AN ARITHMETIC ANALYSIS OF CLOSED SURFACES

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Abstract. From a computability-theoretic standpoint, we consider the following problem: Given a closed surface, as a topological space, how hard is it to recover an atlas? We prove that every computable Polish space homeomorphic to a closed surface admits an arithmetic atlas, and indeed an arithmetic triangulation. This is as simple as one could reasonably hope for; essentially, the locally Euclidean structure of a surface can be recovered from the topological structure in a first-order way, i.e., without reference to curves or homeomorphisms or other higher-order objects.

It follows that given two computable presentations of the same closed surface, there is an arithmetic homeomorphism between them. Moreover, the homeomorphism problem for closed surfaces, presented as topological spaces, is arithmetic. From the algorithmic and definability-theoretic standpoint, this improves Kline's conjecture proved by Bing in the 1940s. We also consider $\mathbb{R}^2$ and the closed unit ball.

1. Introduction

A topological manifold is a topological space for which there *exists* some atlas of coordinate charts; but the coordinate charts do not really form part of the structure of the manifold. In this paper, we ask:

> *How difficult is it to recover an atlas of coordinate charts from the topological structure of a topological manifold?*

We develop a general machinery which allows us to attack this and similar questions and measure the complexity of various problems in topology and geometry. For the most part in this paper we consider surfaces, where the arguments are already quite complicated. Our main results say that, for a closed surface, one can *arithmetically* recover an atlas and a triangulation from the topological structure; we will clarify what this means shortly, but we note that 'arithmetic' here is as closest to 'computable' as one could possibly hope for. Since closed surfaces are easily understood given a triangulation, we also get that any two presentations, as topological spaces, of a closed surface are arithmetically homeomorphic, and that the characterization problem for closed surfaces among all Polish spaces is arithmetic. We now give some background and details.

1.1. **Computability in analysis and topology.** Our approach for measuring difficulty is that of computability theory. The study of undecidability in manifolds mostly branched off from computability in the 1960's, and with a few exceptions (e.g., [Soa04, CS06, NW00]) the two have not really been reunited since.

Topologists have mainly been focused on analysing decision problems about manifolds represented as finite simplicial complexes to determine whether the decision problem is decidable or undecidable. On the other hand, computability theorists have developed many tools for dealing with *countably* representable objects such as separable metric spaces ([Wei00, PER89,

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BHW08]) and also tools for measuring exactly how undecidable a problem is ([Soa87, AK00]). These tools have not yet been applied to manifolds.

This paper seeks to bring these modern tools to bear on the study of *topological* manifolds. There is a huge unexplored area here. In an attempt to amend this in Section 2 we will give a brief history of the computability-theoretic study of manifolds, with the additional goals of (a) giving the geometer some idea of the kinds of questions computability theory can answer, as well as some idea of what it means to be arithmetic, (b) introducing the computability theorist to the existing literature and some interesting problems about manifolds, and (c) posing some questions for further work.

To state our results formally we need to explain the main definitions. We represent the topological space of a surface as the completion $X = \overline{M}$ of a countable metric space M, representing points in the topological space by Cauchy sequences from M just as one might compute with the reals $\mathbb{R} = \overline{\mathbb{Q}}$ by considering rational approximations. We will clarify this formally in Section 3. Such representations go back to Turing [Tur36, Tur37] and have been quite fruitful in using computability theory to study topology [PER89, Wei00, MN13, NS15, Sel20, HKS, HTMN]. Given a computably metrized topological space homeomorphic to a manifold, we consider the computational problem of constructing an atlas. This will not be computable in general, but we can measure the difficulty using computability-theoretic hierarchies which are the computability-theoretic equivalents of complexity-theoretic hierarchies such as P, NP, and the polynomial hierarchy.

If a problem is decidable, it is not necessarily efficiently decidable; we can make a finer analysis by measuring the time or space complexity. Similarly, if a problem is undecidable, it is not necessarily completely intractable *especially* if we are dealing with infinite presentations. For example, Malcev [Mal61] observed that linear independence in a countable vector space over $\mathbb{Q}$ does not have to be decidable, and this is easy to see that the isomorphism problem for such spaces is undecidable. Of course all such spaces are well-understood and are fully classified by their dimension; this is reflected by the fact that these problems have arithmetic solutions. Compare this to a result of Downey and Montalbán [DM08] who proved that deciding whether two countable torsion-free abelian groups are isomorphic is an analytic complete problem, and hence far from arithmetic. The result essentially says that the class of torsion-free abelian groups lacks useful invariants because to check if two such groups are isomorphic one has to go through all potential isomorphisms and check whether any such map works. We cite [GK02] for discussion of the approach and [Mel18, BMM] for more recent applications to classification problems in topology.

