The Property “Arithmetic-is-Recursive” on a Cone

Uri Andrews[*] Matthew Harrison-Trainor[†] Noah Schweber

November 10, 2020

Abstract

We say that a theory T satisfies arithmetic-is-recursive if any X'-computable model of T has an X-computable copy; that is, the models of T satisfy a sort of jump inversion. We give an example of a theory satisfying arithmetic-is-recursive non-trivially and prove that the theories satisfying arithmetic-is-recursive on a cone are exactly those theories with countably many ω-back-and-forth types.

1 Introduction

Vaught's conjecture states that an elementary first-order theory has either countably-many or continuum-many countable models. Conjectured by Vaught in 1961 [Vau61], it is one of the oldest and most well-known open problems in logic. Morley showed that the number of countable models is either $\aleph_0$, $\aleph_1$, or continuum. He used what has become known as the Morley analysis, a variant of which is as follows. The Morley analysis introduces infinitary formulas of $\mathcal{L}_{\omega_1\omega}$. Throughout this paper, we will consider a theory to be an elementary first-order theory, and whenever we talk about Σ_α or Π_α formulas we mean formulas of $\mathcal{L}_{\omega_1\omega}$.

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures and let $\bar{a} \in \mathcal{A}$ and $\bar{b} \in \mathcal{B}$ be tuples of length k. Then $(\mathcal{A}, \bar{a})$ and $(\mathcal{B}, \bar{b})$ are α-back-and-forth equivalent, and we write $(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b})$, if $\bar{a}$ and $\bar{b}$ satisfy the same Σ_α formulas of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Definition 1.2. The α-back-and-forth types (shortened as α-bftypes) of a theory T are the equivalence classes of tuples modulo α-back-and-forth equivalence:

$$\mathbf{bf}_\alpha(T) = \frac{\{(\mathcal{A}, \bar{a}) : \mathcal{A} \vDash T, \bar{a} \in \mathcal{A}\}}{\equiv_\alpha}.$$

Silver's theorem that every coanalytic equivalence relation has either countably or perfectly many equivalence classes [Sil80] implies that for each countable α there are either countably-many or continuum-many α-back-and-forth types. So for a given structure, we have three possibilities. First, it might be that there are continuum-many α-back-and-forth types for some α, in which case there are continuum-many countable models of T. Second, it might

[*]Supported by NSF DMS-1600228.
[†]Supported by an NSERC Banting fellowship.

be that there are countably-many α-back-and-forth types for each α and countably many countable models of T. Third, there might be countably-many α-back-and-forth types for each α, but uncountably many countable models of T. Consider this third case. Scott [Sco65] showed that each countable structure $\mathcal{A}$ has a Scott sentence, that is, an $\mathcal{L}_{\omega_1\omega}$ sentence φ such that $\mathcal{A}$ is the only countable model of φ. Let α be a countable ordinal such that the Scott sentence is Σ_α; then $\mathcal{A}$ is characterized by its α-back-and-forth type. Since there are only $\aleph_1$-many countable ordinals, and so $\aleph_1$-many α-back-and-forth types among all countable ordinals α, there are then at most $\aleph_1$-many (and hence exactly $\aleph_1$-many) models of T. If $\aleph_1 < 2^{\aleph_0}$, then this third case consists of exactly the counterexamples to Vaught's conjecture. (Because it might be that $\aleph_1 = 2^{\aleph_0}$, Vaught's conjecture is often stated in a way that is absolute, namely that every elementary first-order theory has either countably many countable models or a perfect set of countable models. Whatever the status of the continuum hypothesis, the counterexamples to this formulation of the conjecture are exactly the theories with countably many α-back-and-forth types for each countable α, but uncountably many models.)

For a theory T with uncountably-many countable models, we can measure how close T is to being a counterexample to Vaught's conjecture by the least countable ordinal α such that T has continuum-many α-back-and-forth types.

Definition 1.3 (Montalbán, see [Mon12, CK15])**.** Let T be a theory. The back-and-forth ordinal of T is the least countable ordinal α such that there are continuum-many α-bftypes. (If no such ordinal exists, the back-and-forth ordinal is ∞.)

Counterexamples to Vaught's conjecture are exactly the theories with uncountably many countable models but back-and-forth ordinal ∞.

Montalbán [Mon13] showed that a first-order theory T is a counter-example to Vaught's conjecture if and only if T has uncountably many models and satisfies "hyperarithmetic-is-recursive on a cone", that is, there is some set Z so that for any $X \geq_T Z$, if there is a model $\mathcal{A} \models T$ which is hyperarithmetic relative to X, then there is an isomorphic copy of $\mathcal{A}$ which is computable from X. Thus Vaught's conjecture has a formulation purely in terms of computable structure theory. At the 2015 Vaught's Conjecture Workshop in Berkeley, the third author asked for a first-order theory T non-trivially satisfying "arithmetic-is-recursive on a cone". The formal definition of this property is as follows:

Definition 1.4. Let T be a first-order theory.

- T satisfies arithmetic-is-recursive if for every set X, every X'-computable model of T has an X-computable copy.
- T satisfies arithmetic-is-recursive on a cone if there is a set Z such that for all $X \geq_T Z$, every X'-computable model of T has an X-computable copy.

If T satisfies arithmetic-is-recursive, then every $X^{(n)}$-computable model of T has an $X^{(n-1)}$-computable copy, and hence an $X^{(n-2)}$-computable copy, and so on; so every X-arithmetic model of T has an X-computable copy, hence the name.

The main result of this paper is a characterization of the structures satisfying arithmetic-is-recursive on a cone:

Theorem 1.5. *Let T be a first-order theory. The following are equivalent:*

(1) T has countably many ω-bftypes.

(2) T satisfies arithmetic-is-recursive on a cone.

If T has countably many countable models, then T satisfies both properties in a trivial way. It is immediate that such a T has back-and-forth ordinal ∞. Moreover since T has only countably many countable models, there is a set X which can compute a copy of every model of T; then any X'-computable model of T is isomorphic to an X-computable model, and T satisfies arithmetic-is-recursive on the cone above X. So we say that T satisfies arithmetic-is-recursive on a cone non-trivially if T satisfies arithmetic-is-recursive on a cone but has uncountably many countable models.

One way of thinking of theories non-trivially satisfying arithmetic-is-recursive on a cone is as theories which are on their way to being counter-examples to Vaught's conjecture. Unlike counterexamples to Vaught's conjecture, we can give an example of a theory non-trivially satisfying arithmetic-is-recursive on a cone. In particular, we show:

Theorem 1.6. *There is a first-order theory T with uncountably many countable models such that:*

(1) T has back-and-forth ordinal $\omega + 1$, and

(2) T satisfies arithmetic-is-recursive.

A natural line of inquiry is to try and build theories which are closer and closer to being counterexamples to Vaught's conjecture, in the sense that they have higher and higher back-and-forth ordinals. The example we give above is the highest known back-and-forth ordinal less than ∞; indeed this is related to the following unresolved conjecture of Martin:

Martin's Conjecture. *Let T be a first-order theory with fewer than continuum many countable models. Then T has countably many complete types. For each such type, add a new relation symbol R_i which holds of the tuples which have that type. Then for $\mathcal{A} \models T$, the theory of $\mathcal{A}$ in the expanded language with the symbols R_i is countably categorical.*

Suppose $\aleph_1 < 2^{\aleph_0}$. Then Martin's conjecture would imply that any first-order theory with back-and-forth ordinal $< \infty$ has back-and-forth ordinal $\leq \omega + 1$. So Martin's conjecture would imply Vaught's conjecture. In the language of back-and-forth ordinals, we ask:

Question 1.7. Is there a first-order theory with uncountably many countable models and back-and-forth ordinal at least $\omega + 2$?

2 Back-and-forth Types

Fix for this section a class $\mathbb{K}$ of structures for which we will develop the theory of back-and-forth types. Throughout this paper, $\mathbb{K}$ will be the class of models of an elementary first-order theory T. We recommend Section 15 of [AK00] for a good reference on back-and-forth relations.

