem in complexity theory

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Theorem (Babai, 2015)

Graph isomorphism can be solved in quasipolynomial time $2^{O((\log n)^c)}$.

Graph isomorphism is a good candidate for a natural problem intermediate between P and NP.

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While the graph isomorphism problem is analogous to the isomorphism problem for countable structures, in practice we always use a reduction like the polynomial-time version of a Borel reduction, and in fact (as far as I know) the constructions are always of the universal type.

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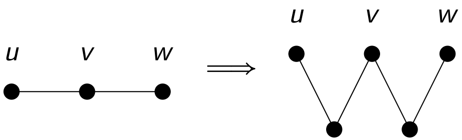
Connect an edge $e = (u, v)$ to both u and v.



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Theorem (Booth, Colbourn)

Let $\mathcal{C}$ be the class of graphs with no induced copy of H.

Then $\mathcal{C}$ is GI-complete if and only if H is not induced subgraph of the path on four vertices.

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Definition

A graph is a cograph if and only if it has no induced P_4.

Theorem (Many people?)

The finite cographs are the smallest class of graphs satisfying:

- *The graph with one vertex is a cograph.*
- *The disjoint union of two cographs is a cograph.*
- *The complement of a cograph is a cograph.*

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- *The graph with one vertex is a cograph.*
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Each cograph has a unique tree decomposition in normal form. Thus we can reduce checking isomorphism for cographs to isomorphism for (labeled) trees.

Isomorphism for (labeled) trees is computable in linear time using an algorithm by Aho, Hopcroft, and Ullman. Importantly, this uses counting and so is unique to the finite realm.

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Theorem

The isomorphism problem for cographs is in polynomial time.

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Observation (Harrison-Trainor, Ko)

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Given a countable tree, we can transform it into a cograph by using it as the tree decomposition. Because trees are Borel complete, countable cographs are Borel complete.

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Given a countable tree, we can transform it into a cograph by using it as the tree decomposition. Because trees are Borel complete, countable cographs are Borel complete.

The cyclic group of order 3 is not the automorphism group of any cograph. This follows from the modular decomposition which adds to the tree decomposition for finite cographs the fact that countable cographs are also closed under nested unions.

Question

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Question

Is there any class $\mathcal{C}$ of finite structures which is analogous to p-groups in that it is GI-complete, but there is no map Φ from finite graphs to $\mathcal{C}$ such that

$$G \cong H \iff \Phi(G) \cong \Phi(H).$$

Torsion-free abelian groups

One of the first examples we considered was rank 1 torsion-free abelian groups, or subgroups of $\mathbb{Q}$.

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We have

$$TFAG_1 \leq_B TFAG_2 \leq_B TFAG_3 \leq_B \cdots.$$

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By Cohn and Walker's cancellation for $\mathbb{Z}$ from direct sums:

$$G \cong H \iff G \oplus \mathbb{Z} \cong H \oplus \mathbb{Z}$$

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For Borel reducibility, Hjorth and Thomas proved that these are all strict.

Theorem (Hjorth, Thomas)

$$TFAG_1 <_B TFAG_2 <_B TFAG_3 <_B \cdots.$$

This is a hard theorem. (Thomas's paper was in JAMS.)

Question (Ho, Knight, Miller, (Dittman))

Let TD_n be the class of fields of transcendence degree n (in characteristic zero). Then

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(Note that $K \mapsto K(x)$ does not work!)

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Building on ideas of Hjorth, Downey and Montalbán proved:

Theorem (Downey, Montalbán)

The isomorphism problem for torsion-free abelian groups is analytic complete.

Recall that this is a consequence of Borel completeness.

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Theorem

Torsion-free abelian groups are Borel complete.

The Shelah-Paolini proof is highly technical:

Definition 3.5. *In the context of Hyp. 3.2, let* $\mathrm{K}_1^{\mathrm{bo}}(M)$ *be the class of objects* $\mathfrak{m}(M) = \mathfrak{m} = (X^{\mathfrak{m}}, \bar{X}^{\mathfrak{m}}, I^{\mathfrak{m}}, \bar{I}^{\mathfrak{m}}, \bar{f}^{\mathfrak{m}}, \bar{E}^{\mathfrak{m}}, Y_{\mathfrak{m}}) = (X, \bar{X}, I, \bar{I}, \bar{f}, \bar{E}, Y)$ *s.t.:*

(1) X *is an infinite countable set and* $X \subseteq \omega$*;*

(2) (a) $(X'_s : s \subseteq_1 M)$ *is a partition of* X *into infinite sets;*

(b) for $s \subseteq_\omega M$*, let* $X_s = \bigcup_{t \subseteq_1 s} X'_t$*;*

(c) $\bar{X} = (X_s : s \subseteq_\omega M)$ *and so* $s \subseteq t \subseteq_\omega M$ *implies* $X_s \subseteq X_t$*;*