In Section 2 we will explain what it means for a decision problem to be arithmetic in more detail, and also how it is related to the iterated Halting problem $0^{(n)}$. At this stage we only mention that, for a problem coded as a subset of natural numbers, being arithmetic is the same as being definable by a first order formula in the semi-ring of natural numbers (thus, the name). Indeed, such a problem is decidable relative to $0^{(n)}$ if and only if both the problem and its complement are definable in the semi-ring $\mathbb{N}$ by a formula beginning with $\exists$ and having n unbounded quantifiers ranging over natural numbers; this follows from the celebrated solution [Mat93] to Hilbert's 10^{th} problem. Essentially what it means to be arithmetic in the context of a Polish space $X = \overline{M}$ is that there is an algorithm which requires answers to *first-order* questions only, that is, questions quantifying over points from the fixed dense set M, but not over higher-order objects such as arbitrary points in the completion $X = \overline{M}$, curves, subsets, etc.[1]

1.2. **The main results.** We return to the question about finding atlases and triangulations of manifolds raised at the beginning of the introduction. At first glance the property of having an arithmetic atlas is not first-order/arithmetic; indeed, it seems to require an exhaustive search through all possible homeomorphic embeddings of the unit disc into the given space. It is thus perhaps unexpected that an atlas of a compact surface can be constructed arithmetically.

Theorem 1.1. *Every computable surface without boundary has an arithmetic atlas.*

The proof of this theorem is in Section 4. The upper bound on the complexity that can be extracted from the proof is $0^{(20)}$; this means that one can construct an atlas by understanding questions involving 19 alternations of first-order quantifiers. We conjecture that this upper bound is not optimal, but we also suspect the complexity cannot be improved too much in general (e.g., we expect that it cannot be improved to something like $0^{(3)}$); we leave this open. On the other hand, this relatively high arithmetic complexity of the atlas comes from repeatedly checking various local properties, e.g., verifying if two points are connected, within some fixed open set and for every ϵ, by a sequence of points of distance ϵ from each other. In other words, the result and the proof illustrate that *all pathologies and obstacles that one faces when trying to build an atlas are local in their nature.*

In our main theorem, Theorem 1.1, we represent a manifold as the completion of a countable metric space M. One would not expect a decision problem involving such representations to be decidable, as at any finite amount of time, a computer program will only have looked at finitely many points from M. So for such decision problems, having an arithmetic solution is the best we could hope for. Of course, there are many possible representations of manifolds. Other than representing a manifold as the completion of a computable metric space, what are some of the most compelling alternate notions of computability for a manifold?

- As the completion of a countable metric space together with an atlas. This type of notion has been recently introduced and studied in [AC17].
- As a closed subset of $\mathbb{R}^q$. It is well-known that every closed manifold can be realised as a subset of a suitable power of $\mathbb{R}$.
- As a simplicial complex, i.e., S is given by a computable triangulation. This is certainly the strongest notion, and of course it is highly practical. Note, however, that some manifolds cannot be triangulated (e.g., [Man16a]), but this would not be an issue for closed surfaces. Nonetheless, all classical triangulation proofs for surfaces that we are aware of are not even close to being arithmetic, let alone computable, so one could argue that this notion is way too strong.

[1]For example, asking "*Are there infinitely many points $x \in M$ within ϵ of a?*" is a first-order question. In contrast, analytic complete problems (such as the aforementioned [DM08]) require an exhaustive search through the uncountable space of functions $f : \mathbb{N} \to \mathbb{N}$. Equivalently, such problems require second-order quantification over *subsets* of natural numbers. In the context of Polish spaces, "*Is there a path from a to b?*" and "*Is there a homeomorphic embedding of an 3-sphere to X?*" are not first-order questions, at least as stated.

How different are these representations from each other? Relying mainly on Theorem 1.1 to do the heavy lifting, we show that for a closed surface we can pass between these representations in an arithmetic way.

Theorem 1.2. *For a closed surface S, we can pass between the following representations arithmetically, including computing homeomorphisms between the two representations:*

(1) S is represented as the completion of a countable metric space.
(2) S is represented as the completion of a countable metric space with an atlas.
(3) S is represented as a simplicial complex.
(4) S is represented as a closed subset of $\mathbb{R}^5$.

The proof of this theorem is throughout the paper but is finally put together in Section 6. We note that (1) → (3), which says that every computably metrized surface has an arithmetic triangulation, is of independent interest. This fact is proved as Theorem 5.1. All the triangulation proofs that we are aware of are far from being algorithmic, so we had to design a new apparatus based on definability up to homeomorphism. Since our results are based on these definability techniques, they are of course relativizable to any oracle.