Definition 2.1. We define the α-back-and-forth relations on $\mathbb{K}$ by induction on α. Let $\mathcal{A}$ and $\mathcal{B}$ be structures in $\mathbb{K}$ and let $\bar{a} \in \mathcal{A}$ and $\bar{b} \in \mathcal{B}$ be tuples with $\text{length}(\bar{a}) \leq \text{length}(\bar{b})$. Then:

- $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if every atomic or negated atomic formula in the first $\text{length}(\bar{a})$ variables and with Gödel number at most $\text{length}(\bar{a})$ true of $\bar{a}$ is also true of $\bar{b}$.
- $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if for every $\bar{c} \in \mathcal{B}$ and $\beta < \alpha$ there is $\bar{d} \in \mathcal{A}$ such that $(\mathcal{B}, \bar{b}\bar{c}) \leq_\beta (\mathcal{A}, \bar{a}\bar{d})$.
- $(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b})$ if $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ and $(\mathcal{B}, \bar{b}) \leq_\alpha (\mathcal{A}, \bar{a})$.

The back-and-forth relations can also be expressed in terms of infinitary $\mathcal{L}_{\omega_1\omega}$ formulas.

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be structures in $\mathbb{K}$, $\alpha \geq 1$, and let $\bar{a} \in \mathcal{A}$ and $\bar{b} \in \mathcal{B}$ be tuples of the same length.*

- $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ *if and only if every Π_α formula true of $\bar{a}$ in $\mathcal{A}$ is true of $\bar{b}$ in $\mathcal{B}$ if and only if every Σ_α formula true of $\bar{b}$ in $\mathcal{B}$ is true of $\bar{a}$ in $\mathcal{A}$.*
- $(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b})$ *if and only if $\bar{a}$ and $\bar{b}$ satisfy the same Σ_α formulas in $\mathcal{A}$ and $\mathcal{B}$ respectively.*

When $|\bar{a}| < |\bar{b}|$, if we write $\bar{b} = \bar{b}'\bar{b}''$ with $|\bar{a}| = |\bar{b}'|$ then $\bar{a} \leq_\alpha \bar{b}$ if and only if $\bar{a} \leq_\alpha \bar{b}'$.

The relations $\leq_\alpha$ are partial pre-orders on $\{(\mathcal{A}, \bar{a}) \mid \mathcal{A} \in \mathbb{K}, \bar{a} \in \mathcal{A}\}$. We can quotient out by $\equiv_\alpha$ to get the partial ordering

$$\mathbf{bf}_\alpha(\mathbb{K}) = \frac{\{(\mathcal{A}, \bar{a}) : \mathcal{A} \in \mathbb{K}, \bar{a} \in \mathcal{A}\}}{\equiv_\alpha}.$$

We call these equivalence classes the α-back-and-forth types (α-bftypes) of $\mathbb{K}$. For $\beta \leq \alpha$, this is partially ordered by $\leq_\beta$ in the obvious way. For $\xi \in \mathbf{bf}_\alpha(\mathbb{K})$, put $|\xi| = k$ and say the arity of ξ is k if ξ is the bftype of a k-tuple. Write $\mathbf{bf}_{\alpha,k}(\mathbb{K})$ for the set of α-bftypes of arity k.

Using Lemma 2.2, we can identify each α-bftype ξ, $\alpha \geq 1$, with the set of Σ_α and Π_α formulas true of a representative for ξ as an $\equiv_\alpha$-equivalence class; we use variables $x_1, \ldots, x_{|\xi|}$, and write $\xi(\bar{x})$ when we want to be explicit. The α-bftypes of arity n are thus exactly the complete sets of Σ_α and Π_α formulas, in n variables, which are realized in some model of $\mathbb{K}$. For $\alpha = 0$, we can identify an α-bftype of arity k with the complete sets of atomic and negated atomic formulas with Gödel number at most k.

Note that since $\equiv_\alpha$ is a Borel equivalence relation, Silver's theorem [Sil80] implies that if $\mathbb{K}$ is a Borel class of structures, then $\mathbf{bf}_\alpha(\mathbb{K})$ is either countable or size continuum.

Given $\xi, \zeta \in \mathbf{bf}_\alpha(\mathbb{K})$, with $|\xi| \leq |\zeta|$, we say $\xi \subseteq \zeta$ if for every (equivalently some) $(\mathcal{A}, b_1, \ldots, b_{|\zeta|})$ of α-bftype ζ, $(\mathcal{A}, b_1, \ldots, b_{|\xi|})$ has α-bftype ξ. Equivalently, $\xi \subseteq \zeta$ as sets of Σ_α and Π_α formulas.

Given $\xi \in \mathbf{bf}_\alpha(\mathbb{K})$ and $\beta < \alpha$, we write $\xi|_\beta$ for the restriction of ξ to a β-bftype, i.e. $\xi|_\beta$ is the β-bftype of a tuple with α-bftype ξ.

Given $\xi \in \mathbf{bf}_\alpha(\mathbb{K})$ and $\beta < \alpha$, define $\text{ext}_\beta(\xi) \subseteq \mathbf{bf}_\beta(\mathbb{K})$ to be the set of ζ such that for every—equivalently some—$(\mathcal{A}, \bar{a})$ of α-bftype ξ, there is some $\bar{d} \in \mathcal{A}$ such that $(\mathcal{A}, \bar{a}\bar{d})$ has β-bftype $\geq_\beta \zeta$. Note that $\text{ext}(\xi)$ is closed downwards under $\leq_\beta$. For $\xi, \zeta \in \mathbf{bf}_\alpha(\mathbb{K})$, $\xi \leq_\alpha \zeta$ if and only if for each $\beta < \alpha$, $\text{ext}_\beta(\zeta) \subseteq \text{ext}_\beta(\xi)$.

The following useful lemma says that in a situation with countably many $(\alpha-1)$-bftypes, we can express the relation $\leq_\alpha$ by a single Π_α formula.

Lemma 2.3 (Montalbán, Lemma of 2.2 [Mon12]). *If $\mathbf{bf}_{\alpha-1}(\mathbb{K})$ is countable, then for each $\xi \in \mathbf{bf}_\alpha(\mathbb{K})$ there is a Π_α formula $\varphi_\xi(\bar{x})$ with $|\bar{x}| = |\xi|$ such that, for all $(\mathcal{B}, \bar{b}) \in \mathbb{K}$,*

$$\xi \leq_\alpha (\mathcal{B}, \bar{b}) \iff \mathcal{B} \models \varphi_\xi(\bar{b}\restriction_{|\xi|}).$$

3 True Stage Constructions

In our constructions we will need Ash's α-systems. These were introduced in [Ash86a, Ash86b, Ash90] but we follow Ash and Knight's book [AK00]. For some of our constructions we will need "special" ω-systems. We give the required definitions below, but we suggest that the reader who is unfamiliar with α-systems read the relevant parts of [AK00] for more details.

Given sets L and U, an *alternating tree* P on L and U is a tree consisting of non-empty finite sequences $\ell_0 u_1 \ell_1 u_2 \cdots$ with $\ell_i \in L$ and $u_i \in U$. An *instruction function* for P is a function q which assigns to each sequence $\sigma \in P$ of odd length (i.e., a sequence whose last term is in L) an element $u \in U$, such that $\sigma u \in P$. A *run* of (P, q) is a path $\ell_0 u_1 \ell_1 u_2 \ell_2 \cdots$ in P such that for all n, $u_{n+1} = q(\ell_0 u_1 \ell_1 \cdots \ell_n)$.

An *enumeration function* is a c.e. set $E \subseteq L \times \omega$ which assigns to each $\ell \in L$ the set $E(\ell) = \{k \in \omega \mid (\ell, k) \in E\}$. Given a run $\pi = \ell_0 u_1 \ell_1 u_2 \ell_2 \cdots$ of (P, q), we define $E(\pi) = \bigcup_{i \in \omega} E(\ell_i)$.