(3) for $\mathcal{U} \subseteq M$ *let* $X_{\mathcal{U}} = \bigcup \{X_s : s \subseteq_1 \mathcal{U}\}$ *and so* $X = X_M = \bigcup \{X_s : s \subseteq_1 M\}$*;*

(4) (a) $\bar{I} = (I_n : n < \omega) = (I_n^{\mathfrak{m}} : n < \omega)$ *are pairwise disjoint;*

(b) $\bar{g} \in I_n$ *implies* $\bar{g} \in \mathcal{G}_*^m$ *for some* $m \leqslant n$*;*

(c) I_n *is finite;*

(5) if $\bar{g}' \lhd \bar{g} \in I_n$*, then* $\bar{g}' \in I_{<n} := \bigcup_{\ell < n} I_\ell$*;*

(6) $I = I^{\mathfrak{m}} = \bigcup_{n<\omega} I_n$*;*

⋮

(10) (a) $\bar{E}^{\mathfrak{m}} = \bar{\bar{E}} = (\bar{E}_n : n < \omega) = (E_n^{\mathfrak{m}} : n < \omega)$*, and, for* $n < \omega$*,* E_n *is the equivalence relation corresponding to the partition of* $\mathrm{seq}_n(X)$ *given by the connected components of the graph* $(\mathrm{seq}_n(X), R_n)$*;*

(b) $Y = Y_{\mathfrak{m}}$ *is a non-empty subset of* X *which* *<u>includes</u>* *the following set:*

$$\{x \in X : \text{ for some } \bar{g} \in I, \, x \in \mathrm{dom}(f_{\bar{g}})\},$$

notice that this inclusion may very well be proper;

(c) $\mathrm{seq}_k(\mathfrak{m}) = \{\bar{x} \in \mathrm{seq}_k(X) : \text{ for some } \bar{g} \in I, \, \bar{x} \subseteq \mathrm{dom}(f_{\bar{g}})\}$*, notice* $\mathrm{seq}_k(\mathfrak{m}) \subseteq \mathrm{seq}_k(Y_{\mathfrak{m}})$ *but the converse need not hold;*

(11) if p *is a prime,* $k \geqslant 2$*,* $\bar{x} \in \mathrm{seq}_k(X)$*,* $\bar{q} \in (\mathbb{Q}_p)^k$*,* $\mathfrak{s} = (p, k, \bar{x}, \bar{q})$ *and* $\bar{a} \in \mathcal{A}_{\mathfrak{s}}$*, then* $\mathrm{supp}_p(\bar{a})$ *is not a singleton,* *<u>where</u>* *we define* $\mathcal{A}_{\mathfrak{s}}$*,* $\mathcal{A}_{\mathfrak{m}}$ *and* $\mathrm{supp}_p(\bar{a})$ *as follows:*

(a) $\mathcal{A}_{\mathfrak{s}} \subseteq \mathcal{A}_{\mathfrak{m}} = \{(a_y : y \in Z) : Z \subseteq_\omega X \text{ and } a_y \in \mathbb{Q}\}$*;*

(b) if $\bar{a} \in \mathcal{A}_{\mathfrak{m}}$*, then we let:*

$$\mathrm{supp}_p(\bar{a}) = \{y \in \mathrm{dom}(\bar{a}) : a_y \notin \mathbb{Q}_p\};$$

⋮

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To finish the tutorial, I will highlight the main ideas from the Shelah-Paolini proof, and in particular the two key ideas.

The proof breaks up into two parts:

- The combinatorial part.
- The group-theoretic part.

The two are intertwined in the sense that each of them influences the other.

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But then if there are edges $s - t - u - v$ then

$$x_s + x_v = (x_s + x_t) - (x_t + x_u) + (x_u + x_v)$$

and so the divisibility relation would suggest that there is an edge between s and v.

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For each $a \in \mathcal{A}$, we have an infinite set X_a. Our group $G(\mathcal{A})$ will have $\mathbb{Z}$-basis $X = \bigcup_{a \in \mathcal{A}} X_a$:

$$\sum_{x \in X} \mathbb{Z}a \subseteq G(\mathcal{A}) \subseteq \sum_{x \in X} \mathbb{Q}x.$$

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We can think of an equivalence relation $\mathbb{X}$ on X of being in the same X_a.

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We call an automorphism fixing each $\mathbb{E}_n$ class a *strong automorphism*.