1.3. **Applications to the classification problem in topology.** Finally, we draw a connection with work on topological characterisations of particular spaces. Suppose we are given $X = \overline{M}$. How hard is it to tell whether X is homeomorphic to a 2-dimensional sphere? Can we characterise the homeomorphism type of the sphere? Such investigations can be traced back to the beginning of the 20th century. For instance, in 1919, Moore [Moo19] gave a complex axiomatic characterisation of the Euclidean plane up to homeomorphism. Almost 30 years later, Bing [Bin46] confirmed Kline's conjecture by showing that the 2-sphere S^2 is described, up to homeomorphism, by the following property: It is the only locally connected metric continuum separated by any simple closed curve but by no pair of points. In 1992 Thomassen [Tho92a] gave a similar characterisation of the unit 2-sphere: It is a compact arcwise-connected metric space X satisfying the following two conditions: (1) if J is an arc in X, then $X - J$ is arcwise connected, and (2) if J is a simple closed curve in X, then $X - J$ is not connected. There are also neat characterisations of n-spheres in terms of curvatures of arcs in the space; see [BKK03].

From a logician's point of view, we want to measure how good these characterisations are, and to measure the inherent complexity of characterising these spaces. A good characterisation would be *syntactically* or *computationally* simpler than just checking that a given space is homeomorphic to S^2 by going through all potential homeomorphisms. Going through all uncountably many potential homeomorphisms would not be arithmetic, for example, so we hope for an arithmetic characterisation. The characterisations described above are not inherently arithmetic, because they refer to arbitrary points and to Jordan curves, of which there are uncountably many.

Computability theory can be used to formally clarify this intuition. More specifically, recently there have been several successful applications of computability-theoretic techniques to classification and characterisation problems in Polish spaces, Polish groups, and separable Banach spaces. Given a property P of a Polish space, we can measure the complexity of the index set of P, $\{M : \overline{M} \text{ has property } P\}$, where M is a countable metric space. For example, Nies and Solecki [NS15] showed that *the characterization problem* for locally compact spaces $\{M : \overline{M} \text{ is locally compact}\}$ is Π^1_1-complete, i.e., it is complete among all problems which require one universal set-quantifier, and thus there is no characterisation of such spaces which would be simpler than the brute-force definition. In contrast, building on the work of Gromov [Gro07], Melnikov and Nies [MN13] showed that and the characterization problem for compact metric spaces and *the isometry problem* $\{(M,N) : \overline{M} \cong \overline{N} \text{ and } \overline{M}, \overline{N} \text{ are compact}\}$ are both arithmetic. It is also known that, for instance, the (topological) isomorphism problems for compact connected and profinite abelian groups are both Σ^1_1-complete [Mel18], and that Lebesgue spaces admit an arithmetical characterisation among all separable Banach spaces [BMM]. The general pattern here is that tractable classes tend to have their index sets arithmetical, while classes where no reasonable invariants are known or expected usually have either the characterization problem or the isomorphism problem Σ^1_1- or Π^1_1-hard. All these results discussed above can be relativised to an arbitrary oracle, and therefore are not really restricted to computable members of the respective classes. We cite the recent survey [DM20] for a detailed discussion of this approach, and for further results and open questions.

How hard is it to tell whether X is homeomorphic to a compact surface? How hard it is to tell if two topologically represented compact surfaces are homeomorphic? Using among other tools (1)→(3) of Theorem 1.2 and the classification of closed surfaces, we get that:

Theorem 1.3.

(1) *The characterization problem for compact surfaces among all Polish spaces is arithmetic. That is, the set*
$$\{M : \overline{M} \text{ is a compact surface}\}$$
is arithmetic.

(2) *The homeomorphism problem for compact surfaces, represented as Polish spaces, is arithmetic. That is, the set of pairs*
$$\{(M,N) : \overline{M} \text{ and } \overline{N} \text{ are homeomorphic compact surfaces}\}$$
is arithmetic.

The proof of this theorem is in Section 5. Theorem 1.2 says that the result above is robust in the sense that it is stable under a change of representation of spaces. Thus, there is nothing special about using Polish representations here. Note that (1) of Theorem 1.3 says that there is a first-order (arithmetical) way to tell whether a given Polish space is homeomorphic to a compact surface. Since there is a complete and computable list of all homeomorphism types of compact surfaces, each compact surface from this list also has an arithmetic characterization as well; for instance, the 2-sphere S^2. From the logician's standpoint, in the special case of S^2, this improves the above-mentioned Kline's characterization proved by Bing.