An α-system is a structure of the form $(L, U, \hat{\ell}, P, E, (\leq_\beta)_{\beta<\alpha})$, where L and U are c.e. sets, P is a c.e. alternating tree on L and U in which all of the sequences start with $\hat{\ell}$, E is a partial computable enumeration function defined on L, and for each $\beta < \alpha$, $\leq_\beta$ is a c.e. binary relation on L such that the following conditions hold:

(1) $\leq_\beta$ is reflexive and transitive,

(2) for $\beta < \gamma < \alpha$, if $\ell \leq_\gamma \ell'$ then $\ell \leq_\beta \ell'$,

(3) if $\ell \leq_0 \ell'$ then $E(\ell) \subseteq E(\ell')$,

(4) if $\sigma u \in P$, where σ ends in $\ell^0 \in L$, and

$$\ell^0 \leq_{\beta_0} \ell^1 \leq_{\beta_1} \cdots \leq_{\beta_{k-1}} \ell^k$$

for $\alpha > \beta_0 > \cdots > \beta_k$, then there exists $\ell^* \in L$ such that $\sigma u \ell^* \in P$ and $\ell^i \leq_{\beta_i} \ell^*$ for all $i \leq k$.

Ash's metatheorem is:

Theorem 3.1 (Theorem 14.1 of [AK00]). *Let $(L, U, \hat{\ell}, P, E, (\leq_\beta)_{\beta<\alpha})$ be an α-system. Then for any Δ^0_α instruction function q, there is a run π of (P, q) such that $E(\pi)$ is c.e., while π itself is Δ^0_α.*

For limit ordinals, we will need a slightly weaker version of condition (4). If q is a Δ^0_α instruction function for α a limit ordinal, then every computation of q uses only finitely many questions to Δ^0_β for various $\beta < \alpha$. Section 14.5 of [AK00] introduces a variant of the α-systems described above which takes this into account. We will need only the case $\alpha = \omega$ and we give the definition only for this case (using $1, 2, 3, \ldots$ as the increasing sequence of ordinals with limit ω). A special ω-system is defined as above except that (4) is replaced by (4′):

(4′) if $\sigma u \in P$, where σ has length $2n+1$ and ends in $\ell^0 \in L$, and

$$\ell^0 \leq_{\beta_0} \ell^1 \leq_{\beta_1} \cdots \leq_{\beta_{k-1}} \ell^k$$

for $n \geq \beta_0 > \cdots > \beta_k$, then there exists $\ell^* \in L$ such that $\sigma u \ell^* \in P$ and $\ell^i \leq_{\beta_i} \ell^*$ for all $i \leq k$.

A special Δ^0_ω instruction function for P is an instruction function q whose restriction to sequences in P of length $2n+1$ is Δ^0_{n+1} uniformly in n. Then the metatheorem is:

Theorem 3.2 (Special case of Theorem 14.4 of [AK00])**.** *Let $(L, U, \hat{\ell}, P, E, (\leq_\beta)_{\beta<\alpha})$ be a special ω-system. Then for any special Δ^0_ω instruction function q, there is a run π of (P, q) such that $E(\pi)$ is c.e., while π itself is Δ^0_ω.*

4 Models Satisfying Arithmetic-Is-Recursive

In this section we prove the direction (2)⇒(1) of Theorem 1.5 by proving the contrapositive. Suppose that T is a theory with uncountably many ω-bftypes. We will show that T does not satisfy arithmetic-is-recursive on a cone. The proof splits into two cases depending on whether there is $\alpha < \omega$ such that T has countably many α-bftypes, or whether T has uncountably many α-bftypes for each $\alpha < \omega$ but countably many ω-bftypes. The former case will be simplest, just using a degree-theoretic jump inversion argument, while the latter case will use special ω-systems. We begin with the first case.

Theorem 4.1. *Suppose that T has uncountably many α-bftypes for some $\alpha < \omega$. Then T does not satisfy arithmetic-is-recursive on a cone.*

Proof. Fix a set X. We will show by contradiction that T does not satisfy arithmetic-is-recursive on the cone above X. So assume that T does satisfy arithmetic-is-recursive on the cone above X, and assume without loss of generality that $X \geq_T T$.

Since there are uncountably many α-bftypes, there is one, say ξ, which is not X-arithmetic. Let $\mathcal{M}$ be a model of T realizing ξ. We use the following cone avoiding version of Friedberg jump inversion; see Exercise 4.18 of [Ler17]:

Fact. *Given sets A, B, and $C >_T A$, there is $G \geq_T A$ such that $G' \equiv_T G \oplus A' \equiv_T B \oplus A'$ and $G \not\geq_T C$.*

Begin with $Y_0 = \mathcal{M} \oplus X^{(\alpha+1)}$. Applying the jump inversion theorem with $A = X^{(\alpha)}$, $B = \mathcal{M}$, and $C = \xi \oplus X^{(\alpha)}$, we get $Y_1 \geq_T X^{(\alpha)}$ with $Y_1' \equiv_T Y_1 \oplus X^{(\alpha+1)} \equiv_T \mathcal{M} \oplus X^{(\alpha+1)} \equiv_T Y_0$, and $Y_1 \not\geq_T \xi$. (This is where we use the fact that ξ is not X-hyperarithmetic, so that $C >_T A$.) Then, having defined Y_i, we can define $Y_{i+1} \geq X^{(\alpha+1-i)}$ with $Y_{i+1}' \equiv_T Y_i$ and $Y_{i+1} \not\geq_T \xi$. So we get $Y_{\alpha+1} \geq X$ with $Y_{\alpha+1}^{(\alpha)} \equiv Y_1$ and $Y_{\alpha+1}^{(\alpha+1)} \equiv_T Y_0 \equiv_T \mathcal{M} \oplus X^{(\alpha+1)}$.

Since T satisfies arithmetic-is-recursive on the cone above X, $\mathcal{M}$ has a $Y_{\alpha+1}$-computable copy $\mathcal{N}$. Let $\bar{a} \in \mathcal{N}$ realize ξ. Given a Σ_α formula φ, we can use $Y_1 \equiv_T Y_{\alpha+1}^{(\alpha)}$ to decide whether or not $\mathcal{N} \models \varphi(\bar{a})$; so Y_1 can compute ξ. This yields a contradiction as Y_1 was chosen using the jump inversion theorem to have $Y_1 \not\geq_T \xi$. ☐

Now consider the second case: there are countably many α-bftypes for each $\alpha < \omega$ but uncountably many ω-bftypes. We will show that T does not satisfy arithmetic-is-recursive on a cone by constructing, for any set $X \subseteq \omega$, a $Y \geq_T X$ and a Y'''-computable $\mathcal{M} \models T$ with no Y-computable copy. To build $\mathcal{M}$, we use a Y'''-computable Henkin-style construction while diagonalizing against Y-computable structures.

Informally, one should think of the idea of the construction as follows. As we build the Y'''-computable $\mathcal{M}$, we Y'''-computably guess at whether $\mathcal{A}_i$ satisfies some Σ_α sentence, $\alpha < \omega$; since $\mathcal{A}_i$ is Y-computable but we are building $\mathcal{M}$ using Y''', we have a couple extra jumps of Y to take advantage of while building $\mathcal{M}$. So we can ensure that $\mathcal{M}$ satisfies this sentence if and only if $\mathcal{A}_i$ does not.

When we construct our model $\mathcal{M}$ by stages, at each stage s we will determine the α_s-bftype of an initial segment of $\mathcal{M}$ for an increasing sequence $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$. In order to diagonalize, we need the following lemma.