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(For technical reasons, we will have $p_{\bar{q},e} \nmid q_i$.)

Define $G(\mathcal{A})$ by putting, for $\bar{x} \in e$,

$$\forall n \quad p^n_{\bar{q},e} \text{ divides } \sum q_i x_i.$$

Then strong automorphisms of the combinatorial structure induce automorphisms of $G(\mathcal{A})$, and essentially we want to make sure that this works in the other direction as well.

Here's an example of how this works. Given $x \in X_A$, we want to show that x does not satisfy the same divisibility as any element $\sum_{i=1}^{\ell} q_i y_i$ for any $\ell \geq 2$. This is because we want to distinguish the single elements of X from tuples.

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Suppose

$$x = \sum_{i=1}^{\ell} q_i y_i .$$

Let e be the equivalence class of $\bar{y}$. Let $p = p_{\bar{q},e}$. Then $p^\infty \mid x$.

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But maybe this could be true?

Third main idea: Given $\bar{x}_1, \ldots, \bar{x}_k \in X^n$ in the same $\mathbb{E}_n$-equivalence class, there are $x_{i_1}^{j_1}, x_{i_2}^{j_2}$ such that

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Then, in the right-hand-side of

$$x = \sum r_j \sum_{i=1}^{\ell} q_i z_{i,j}$$

there must be at least two elements which only show up once!

The heart of the combinatorics can be captured in the following construction:

Theorem

There is a structure $\mathcal{M}$ with an equivalence relation $\mathbb{X}$ and equivalence relations $\mathbb{E}_n$ on n-tuples such that:

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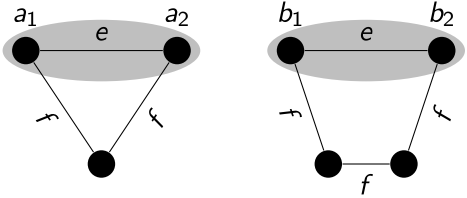
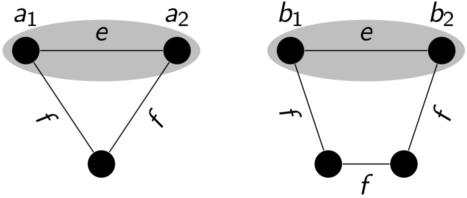
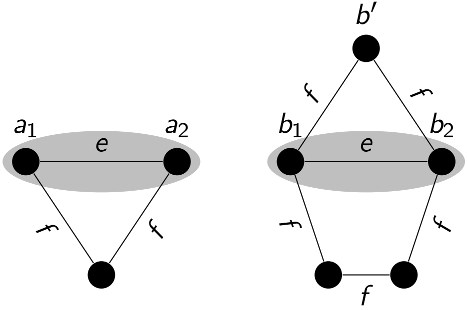
The problem is two different tuples $\bar{a}$ and $\bar{b}$ might be extended in two different, but incompatible, ways.



a_1
a_2
e
f
f
b_1
b_2
e
f
f
f



If we have a map taking $\bar{a} = (a_1, a_2)$ to $\bar{b} = (b_1, b_2)$, then we would have to have to add a node b' as in the following diagram:



The way to think of this is that for a tuple $\bar{a}$, there are "potential problems". If two tuples have the same atomic type but incompatible "potential problems" then we should not try to homogenize them.

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Think of each tuple $\bar{a}$ as having associated to it a larger set $\mathrm{Cl}(\bar{a})$ which contains all of the potential problems. This will be a closure operator. We can think of $\mathrm{Cl}(\bar{a})$ as "guarding" $\bar{a}$ from problems.

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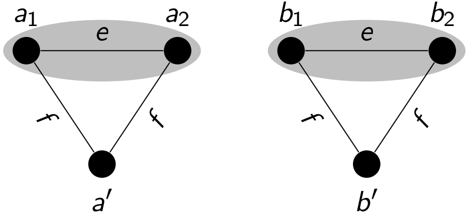

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A *guarded structure* is an $\mathcal{M}$ equipped with a guard map $\mathrm{Cl}\colon [M]^{<\omega} \to [M]^{<\omega}$ such that:

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In the language of generalized Fraisse limits, the structure we build will be *weakly homogeneous* and its age will have the *cofinal amalgamation property*.

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Fourth main idea: The class of structures with two equivalence relations is Borel-complete, so we can reduce this to groups. Equivalence relations are easier to incorporate into the construction.

Of course, there are many more details and a lot more to check, but that is the general idea of the argument.

Theorem

Torsion-free abelian groups are Borel complete.

Takeaway: Structures which are Borel complete but not universal are some of the most interesting classes of structures in computable structure theory.

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Thanks for listening!