We expect that, using the machinery in this paper, one should be able to interpret the characterisations given in (1) of the theorem above in a way that is arithmetic without passing through Theorem 1.1, e.g., by considering (for some fixed n) only $0^{(n)}$-computable Jordan curves instead of arbitrary ones, in the spirit of the result of Bing.

The proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3 are intertwined throughout the paper. The implication (4) → (1) of Theorem 1.2 is of course a triviality; see Fact 6.1. The most technical theorem of this article is Theorem 1.1 which gives (1) → (2) of Theorem 1.2; it will also be one of the crucial steps in the proof of Theorem 1.1. Theorem 5.1 shows (2) → (3) of Theorem 1.2, and it will also be used in the proof of Theorem 1.3. We give two proofs of (3) → (4) of Theorem 1.2; one of these proofs works for arbitrary compact spaces of finite dimension thus really showing (1) → (4) for such spaces. We believe that this result, although it is perhaps not surprising, is new. The second proof shows (2) → (4); it is very short and almost literally follows the textbook embedding proof for manifolds. This second proof can be found in [AC17], but we give this proof for the sake of exposition.

2. Background, history, and open problems

2.1. **Undecidable problems in group theory and topology.** In the 1910's Dehn suggested asking, for a given finitely presented group $G = \langle X|R\rangle$, whether there is an algorithm that decides whether two given words in X are equal in G. If so, we say that the word problem is decidable. Such questions are, of course, intrinsically tied to geometry via the study of the fundamental group of a manifold. Dehn [Deh12] showed, for example, that the word problem is decidable for the fundamental groups of closed orientable two-dimensional manifolds of genus greater than or equal to 2.

In the 1950's, Novikov [Nov55] and Boone [Boo59] independently showed that there are finitely presented groups with undecidable word problem. They did this by constructing a group $G = \langle X|R\rangle$ which codes the *halting problem* into the word problem of the group. The halting problem is the decision problem which asks, given a program P, whether P ever halts. Turing [Tur36] showed that there is no algorithm to decide this. The coding given by Novikov and Boone is a computable transformation of programs P into words w_P such that

$$P \text{ halts} \iff w_P = e.$$

If the word problem were decidable, then we could also decide the halting problem: given a program P, compute w_P, and decide whether $w_P = e$; this tells us whether or not P halts. Similarly, Adyan [Ady57a, Ady57b] and Rabin [Rab58] showed that it is undecidable, given two finitely presented groups, whether the two are isomorphic.

Soon after, Markov [Mar58] used these results to show that it is undecidable whether two (closed) manifolds of dimension 4 or higher, given as finite simplicial complexes, are homeomorphic. The proof again uses a coding, this time of finitely presented groups into manifolds. Fix $n \geq 4$. Markov shows how to transform a finite presentation of a group $\langle X|R\rangle$ into an n-manifold $M_{\langle X|R\rangle}$ with fundamental group $\langle X|R\rangle$, and moreover, so that given two different finite presentations of the same group, the corresponding manifolds are homeomorphic:

$$\langle X|R\rangle \cong \langle Y|S\rangle \iff M_{\langle X|R\rangle} \cong M_{\langle Y|S\rangle}.$$

This gives a reduction of the problem of deciding whether two finite presentations of groups are isomorphic to deciding whether two manifolds are homeomorphic, showing that the latter problem is also undecidable. One can think of this undecidability result as a non-classification result: a sufficiently powerful classification result would yield an algorithm for the homeomorphism problem. Indeed, for $d \leq 3$ the homeomorphism problem for manifolds, presented as finite simplicial complexes, is decidable because of classification theorems; for $d = 3$ this uses the work of G. Perelman on W. Thurston's geometrization conjecture (see [Kup19]).

There are also other examples of decidability/undecidability results in geometry and topology, for example:

- S.P. Novikov (see the appendix to [VKF74]) showed that every n-manifold M, $n \geq 5$, is unrecognizable: the problem of deciding whether a given n-manifold is homeomorphic to M is undecidable. It is an open question whether S^4 is unrecognizable.

- It is also undecidable whether the geometric realization of a finite simplicial complex is a manifold. (See [Poo14].)
- There is an algorithm which, given two knots (represented as finite sequences of rational points in $\mathbb{R}^3$, which one should think of as being connected by straight lines), determines whether they are equivalent [Hak61, Hem79]. If $n \geq 3$, the problem of deciding whether two embeddings of S^n in $\mathbb{R}^{n+2}$ are equivalent is undecidable [NW96]. For $n = 2$, the answer is unknown.