Lemma 4.2. *Suppose that T has countably many n-bftypes for each $n < \omega$ but uncountably many 0-ary ω-bftypes. There is a perfect binary tree $S \subseteq 2^{<\omega}$ with no dead ends and a collection $\{\eta_\sigma : \sigma \in S\}$ of 0-ary bftypes such that η_σ is a $(|\sigma|+2)$-bftype and:*

- *if $\sigma \prec \tau \in S$ then $\eta_\sigma = \eta_\tau|_{|\sigma|+2}$ (η_τ is an extension of the $(|\sigma|+2)$-bftype η_σ to a $(|\sigma|+3)$-bftype);*
- *if $\sigma, \tau \in S$, $\sigma \neq \tau$, and $|\sigma| = |\tau|$ then $\eta_\sigma \neq \eta_\tau$.*

Proof. Choose a 2-bftype $\eta_\varnothing$ such that there are uncountably many ω-bftypes extending $\eta_\varnothing$. Suppose that we have defined η_σ for $|\sigma| = i$, and that there are uncountably many ω-bftypes extending η_σ. If there are $(i+3)$-bftypes $\zeta_0 \neq \zeta_1$ extending η_σ, with uncountably many ω-bftypes extending each of them, then put $\sigma 0, \sigma 1 \in S$ and set $\eta_{\sigma 0} = \zeta_0$ and $\eta_{\sigma 1} = \zeta_1$. Since there are countably many $(i+3)$-bftypes, if we cannot find such ζ_0 and ζ_1 then there must be a unique $(i+3)$-bftype ζ extending η_σ with uncountably many extensions to an ω-bftype. In this case, put $\sigma 0 \in S$ and $\sigma 1 \notin S$, and set $\eta_{\sigma 0} = \zeta$.

We just have to argue that S is perfect. If not, then for some $\sigma \in S$ there is a unique path $\sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma = \sigma_i \subseteq \sigma_{i+1} \subseteq \cdots$ through S. Then $\bigcup_j \eta_{\sigma_j}$ is an ω-bftype. But then there are only countably many ω-bftypes extending η_σ not equal to $\bigcup_j \eta_{\sigma_j}$, since for each $j > i$ there are only countably many ω-bftypes extending η_σ but not extending η_{σ_j}. This contradicts the fact that $\sigma = \sigma_i$ was constructed to have uncountably many extensions to an ω-bftype. ☐

Theorem 4.3. *Suppose that T has uncountably many ω-bftypes but countably many n-bftypes for each $n < \omega$. Then T does not satisfy arithmetic-is-recursive on a cone.*

In particular, there is a set $X \subseteq \omega$ such that for every $Y \geq_T X$ there is a Y'''-computable model of T with no Y-computable copy.

Proof. Let k be such that there are uncountably many k-ary ω-bftypes. Adding to the language of T a k-tuple of constants, we may assume that there are countably many n-bftypes for each $n < \omega$ but uncountably many 0-ary ω-bftypes. Let S be a perfect binary tree and let $\{\eta_\sigma : \sigma \in S\}$ be as in Lemma 4.2.

Let $C \subseteq \omega$ be such that whenever $\xi, \zeta \in \mathbf{bf}_\alpha(\mathbb{K})$, $\alpha < \omega$, there is a Σ^C_α formula on which they differ; moreover, choose C such that all of the formulas from Lemma 2.3 are Π^C_α. Let $X \subseteq \omega$ be such that:

(1) $X \geq_T C$.

(2) X computes an indexed list of the α-bftypes, of each arity, for $\alpha < \omega$, and the relations on them.

(3) For each $\xi \in \mathbf{bf}_{<\omega}(\mathbb{K})$, X can decide which C-computable formulas are in ξ.

(4) X computes the tree S and the bftypes η_σ.

To show that T does not satisfy arithmetic-is-recursive on any cone, given any $Y \geq_T X$, we will build a Y'''-computable structure $\mathcal{M}$ which is not isomorphic to any Y-computable structure. (It is likely that three jumps is not optimal, and that one could prove the result for two or even one jump, but allowing ourselves three jumps streamlines the proof.) Let $\{\mathcal{A}_i \mid i \in \omega\}$ be a Y-computable list of the (possibly partial) Y-computable structure against which we will diagonalize. We will build $\mathcal{M}$ using a true stage construction with a Y'''-computable special ω-system. $\mathcal{M}$ will be built as a quotient structure of the variables modulo equality.

Our special ω-system will be $(L, U, \hat{\ell}, P, E, (\leq_n)_{n<\omega})$, defined as follows:

- L is the set of pairs (n, ξ, ζ) where ξ is a 0-ary $(n+2)$-bftype, $n < \omega$, and ζ is a k-ary n-bftype such that $\xi \models \exists \bar{x} \zeta(\bar{x})$, i.e.,

$$\xi \models \exists \bar{x} \bigwedge_{\varphi \in \zeta} \varphi(\bar{x})$$

  where φ ranges over Σ^C_n and Π^C_n formulas.

- $U = \mathbf{bf}_{<\omega,0}(T)$.

- $\hat{\ell}$ is $(0, \eta_\varnothing, \varnothing)$.

- P is the set of paths $\ell_0 u_1 \ell_1 u_2 \ell_2 \cdots$ where:

  - $\ell_i = (i, \cdot, \cdot)$.
  - $u_i = \eta_\sigma$ where $\sigma \in S$ has $|\sigma| = i$, so that η_σ is an $(i+2)$-bftype.

- if $\ell_i = (i, \xi, \zeta)$, $u_{i+1} = \eta_\sigma$, and $\ell_{i+1} = (i+1, \xi^*, \zeta^*)$, then:
    * $\zeta \subseteq \zeta^*|_i$, i.e. ζ^* is an extension of the i-bftype ζ to an $(i+1)$-bftype of higher arity;
    * $\xi^* = \eta_\sigma$;
    * $\xi = \eta_\sigma|_{i+2} = \xi^*|_{i+2}$;
  - if $\ell_i = (i, \xi, \zeta)$ and $\ell_{i+1} = (i+1, \xi^*, \zeta^*)$, then if $\varphi \in \zeta$ is among the first i-many Σ_i^C formulas, where $\varphi = \bigvee\!\!\!\bigvee_j (\exists \bar{x}) \psi_j(\bar{x})$, then for some j and tuple $\bar{y}$, $\zeta^* \models \psi_j(\bar{y})$,
  - if $\ell_i = (i, \xi, \zeta)$ and $\ell_{i+1} = (i+1, \xi^*, \zeta^*)$, then if $\varphi \in \zeta$ is among the first i-many Π_i^C formulas, where $\varphi = \bigwedge\!\!\!\bigwedge_j (\forall \bar{x}) \psi_i(\bar{x})$, then for all $j < i$ and the first i appropriate tuples $\bar{y}$, $\zeta^* \models \psi_j(\bar{y})$,
  - if $\ell_i = (i, \xi, \zeta)$, then $|\zeta| \geq i$.

- Let $E(\ell)$ be the atomic diagram of ζ, where $\ell = (i, \xi, \zeta)$.
- We define $(i, \xi, \zeta) \leq_n (j, \xi^*, \zeta^*)$, for $\alpha < \omega$, if $i \leq j$ and $\zeta \leq_\alpha \zeta^*$.

Each of these are Y'''-c.e. (in fact Y-c.e.) by choice of $X \leq_T Y$. Conditions (1), (2), and (3) of being a special ω-system are clear. We must check condition (4′).

Claim 1. $(L, U, \hat{\ell}, P, E, (\leq_\beta)_{n<\omega})$ *satisfies the special extendability condition (4′).*

Proof. Suppose that $\sigma u \in P$, where σ has length $2n+1$ and ends in $\ell^0 \in L$, and

$$\ell^0 \leq_{\beta_0} \ell^1 \leq_{\beta_1} \cdots \leq_{\beta_{k-1}} \ell^k$$

for $n \geq \beta_0 > \cdots > \beta_k$. We must show that there exists $\ell^* \in L$ such that $\sigma u \ell^* \in P$ and $\ell^i \leq_{\beta_i} \ell^*$ for all $i \leq k$. Let $\ell^i = (n_i, \xi_i, \zeta_i)$. Note that $n_0 = n$.

Since $\zeta_{k-1} \leq_{\beta_{k-1}} \zeta_k$, there is $\zeta'_{k-1} \supseteq \zeta_{k-1}$ with $\zeta_k \leq_{\beta_k} \zeta'_{k-1}$. Then $\zeta_{k-2} \leq_{\beta_{k-2}} \zeta_{k-1} \subseteq \zeta'_{k-1}$, and so there is $\zeta'_{k-2} \supseteq \zeta_{k-2}$ such that $\zeta'_{k-1} \leq_{\beta_{k-1}} \zeta'_{k-2}$. Note that as $\zeta_k \leq_{\beta_k} \zeta'_{k-1} \leq_{\beta_{k-1}} \zeta_{k-2}$, we have $\zeta_k \leq_{\beta_k} \zeta'_{k-2}$. Continuing in this way, we get $\zeta'_1 \supseteq \zeta_1$ such that for each $i \geq 1$ $\zeta_i \leq_{\beta_i} \zeta'_1$.