2.2. **Turing reducibility and the arithmetic hierarchy.** In all of these classical results discussed above, the distinction made is between decidable and undecidable problems. When a problem is decidable, we can make a finer analysis by measuring how efficient the algorithms are, i.e., how long they take to run. This is the domain of complexity theory, by placing them in certain complexity classes like P and NP. Similarly, for undecidable decision problems, computability theory has ways of measuring how undecidable they are. The central notion is that of a Turing reduction. Informally, say that one decision problem D is Turing reducible to another E, written $D \leq_T E$, if we can computably transform solutions to the decision problem E into a solution for D. (Imagine, for example, that one has a computer together with a magic box, or *oracle*, that provides answers to instances of the decision problem E; then $D \leq_T E$ if this computer, running some program with access to the oracle, can solve D.) Think of $D \leq_T E$ as saying that D is easier to compute than E (though both may be non-computable). D and E have the same Turing degree, written $D \equiv_T E$, if $D \leq_T E$ and $E \leq_T D$. We also write $D <_T E$ if $D \leq_T E$ but $E \nleq_T D$; D is of Turing degree strictly less than E. All decidable problems have the same Turing degree, which is the least Turing degree.

For example, as described above, in showing that there is a finitely presented group G with undecidable word problem Novikov and Boone gave a Turing reduction from the Halting problem to the word problem for G:

$$\text{HaltingProblem} \leq_T \text{WordProblem}(G).$$

In fact, they have the same Turing degree, as we can make a Turing reduction from the word problem for G to the halting problem: Given a word w in the generators of G, one can write a program P_w that searches for a series of relators showing that $w = e$, and if it every finds such a relation, halts. Then P halts if and only if $w = e$. Note that both the halting problem and the word problem are both essentially infinite searches: access to the halting problem allows us to decide whether such a search will ever end.

There are natural problems which the halting problem cannot solve. For example, given a finitely presented group $G = \langle X | R \rangle$, suppose that we want to decide whether G is torsion-free. Lempp [Lem97] showed that this decision problem is not reducible to the halting problem. Deciding whether a given element is torsion is an infinite search for some power which is equal to the identity; but to check whether the group is torsion, we must ask whether for every element there is some power equal to the identity. This is not a single search, but infinitely many searches.

Given that this decision problem is not Turing reducible to the halting problem, how difficult is it? To answer such a question we want a natural "measuring stick" to measure such problems against. One can get this using the *Turing jump*. The Turing jump assigns to each decision problem D a harder decision problem D', such that D' cannot be solved by a computer with oracle D. D' is the halting problem relative to D: the problem of deciding whether a program P, run on a computer with oracle D, halts. Turing showed that $D < D'$.

We can iterate the Turing jump, writing $D = D^{(0)}, D', D'', D^{(3)}, D^{(4)}, \ldots$ for the successive jumps of D. We also write 0 for the empty set, so that $0'$ is the halting problem, $0''$ is the jump of the halting problem, and so on. The iterates of the Turing jump of 0 are part of the *arithmetic hierarchy*. We say that X is *arithmetic* if $X \leq_T 0^{(n)}$ for some $n \in \mathbb{N}$.

Let us return to the problem of determining whether a finitely presented group $G = \langle X | R \rangle$ is torsion. Lempp [Lem97] showed that this decision problem has Turing degree $0''$. Essentially, this is because it is a decision problem involving one alternation of quantifiers: for every word w in X, is there some n and a sequence of relators witnessing that w^n represents 1? To give some intuition about what kinds of decision problems $0''$ can be used to solve, we will explain why this decision problem is Turing reducible to $0''$. Using the halting problem, for each word w in X, we can write a program P_w that searches for some n and a series of relators witnessing that w^n represents 1. Then P_w halts if and only if w represents a torsion element. Using the halting problem we can determine, for each w, whether or not P_w halts. Now, for each presentation $\langle X | R \rangle$, using the halting problem as an oracle, we can write a program $P_{\langle X | R \rangle}$ which searches for a word w such that P_w does not halt and w does not represent a torsion element. Then $P_{\langle X | R \rangle}$ halts if and only if $\langle X | R \rangle$ is not a torsion group. And $0''$ can determine whether or not $P_{\langle X | R \rangle}$ halts.

In general, the more quantifiers required to solve a decision problem, the higher in the arithmetic hierarchy it lies. Thus knowing at what level a decision problem lies tells us something about what a solution to the problem must involve. In general, a decision problem is arithmetic if it can be solved using some finite number of existential and universal quantifiers; the level in the arithmetic hierarchy is determined by the number of alternations between existential quantifiers and universal quantifiers. A technical note is that all of these quantifiers must be over natural numbers or other objects, such as words, which can be *represented* by natural numbers. The halting problem cannot decide, for example, whether there is a real number satisfying some equation, because there are uncountably many real numbers. So knowing that a decision problem is arithmetic says that a solution does not need to consider such higher-order objects.

2.3. **Some open problems.** Consider again the homeomorphism problem. When Markov showed that, for $n \geq 4$, the homeomorphism problem for n-manifolds is undecidable, he showed that the halting problem is Turing reducible to the homeomorphism problem for n-manifolds:

$$0' \leq_T \mathrm{HomeomorphismProblem}(n).$$

But we do not know whether the halting problem can be used to solve the homeomorphism problem; the homeomorphism problem might be of higher Turing degree.[2]

Question 1. What is the Turing degree of the homeomorphism problem for n-manifolds, presented as finite simplicial complexes, $n \geq 4$?

[2]Note that it is not true, for $n \geq 4$, that two n-manifolds, represented as finite simplicial complexes, are homeomorphic if and only if the simplicial complexes have combinatorially equivalent subdivisions. This claim was originally formed as a conjecture by Steinitz and Tietze and is known as the Hauptvermutung. It is true for $n \leq 3$ [Moi52], but was disproved in general by Milnor [Mil61]. If the Hauptvermutung were true, then the homeomorphism problem for manifolds, represented as simplicial complexes, would have the same degree as the halting problem because to determine whether two complexes are homeomorphic, we can search for combinatorially equivalent refinements; the halting problem can decide whether such a search is successful.

For a given n, knowing the complexity of the homeomorphism problem for n-manifolds tells us something about how one can classify manifolds. For $n \geq 5$, n-manifolds can be classified to some extent by surgery theory. A classification theorem for a class of objects can lead to an improved upper bound on the computational complexity of the homeomorphism/isomorphism problem for those objects; so we propose (a) studying what sort of upper bound on the homeomorphism problem can be obtained from surgery theory, and (b) determining whether or not this upper bound is sharp. If the upper bound is sharp, then this would mean that surgery theory is in some sense the best possible classification of n-manifolds. (If the upper bound is not sharp then this would mean that surgery theory does not yield the best possible classification of n-manifolds, though any reasonable proof that it is not sharp would probably consist of giving a better classification.) For topological 4-manifolds, surgery theory does not give a classification. Can we show that the homeomorphism problem for 4-manifolds is very complicated, e.g., not arithmetic? A result like this would show that 4-manifolds are inherently too complicated to have a good classification theorem.

Another phenomenon that shows up in dimension ≥ 4 is that not all closed manifolds can be represented as simplicial complexes [AM90, Man16b]. This issue of representations is sometimes an important one in computability theory. It might be, for example, that it is easier to classify closed manifolds which can be triangulated than it is to classify arbitrary closed manifolds; if this were the case, then in order to truly measure the difficulty of classifying arbitrary closed manifolds, one must consider non-triangulable closed manifolds. There are also questions that only make sense to ask if we use other representations, such as:

Question 2. Given a closed n-manifold, $n \geq 4$, is it arithmetic to decide whether it is triangulable?

In dimension 3, as in dimension 2, every closed manifold is triangulable, and so one could still hope to prove the analog of Theorem 1.1. That construction crucially uses the Schoenflies theorem, which says that the region bounded by any Jordan curve is homeomorphic to the open unit disk. This fact does not generalise to three dimensions, and this is a barrier to extending our results. We leave open:

Question 3. Does every closed 3-manifold have an arithmetic atlas?

We list some further open questions:

Question 4. Is it arithmetic to decide whether a given topological manifold admits a smooth structure?

Question 5. Is it arithmetic to decide whether two embeddings of S^n in $\mathbb{R}^{n+2}$ are equivalent, $n \geq 2$?

Finally, we suspect that our results have natural analogs in reverse mathematics (see, e.g., [Sim09]). For instance, perhaps ACA_0 proves that every compact surface can be triangulated. We leave the verification of this as an open problem.

3. Technical preliminaries

3.1. **Computable Polish spaces.** Recall that a Polish space is a separable completely metrizable topological space; so every manifold is in particular a Polish space. One performs computations on objects via *representations* of those objects encoded as natural numbers. Since Polish spaces typically have uncountably many points, we cannot represent each point by a natural number. Turing had already realized this long ago when computing with the real numbers; one must instead compute with approximations to a real number.

Definition 3.1. A real number $\alpha \in \mathbb{R}$ is *computable* (Turing [Tur36, Tur37]) if there exists a Turing machine that, given $n \in \mathbb{N}$, outputs a rational r within 2^{-n} of α.