Let $|\zeta_0| = m$ and $|\zeta'_1| = n$. Let $\varphi(x_1, \ldots, x_n)$ be a Π_{β_1} formula as in Lemma 2.3 for ζ'_1 and β_1: for all $(\mathcal{B}; b_1, \ldots, b_n) \in \mathbb{K}$,

$$\zeta'_1 \leq_{\beta_1} (\mathcal{B}; b_1, \ldots, b_n) \Longleftrightarrow \mathcal{B} \models \varphi(b_1, \ldots, b_n).$$

Then

$$\zeta'_1 \models \varphi(x_1, \ldots, x_n)$$

and so

$$\zeta'_1 \models (\exists x_{m+1}, \ldots, x_n) \varphi(x_1, \ldots, x_n).$$

This is Σ_{β_0}, so since $\zeta_0 \leq_{\beta_0} \zeta_1 \subseteq \zeta'_1$,

$$\zeta_0 \models (\exists x_{m+1}, \ldots, x_n) \varphi(x_1, \ldots, x_n).$$

Now let $u = \eta$; this is an $(n+3)$-bftype, while ξ_0 is an $(n+2)$-bftype. We have $\xi_0 = \eta|_{n+2}$ and $\xi_0 \models \exists \bar{x}\ \zeta_0(\bar{x})$, so $\eta \models \exists \bar{x}\ \zeta_0(\bar{x})$. So there is an n-bftype $\zeta'_0 \supseteq \zeta_0$ consistent with η with

$$\zeta'_0 \models \varphi(x_1, \ldots, x_n).$$

We can extend ζ_0' to an $(n+1)$-bftype ζ_0'' consistent with η. We have $\zeta_1' \leq_{\beta_1} \zeta_0''$. Since ζ_0'' is an $(n+1)$-bftype consistent with the $(n+3)$-bftype η, $\eta \models (\exists \bar{x})\zeta_0''(\bar{x})$.

Moreover, we can choose ζ_0' and ζ_0'' so that:

- $|\zeta_0''| \geq n+1$.
- For each Σ_n^C formula $\varphi \in \zeta_0$ from among the first n formulas, where $\varphi = \bigvee\!\!\!\bigvee_i (\exists \bar{x})\psi_i(\bar{x})$, there are $\bar{x}$ and i such that $\zeta_0'' \models \psi_i(\bar{x})$.
- For each Π_n^C formula $\varphi \in \zeta_0$ from among the first n formulas, where $\varphi = \bigwedge\!\!\!\bigwedge_i (\forall \bar{x})\psi_i(\bar{x})$, for each $j < n$ and tuples $\bar{y}$ among the first n appropriate tuples, $\zeta_0'' \models \psi_j(\bar{y})$.

Let $\ell^* = (n+1, \eta, \zeta_0'')$. □

We now define the $\Delta_\omega^0(Y''')$-computable special instruction function q. Given $\ell_0 u_1 \ell_1 u_2 \ell_2 \cdots \ell_m \in P$, let $\ell_m = (m, \xi, \zeta)$ and let $u_m = \eta_\sigma$ for $\sigma \in S$, $|\sigma| = m$. Since q must be a special instruction function, we must compute $q(\pi)$ using $\Delta_{m+1}^0(Y''') = \Delta_{m+4}^0(Y)$. First, we can decide Y-computably whether there are one or two immediate extensions of σ on S.

- If $\sigma 0, \sigma 1 \in S$, choose a Σ_{m+3}^C or Π_{m+3}^C sentence φ such that $\varphi \in \eta_{\sigma 0}$, $\varphi \notin \eta_{\sigma 1}$. Let i be least such that $\mathcal{A}_i \models \eta_\sigma$; we will diagonalize against $\mathcal{A}_i$. Using $\Delta_{m+4}^0(Y)$ we can decide whether $\mathcal{A}_i \models \varphi$.
  - If $\mathcal{A}_i \models \varphi$, then let $q(\pi) = \eta_{\sigma 1}$.
  - If $\mathcal{A}_i \not\models \varphi$, then let $q(\pi) = \eta_{\sigma 0}$.
- If $\sigma 0 \in S$, $\sigma 1 \notin S$, then let $q(\pi) = \eta_{\sigma 0}$.

Thus $(L, U, \hat{\ell}, P, E, (\leq_\beta)_{\beta \leq \alpha})$ is a Y'''-computable special ω-system, and the instruction function q defined above is a special $\Delta_\omega^0(Y''')$ instruction. By Ash's metatheorem for special ω-systems, Theorem 3.2, there is a run π of (P, q) such that $E(\pi)$ is c.e. in Y'''. Let $\mathcal{M}$ be the Y'''-computable structure with diagram $E(\pi)$; the domain of $\mathcal{M}$ is the set of variables modulo the equivalence relation $x \sim y \iff$ "$x = y$" $\in E(\pi)$. Let $\pi = \ell_0 u_1 \ell_1 u_2 \ldots$, with $\ell_i = (i, \xi_i, \zeta_i)$.

Claim 2. *If $\psi \in \zeta_i$ for some i, then $\mathcal{M} \models \psi$.*

Proof. By induction on the complexity of ψ as in Theorem 4.1. □

Claim 3. *$\mathcal{M}$ is a model of T.*

Proof. Since the ζ_i are bftypes for the theory T, for each $\psi \in T$, $\psi \in \zeta_i$ for sufficiently large i. Then by the previous claim, $\mathcal{M} \models \psi$. □

Claim 4. *$\mathcal{M}$ is not isomorphic to any Y-computable structure.*

Proof. Let $u_i = \eta_{\sigma_i}$. By construction, the ω-bftype of $(\mathcal{M}, \varnothing)$ is $\bigcup \eta_{\sigma_i}$. But this is not the ω-bftype of any $(\mathcal{A}_i, \varnothing)$, because at each split in S the instruction function q diagonalizes against another $\mathcal{A}_i$. (Recall that S is perfect.) □

We have constructed a Y'''-computable structure $\mathcal{M} \models T$ which is not isomorphic to any Y-computable structure. This completes the proof. □

5 Models With Few Back-And-Forth Types

In this section we will prove the direction (1)⇒(2) of Theorem 1.5:

Theorem 5.1. *Suppose that T has countably many ω-bftypes. Then T satisfies arithmetic-is-recursive on a cone.*

To prove the theorem we will use a notion of bftypes being isolated over other types. Fix a set $C \subseteq \omega$. Recall that since ω is a limit ordinal, the ω-bftypes can be identified with the Σ_n formulas they contain, for $n < \omega$; we will use the notation $\Sigma_{<\omega}$ to denote the class of all Σ_n formulas, $n < \omega$.

Definition 5.2. Let ξ be an ω-bftype of arity m and $\varphi(x_1, \ldots, x_n)$ a formula with $n > m$. We say that an ω-bftype ζ of arity n is C-isolated over $\xi \cup \{\varphi\}$ if there is a $\Sigma^C_{<\omega}$ formula $\psi(x_1, \ldots, x_n)$ such that $\xi \cup \{\varphi, \psi\} \vDash \zeta$.

The following lemma says that for the right choice of C we can always extend such a $\xi \cup \{\varphi\}$ to a bftype which is C-isolated over it, given that there are countably many bftypes.

Lemma 5.3. *Suppose that the class $\mathbb{K}$ has countably many ω-bftypes. Let $C \subseteq \omega$ be such that each pair of distinct ω-bftypes disagrees on a $\Sigma^C_{<\omega}$ formula. Given an ω-bftype ξ of arity m, and a formula $\varphi(x_1, \ldots, x_n)$, $n > m$, which is consistent with $\xi(x_1, \ldots, x_m)$, there is $\zeta \in \mathbf{bf}_{\omega,n}(\mathbb{K})$ which is C-isolated over $\xi \cup \{\varphi\}$.*

For the proof, we will use results about atomic models in fragments of $\mathcal{L}_{\omega_1\omega}$. The book [Mar16] is a good reference. Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1\omega}$. Recall that a fragment is a set $\mathbb{A}$ of $\mathcal{L}_{\omega_1\omega}$-formulas satisfying the following closure properties:

(1) all atomic formulas are in $\mathbb{A}$,

(2) all subformulas of formulas in $\mathbb{A}$ are also in $\mathbb{A}$,

(3) all substitution instances of formulas in $\mathbb{A}$ are also in $\mathbb{A}$,

(4) all formal negations[1] of formulas in $\mathbb{A}$ are also in $\mathbb{A}$, and

(5) $\mathbb{A}$ is closed under $\neg$, $\wedge$, $\vee$, $\exists v$, and $\forall v$.

Let $T \subseteq \mathbb{A}$ be a satisfiable theory that is $\mathbb{A}$-complete, by which we mean that $T \vDash \varphi$ or $T \nvDash \varphi$ for any $\mathbb{A}$-sentence φ.

Definition 5.4 (Definition 4.12 of [Mar16])**.** A satisfiable formula $\theta(v)$ is $\mathbb{A}$-complete if for any $\mathbb{A}$-formula $\varphi(v)$ either $T \vDash \theta(v) \longrightarrow \varphi(v)$ or $T \vDash \theta(v) \longrightarrow \neg\varphi(v)$.

A formula $\varphi(v)$ is $\mathbb{A}$-completable if there is a complete $\theta(v)$ with $T \vDash \theta(v) \longrightarrow \varphi(v)$.

Definition 5.5 (Definition 4.17 of [Mar16])**.** T is $\mathbb{A}$-atomic if every satisfiable formula in $\mathbb{A}$ is completable.

The following theorem is analogous to the case in elementary first-order logic.

[1] See Definition 1.16 of [Mar16] for the definition of the formal negation.

Theorem 5.6 (Theorem 4.19 of [Mar16]). *If for all $n \in \omega$ there are only countably many n-ary $\mathbb{A}$-types, then T is $\mathbb{A}$-atomic.*

We can now show how Lemma 5.3 follows from this theorem.

Proof of Lemma 5.3. Consider an expanded signature $\tau^+ = \tau \cup \{c_1, \ldots, c_m\}$ with new constant symbols. Let $\mathbb{A}$ be the countable fragment of consisting of the $\Sigma^C_{<\omega}$ formulas. Identifying ξ with the $\Sigma^C_{<\omega}$ formulas it contains, let $T^+ = \xi(c_1, \ldots, c_m)$. T^+ is a complete $\mathbb{A}$-theory in the signature τ^+.

By choice of C, for each n there is a one-to-one identification of the n-ary ω-bftypes ζ extending ξ with the complete $(n-m)$-ary $\mathbb{A}$-types $\zeta(c_1, \ldots, c_m, x_{m+1}, \ldots, x_n)$ in T^+. Thus T^+ has countably many $\mathbb{A}$-types. By Theorem 5.6, T^+ is $\mathbb{A}$-atomic.

The formula $\varphi(x_1, \ldots, x_n)$ is consistent with $\xi(x_1, \ldots, x_m)$, and so it follows that the formula $\varphi(c_1, \ldots, c_m, x_{m+1}, \ldots, x_n)$ is consistent with T^+. Thus $\varphi(c_1, \ldots, c_m, x_{m+1}, \ldots, x_n)$ is completable by a formula $\psi(c_1, \ldots, c_m, x_{m+1}, \ldots, x_n)$. Then $\xi \cup \{\varphi(x_1, \ldots, x_m), \psi(x_1, \ldots, x_n)\}$ extends to a unique ω-bftype ζ which is C-isolated over $\xi \cup \{\varphi\}$. $\square$

In the proof of Theorem 5.1 we will also use embeddings between bftypes.

Definition 5.7. Given $\xi, \zeta \in \mathbf{bf}_\omega(T)$, an embedding from ξ to ζ is a function $f \colon \{0, 1, \ldots, |\xi| - 1\} \to \{0, 1, \ldots, |\zeta| - 1\}$ such that $\xi(x_0, \ldots, x_{|\xi|-1}) = \zeta(x_{f(0)}, \ldots, x_{f(|\xi|-1)})$.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Suppose that T has countably many ω-bftypes. We must show that T satisfies arithmetic-is-recursive on a cone. First we must identify the base of the cone.

Let $C \subseteq \omega$ be such that each pair of distinct ω-bftypes disagrees on a $\Sigma^C_{<\omega}$ formula. Let $X \subseteq \omega$ be a set such that:

(1) $X \geq_T C$.

(2) X computes an indexed list of $\mathbf{bf}_{\leq\omega}(T)$ and the relations (e.g., $\leq_\beta$, $\subseteq$, and so on) on them, as well as the partial isomorphisms between them.

(3) For each ω-bftype ξ and finite set Φ of $\mathcal{L}^C_{\omega_1\omega}$ formulas, X can compute an ω-bftype ζ which is C-isolated over the partial ω-bftype $\xi \cup \Phi$; moreover, given ζ, X can compute an isolating formula.

We claim that T satisfies arithmetic-is-recursive on the cone above X.

Let $Y \geq_T X$ and let $\mathcal{B}$ be a Y'-computable model of T. We must build a Y-computable copy of $\mathcal{B}$. Non-uniformly, fix $\kappa \in \mathbf{bf}_{\omega,0}(T)$ the ω-bftype of $(\mathcal{B}, \varnothing)$. We build an $(\omega+1)$-system $(L, U, \hat{\ell}, P, E, (\leq^L_\xi)_{\xi\leq\omega})$ as follows:

- L is the set of tuples (ξ, ζ, f) where $\xi, \zeta \in \mathbf{bf}_\omega(T)$, $f \colon \zeta \to \xi$ is an embedding, and ξ is C-isolated over $f(\zeta) \subseteq \xi$.
- U is the set of pairs (ζ, g) where $\zeta \in \mathbf{bf}_\omega(T)$ and g is an embedding of ω-bftypes.
- $\hat{\ell}$ is $(\kappa, \kappa, \varnothing)$.

- P is the set of paths $\ell_0 u_1 \ell_1 u_2 \ell_2 \dots$ where, if $\ell_i = (\xi_i, \zeta_i, f_i)$, then
  - $u_i = (\zeta_i, g_i)$ for some embedding $g_i{:}\, \xi_i \to \zeta_{i+1}$,
  - $\xi_i \subseteq \xi_{i+1}$, $\zeta_i \subseteq \zeta_{i+1}$, and $f_i \subseteq f_{i+1}$, and
  - the embeddings $g_i{:}\, \xi_i \to \zeta_{i+1}$ and $f_i{:}\, \zeta_i \subseteq \zeta_{i+1} \to \xi_i$ are compatible, i.e., $g_i \circ f_i{:}\, \zeta_i \to \zeta_{i+1}$ and $f_{i+1} \circ g_i{:}\, \xi_i \to \xi_{i+1}$ are the identity.
- $(\xi, \zeta, f) \leq_\alpha (\xi^*, \zeta^*, g)$ if $\xi \leq_\alpha \xi^*$.
- if $\ell = (\xi, \zeta, f)$, then $E(\ell)$ is the atomic diagram of ξ.

These sets are all Y-c.e. as required. It is clear that this $(\omega+1)$-system satisfies conditions (1)-(3). It remains to show that it satisfies condition (4).

Claim 1. *$(L, U, \hat{\ell}, P, E, (\leq_\beta)_{\beta \leq \omega})$ satisfies the extendability condition.*

Proof. Suppose that $\sigma u \in P$, where σ ends in $\ell^0 \in L$, and

$$\ell^0 \leq_{\beta_0} \ell^1 \leq_{\beta_1} \cdots \leq_{\beta_{k-1}} \ell^k$$

for $\omega \geq \beta_0 > \cdots > \beta_k$. We must show that there exists $\ell^* \in L$ such that $\sigma u \ell^* \in P$ and $\ell^i \leq_{\beta_i} \ell^*$ for all $i \leq k$. Let $\ell^i = (\xi_i, \zeta_i, f_i)$.