Think of a computable real as one which has a computable Cauchy sequence converging to it for which we know the rate of convergence.

In this definition, one is thinking of the reals $\mathbb{R}$ presented as the completion of the rationals $\mathbb{Q}$. We generalize this to arbitrary metric spaces:

Definition 3.2. A *presentation* of a Polish space X is a countable set M of points and a metric d such that X is homeomorphic to the completion $\overline{M}$ of the metric space (M,d).

The presentation is a *computable presentation* (also called a *computable metrization* or a *computable Polish presentation*) if M is a computable set (of natural numbers) and the metric d is computable, that is, for every $a,b \in M$, the distance $d(a,b)$ is a computable real uniformly in a and b, and such that X is homeomorphic to the completion $\overline{M}$ of M [Wei00].

For simple reading, we will define computable representations of various objects. There are also corresponding definitions of $0^{(n)}$-computable representations or arithmetic representations; e.g., a $0^{(2)}$-computable presentation of a Polish space $X = \overline{M}$ is a metric space M such that the metric d is $0^{(2)}$-computable. We will not make these definitions explicitly, but they can easily be inferred by the reader.

Given a presentation $X = \overline{M}$ of a Polish space, we refer to the points of M as *special points*. The presentation $\mathbb{R} = \overline{\mathbb{Q}}$ is a computable presentation of $\mathbb{R}$ using $\mathbb{Q}$ as the special points. There are of course other possible presentations of $\mathbb{R}$, though this is the most natural.

We represent the non-special points of X using fast Cauchy sequences from M. In a metric space, we say that a Cauchy sequence (x_i) is *fast* if $d(x_i, x_{i+1}) < 2^{-i-1}$.

Definition 3.3. A *name* for a point $x \in X = \overline{M}$ is a fast Cauchy sequence (x_i) of points in M converging to x.

In a computably presented Polish space $X = \overline{M}$, a *computable name* for x is a computable sequence of points (x_i) in M which is a fast Cauchy sequence converging to x.

When we talk about the topology on X, we will use the open balls centered in special points with rational radii.

Definition 3.4. While considering a presentation $X = \overline{M}$ of a Polish space, when we refer to a *basic open ball* we mean a ball $B_r(x)$ with $x \in M$ and $r \in \mathbb{Q}$. We use $D_r(x)$ for the *basic closed ball* (which can be different from the closure $\overline{B_r(x)}$ of the open ball).

We also want to talk about continuous functions between spaces. Again, we want to represent them by countable objects.

Definition 3.5. Let f be a continuous function between Polish spaces $X = \overline{M}$ and $Y = \overline{N}$. A *name* of f is any collection of pairs of basic open balls (B,C) of X and Y respectively, with rational radii and centers in M and N respectively, such that $f(B) \subseteq C$, and for every $x \in \overline{M}$ and every $\epsilon > 0$ there exists $(B,C) \in \Psi$ such that $B \ni x$ and $r(C) < \epsilon$.

A function $f : X \to Y$ between computably presented Polish spaces X, Y is *computable* if it possesses a c.e. name. (Similarly, we say that is is A-computable, for $A \subseteq \omega$, if it has an A-c.e. name.)

Every continuous function has a name, and so is computable relative to some oracle. We often abuse notation to write f for a particular name for f, writing, e.g., f' for the Turing jump of the name for f. (We will never use derivatives in this paper, so we reserve f' for the Turing jump.)

The above definition of a computable map is equivalent to saying that f is represented by a Turing functional that maps fast Cauchy sequences to fast Cauchy sequences.

Lemma 3.6. *Fix $k \in \omega$. It is arithmetic ($0^{(k+4)}$-computable) to say that one compact computable Polish space X is $0^{(k)}$-computably homeomorphic to another compact computable Polish space Y and to find such a homeomorphism.*

By this we mean that the homeomorphism, in the one direction $X \to Y$, has a $0^{(k)}$-computable name. We will see in Claim 3.6.1 that the inverse from $Y \to X$ must then be $0^{(k+1)}$-computable.

Proof. Let $X = \overline{M}$ and $Y = \overline{N}$ be computable presentations of Polish spaces X and Y respectively. It is sufficient to say that there is a $0^{(k)}$-computable continuous bijection; any such map between compact spaces must have continuous inverse. Moreover, the inverse is $0^{(k+1)}$-computable.