Subclaim. *There is $\xi^* \supseteq \xi_0$ which is C-isolated over ξ_0 and such that $\xi_i \leq_{\beta_i} \xi^*$ for all i.*

Proof. Since $\xi_{k-1} \leq_{\beta_{k-1}} \xi_k$, there is $\xi'_{k-1} \supseteq \xi_{k-1}$ with $\xi_k \leq_{\beta_k} \xi'_{k-1}$. Then $\xi_{k-2} \leq_{\beta_{k-2}} \xi_{k-1} \subseteq \xi'_{k-1}$, and so there is $\xi'_{k-2} \supseteq \xi_{k-2}$ such that $\xi'_{k-1} \leq_{\beta_{k-1}} \xi'_{k-2}$. Note that as $\xi_k \leq_{\beta_k} \xi'_{k-1} \leq_{\beta_{k-1}} \xi_{k-2}$, we have $\xi_k \leq_{\beta_k} \xi'_{k-2}$. Continuing in this way, we get $\xi'_1 \supseteq \xi_1$ such that for each $i \geq 1$, $\xi_i \leq_{\beta_i} \xi'_1$.

We have $\xi_0 \leq_{\beta_0} \xi_1 \subseteq \xi'_1$. Let $|\xi_0| = m$ and $|\xi'_1| = n \geq m$. Let $\varphi(x_1, \dots, x_n)$ be a $\Pi^C_{\beta_1}$ formula as in Lemma 2.3 for ξ'_1 and β_1: for all $(\mathcal{B}; b_1, \dots, b_n) \in \mathbb{K}$,

$$\xi \leq_{\beta_1} (\mathcal{B}; b_1, \dots, b_n) \Longleftrightarrow \mathcal{B} \models \varphi(b_1, \dots, b_n).$$

Then $\xi'_1 \models (\exists x_{m+1}, \dots, x_n) \varphi(x_1, \dots, x_n)$. This is a Σ_{β_0} formula, and since $\xi_0 \leq_{\beta_0} \xi'_1$, $\xi_0 \models (\exists x_{m+1}, \dots, x_n) \varphi(x_1, \dots, x_n)$.

Thus $\xi_0 \cup \varphi(x_1, \dots, x_n)$ is consistent. By Lemma 5.3, there is $\xi^* \supseteq \xi_0 \cup \varphi(x_1, \dots, x_n)$ which is C-isolated over $\xi_0 \cup \varphi(x_1, \dots, x_n)$. Since φ is $\Sigma^C_{<\omega}$, ξ^* is C-isolated over ξ_0. Also, since $\xi^* \models \varphi(x_1, \dots, x_n)$, $\xi'_1 \leq_{\beta_1} \xi^*$. Thus $\xi^* \supseteq \xi_0$ and $\xi_i \leq_{\beta_i} \xi^*$ for all i. ☐

We now return to the proof of Claim 1. Let $\xi = \xi_0$, $\zeta = \zeta_0$, $f = f_0$, and let $u = (\zeta^*, g)$ where $g{:}\, \xi \to \zeta^*$ is an embedding. Since ξ^* is isolated over ξ, there is $\xi^{**} \supseteq \xi^*$ and an embedding $f^*{:}\, \zeta^* \to \xi^{**}$ compatible with g. Then let $\ell^* = (\xi^{**}, \zeta^*, f^*)$. ☐

We now need to define the $\Delta^0_{\omega+1}(Y)$ instruction function q. Given a finite run $\pi = \ell_0 u_1 \ell_1 u_2 \dots \ell_i \in P$, we define $q(\pi)$. We will define q such that if π is a run according to q, then we have that $u_i = (\zeta_i, g_i)$ and ζ_i is the ω-bftype of an initial segment $\bar{a}_i$ of $\mathcal{B}$. Let $\ell_i = (\xi, \zeta, f)$. Recall that ξ is isolated over $f(\zeta)$ by a Σ^C_α formula φ for some $\alpha < \omega$. Let $\zeta = \zeta(x_1, \dots, x_m)$. Let $\xi = \xi(y_1, \dots, y_m, z_1, \dots, z_n)$ where $y_i = x_{f(i)}$ and $z_1, \dots, z_n$ list the other variables. Then $\xi \models \varphi(y_1, \dots, y_m, z_1, \dots, z_n)$, and so $\xi \models \exists u_1, \dots, u_n \varphi(y_1, \dots, y_m, u_1, \dots, u_n)$.

So $\zeta \models \exists u_1, \dots, u_n \varphi(x_1, \dots, x_m, u_1, \dots, u_n)$. Now ζ is the ω-bftype of an initial segment $\bar{a}$ of $\mathcal{B}$. Choose a longer initial segment $\bar{a}' = (\bar{a}, a'_1, \dots, a'_k)$ such that among $\bar{a}'$ are witnesses $a'_{j_1}, \dots, a'_{j_m}$ to $\mathcal{B} \models \varphi(\bar{a}, a'_{j_1}, \dots, a'_{j_m})$, and such that $\bar{a}'$ includes at least the first $i+1$ elements of $\mathcal{B}$. Since φ isolated ξ over $f(\zeta)$, $(\bar{a}, a'_{j_1}, \dots, a'_{j_n})$ has the same ω-bftype as ξ. Let $q(\pi)$ be the ω-bftype of $\bar{a}'$ with the unique embedding compatible with f. We can compute this using $\Delta^0_{\omega+1}(Y)$.

There is a run π of (P, q) such that $E(\pi)$ is c.e. in Y. Let $\mathcal{M} = E(\pi)$. If $\pi = \ell_0 u_1 \ell_1 u_2 \dots$ where $\ell_i = (\xi_i, \zeta_i, f_i)$, we claim that $f = \bigcup_i f_i$ induces an isomorphism $\mathcal{B} \to \mathcal{M}$. We see from the construction that f_i contains the first i elements of $\mathcal{M}$ in its domain, and the first i elements of $\mathcal{B}$ in its range. Since each f_i is a partial isomorphism, f is thus an isomorphism from $\mathcal{B}$ to $\mathcal{M}$. □

6 An Example

In this section we will show that there is a decidable theory satisfying arithmetic-is-recursive (and with back-and-forth ordinal $\omega + 1$). No cone is required.

Theorem 1.6. *There is a theory T with uncountably many countable models such that:*

(1) T has back-and-forth ordinal $\omega + 1$, and

(2) T satisfies arithmetic-is-recursive.

The fact that T has $2^{\aleph_0}$-many models means that T does not satisfy arithmetic-is-recursive on a cone in a trivial way, e.g. by having the base of the cone compute all models of T.

Proof. The theory will not be too complicated but will require the use of Marker extensions. First we will define a theory T in the language $\mathcal{L} = \{U_i, R_n \mid i, n \in \omega\}$ consisting entirely of unary relations. The theory T will say that:

- the U_i are disjoint infinite sets,
- $R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$,
- there are infinitely many elements in $U_i \cap (R_n \smallsetminus R_{n+1})$ for each i, n.

The principal types of T are isolated by being in $U_i \cap (R_n \smallsetminus R_{n+1})$ for some i and n. There is one non-principal type of an element not contained in any U_i, and for each i, there is the non-principal type of an element contained in $U_i \cap \bigcap_{n \in \omega} R_n$. Thus T has countably many types. On the other hand, T has $2^{\aleph_0}$-many models, because we can independently for each i realise or omit the non-principal type in U_i.

Now let T^* be the theory obtained from T by taking the Σ_n Marker extension of each R_n. This theory T^* will satisfy the conclusion of the theorem. The reader may already see why this is the case, but for completeness we must be more formal.

To take our Marker extensions, we use the back-and-forth trees from [HW02] and the facts proven about them in [CDHTM]. For each n, there are trees $\mathcal{A}_n$ and $\mathcal{E}_n$ such that:

Lemma 6.2 (See Lemma 3.3 of [CDHTM])**.** *For $0 < n$, the structures $\mathcal{A}_n$ and $\mathcal{E}_n$ satisfy:*

(1) *Uniformly in n and an index for a Σ_n set S, there is a computable sequence of structures $\mathcal{C}_x$ such that*

$$x \in S \iff \mathcal{C}_x \cong \mathcal{E}_n$$

and

$$x \notin S \iff \mathcal{C}_x \cong \mathcal{A}_n.$$

(2) *For each n, there is an elementary first-order $\exists_n$-sentence φ_n, computable uniformly in n, such that $\mathcal{E}_n \models \varphi_n$ and $\mathcal{A}_n \not\models \varphi_n$.*

(3) *$\mathcal{A}_n$ and $\mathcal{E}_n$ are prime models of their theories.*

(4) *The theories of $\mathcal{A}_n$ and $\mathcal{E}_n$ are finitely axiomatizable and countably categorical.*

(4) is not proven in [CDHTM] but follows easily from the analysis there.

We will now define the theory T^*. The language of T^* is $\mathcal{L}^* = \{U_i, f_n, P : i, n \in \omega\}$ where the U_i are unary relations, the f_n are unary functions, and P is the binary relation from the signature of the trees $\mathcal{A}_n$ and $\mathcal{E}_n$. The theory T^* will say that:

- the U_i are disjoint infinite sets,
- for each $x \in U_i$ and n, there is an infinite set $f_n^{-1}(x)$, and these sets are all disjoint and not contained in any U_j;
- the relation P puts the structure of a tree on each infinite set $f_n^{-1}(x)$, and this tree is either $\mathcal{A}_n$ or $\mathcal{E}_n$;
- if $f_{n+1}^{-1}(x) \cong \mathcal{E}_{n+1}$, then $f_n^{-1}(x) \cong \mathcal{E}_n$;
- there are infinitely many elements x in U_i with $f_{n+1}^{-1}(x) \cong \mathcal{A}_{n+1}$ and $f_n^{-1}(x) \cong \mathcal{E}_n$.

Note that we use the fact that the theories of $\mathcal{A}_n$ and $\mathcal{E}_n$ are finitely axiomatizable and countably categorical in order to express these statements.

We introduce the definable relations $R_n(x) \iff f_n^{-1}(x) \cong \mathcal{E}_n$. Then the last two statements become

- $R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$,
- there are infinitely many elements in $U_i \cap (R_j \setminus R_{j+1})$ for each i, j.

One can thus see the relation to the theory T introduced earlier. Moreover, to a model of T^* one can naturally associate a model of T, and vice versa, and this association respects isomorphism.

To see that T^* satisfies arithmetic-is-recursive, suppose that $\mathcal{A}^*$ is an X'-computable model of T^*. Let $\mathcal{A}$ be the associated model of T obtained from $\mathcal{A}^*$. Then we can X'-compute the relations U_i on $\mathcal{A}$, and we can X'-compute a Σ_n approximation to the relations R_n on $\mathcal{A}$; so we can X-compute a Σ_{n+1} approximation to the relations R_n on $\mathcal{A}$. Now will build an isomorphic copy $\mathcal{B}$ of $\mathcal{A}$ such that we can X-compute the relations U_i on $\mathcal{B}$, and we can X-compute a Σ_n approximation to the relations R_n on $\mathcal{B}$.

We build $\mathcal{B}$ as follows. We can easily build in $\mathcal{B}$ infinitely many realizations of each non-principal type. In addition, we $\mathcal{B}$ will realise some non-principal types. Let $g_i(a,s)$ be a $\Delta^0_2(X)$ approximation to $U_i(a)$ in $\mathcal{A}$, so that $\mathcal{A} \vDash U_i(a)$ if and only if $\lim_{s\to\infty} g_i(a,s) = 1$. For each triple (a,i,s) such that $g_i(a,s) = 1$ and $g_i(a,s-1) = 0$ (or $s = 0$), we build an element $b_{a,i,s}$ in $\mathcal{B}$ such that $\mathcal{B} \vDash U_i(b_{a,i,s})$, $\mathcal{B} \vDash R_1(b_{a,i,s})$, and for $n \geq 1$

- if for all s' with $s+n \geq s' \geq s$ we have $g_i(a,s') = 1$, then

$$\mathcal{B} \vDash R_{n+1}(b_{a,i,s}) \iff \mathcal{A} \vDash R_n(a),$$

- otherwise, if there is s' with $s+n \geq s' \geq s$ and $g_i(a,s') = 0$,

$$\mathcal{B} \vDash \neg R_{n+1}(b_{a,i,s}).$$

Essentially, when $\mathcal{B}$ copies $\mathcal{A}$, if it is copying a principal type then $\mathcal{B}$ "shifts" the relation that makes the type principal by one to gain an extra jump; so if the type of a is principal because of $\neg R_n(a)$, the type of $b_{a,i,s}$ will be principal because of $\neg R_{n+1}(b_{a,i,s})$. We can X-compute a Σ_n approximation to R_n in $\mathcal{B}$ using the X-computable Σ_n approximation to R_{n-1} in $\mathcal{A}$. If a realises a principal type, then so will $b_{a,i,s}$ for every i,s. If a realises a non-principal type, then there will be exactly one value of i,s such that $b_{a,i,s}$ realises that same non-principal type, namely the i such that $a \in U_i$ and the least s after which $g_i(a,s)$ has stabilised. Thus $\mathcal{B}$ will be isomorphic to $\mathcal{A}$.

Now we can build, using (1) of Lemma 6.2, an X-computable copy of the structure $\mathcal{B}^* \vDash T^*$ associated to $\mathcal{B}$. Then $\mathcal{B}^*$ is an X-computable copy of $\mathcal{A}^*$ as desired.

To see that T has back-and-forth ordinal $\omega+1$, we must show that there are countably many ω-bftypes, and uncountably many $\omega+1$-bftypes. It is not hard to see that the models of T^* are homogeneous, and so the ω-bftypes coincide with the model-theoretic types. There are only countably many of these. But then the $\omega+1$-bftypes determine which non-principal types are realised, and so there are uncountably many of these. □

References

[AK00] Chris J. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[Ash86a] Chris J. Ash. Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees. *Trans. Amer. Math. Soc.*, 298(2):497–514, 1986.

[Ash86b] Chris J. Ash. Stability of recursive structures in arithmetical degrees. *Ann. Pure Appl. Logic*, 32(2):113–135, 1986.

[Ash90] Chris J. Ash. Labelling systems and r.e. structures. *Ann. Pure Appl. Logic*, 47(2):99–119, 1990.

[CDHTM] Barbara F. Csima, Michael Deveau, Matthew Harrison-Trainor, and Mohammad A. Mahmoud. Degrees of categoricity above limit ordinals. Preprint.

[CK15] Barbara F. Csima and Carolyn Knoll. Measuring complexities of classes of structures. *Ann. Pure Appl. Logic*, 166(12):1365–1381, 2015.

[HW02] Denis R. Hirschfeldt and Walker M. White. Realizing levels of the hyperarithmetic hierarchy as degree spectra of relations on computable structures. *Notre Dame J. Formal Logic*, 43(1):51–64 (2003), 2002.

[Ler17] Manuel Lerman. *Degrees of Unsolvability: Local and Global Theory*. Perspectives in Logic. Cambridge University Press, 2017.

[Mar16] David Marker. *Lectures on infinitary model theory*, volume 46 of *Lecture Notes in Logic*. Association for Symbolic Logic, Chicago, IL; Cambridge University Press, Cambridge, 2016.

[Mon12] Antonio Montalbán. Counting the back-and-forth types. *J. Logic Comput.*, 22(4):857–876, 2012.

[Mon13] Antonio Montalbán. A computability theoretic equivalent to Vaught's conjecture. *Adv. Math.*, 235:56–73, 2013.

[Sco65] Dana Scott. Logic with denumerably long formulas and finite strings of quantifiers. In *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, pages 329–341. North-Holland, Amsterdam, 1965.

[Sil80] Jack H. Silver. Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Ann. Math. Logic*, 18(1):1–28, 1980.

[Vau61] Robert L. Vaught. Denumerable models of complete theories. In *Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959)*, pages 303–321. Pergamon, Oxford; Państwowe Wydawnictwo Naukowe, Warsaw, 1961.