Claim 3.6.1. *If $f : X \to Y$ is a computable homeomorphism between compact spaces, then f^{-1} is $0'$-computable.*

Proof. This fact is well-known. (Indeed, under the assumption of effective compactness of Y, one can computably reconstruct f^{-1}, and it takes $0'$ to enumerate all covers.) We give a proof here to make our presentation self-contained. Our task is find a c.e. in $0'$ name for f^{-1}. To do this, it suffices to find (c.e. in $0'$), given $y \in \overline{N}$ and ϵ, a basic open ball $B_\delta(y') \ni y$ such that $f^{-1}[B_\delta(y')] \subseteq B_\epsilon(x)$. Recall that $D_r(x)$ is the basic closed ball or radius r centered at x.

For each $y' \in N$, $\epsilon > 0$, and $\delta \in \mathbb{Q}$, look for $x \in M$, basic open balls $B_{r_1}(x_1), \dots, B_{r_n}(x_n)$ of X, and basic open balls $B_{s_1}(y_1), \dots, B_{s_n}(y_n)$ of Y, such that:

(1) The basic closed balls $D_{r_1}(x_1), \dots, D_{r_n}(x_n)$ cover $X - D_{\epsilon/2}(x)$;
(2) For each i, $B_\delta(y')$ and $B_{s_i}(y_i)$ are formally disjoint ($d(y', y_i) > \delta + s_i$);
(3) For each i, $f[B_{r_i}(x_i)] \subseteq B_{s_i}(y_i)$ as determined by the name for f.

If these exist, then $f^{-1}[B_\delta(y')] \subseteq B_\epsilon(x)$. Collect together, as a name for f^{-1}, all such pairs $(B_\delta(y'), B_\epsilon(x))$. It is for (1) that we use $0'$; if $D_{r_1}(x_1), \dots, D_{r_n}(x_n)$ do not cover $X - D_{\epsilon/2}(x)$, then as $X - D_{\epsilon/2}(x) - D_{r_1}(x_1) - \cdots - D_{r_n}(x_n)$ is open, there must be some element of the dense set M not contained in any of those basic closed balls, which we can recognize by its distance from each of $x, x_1, \dots, x_n$.

Now we must argue that for each $y \in \overline{N}$ and ϵ, there are $y' \in N$, $\delta > 0$, and $x \in M$ such that $y \in B_\delta(y')$ and for which we enumerate $(B_\delta(y'), B_\epsilon(x))$ into the name for f^{-1}. Standard topological arguments show that, taking as y' the point y itself, such an x and δ exist. (Note that $X - B_{\epsilon/2}(x)$ is compact.) Of course, y might not be in N; but taking as y' a point within $\delta/2$ of y, we have that $y \in B_{\delta/2}(y')$ and y' still satisfies (1)-(3) with δ replaced by $\delta/2$ and the same witnesses otherwise, so that $(B_{\delta/2}(y'), B_\epsilon(x))$ is enumerated into the name for f^{-1}. □

Given a continuous function $f{:}\,X \to Y$, we can compute a function $\tilde{f}{:}\,M \times \omega \to N$ such that for each $x \in M$ and $n \in \omega$, $d(f(x,n), f(x,n+1)) < 2^{-n}$. Then f is surjective if and only if $\lim_n \tilde{f}(M,n)$ is dense in N. (This is because continuous images of compact sets are compact, and compact subsets of Polish spaces are closed.) This can be decided using two jumps of $\tilde{f}$.

So now, using $0^{(k+3)}$, search for a $0^{(k)}$-computable continuous function $f{:}\,X \to Y$ and a $0^{(k+1)}$-computable $g{:}\,Y \to X$ such that f and g are both surjective, as described in the previous paragraph, and such that $f \circ g$ and $g \circ f$ are the identity. To see whether $f \circ g$ is the identity, consider the set of points

$$\{y \in Y : f(g(y)) = y\}.$$

This set is closed, and so if $f \circ g \neq \mathrm{Id}_Y$ then this must be witnessed by one of the special points in N. In particular, it is sufficient to check that $f \circ g = \mathrm{Id}_Y$ on N. This is not computable, but it is $\Pi_2^{f \oplus g}$ and thus can be checked with two jumps of f and g.

This search is $0^{(k+3)}$-computable, and so $0^{(k+4)}$ can decide whether it is successful. □

The two major obstacles we will have to face are that, in a Polish space $X = \overline{M}$, the basic open balls may not be connected and may not be simply connected. The results in the rest of this section show that nevertheless we can handle them arithmetically.

Given two basic open balls $B_r(x)$ and $B_t(y)$, there is a notion of formal containment of $B_r(x)$ in $B_t(y)$: that is, $d(x,y) < t - r$, which is an $\exists$-condition. (Similarly, a basic closed ball $D_r(x)$ is formally contained in a basic open ball $B_t(y)$ if $d(x,y) < t - r$.) However, it is possible that $B_r(x) \subseteq B_t(y)$ without formal containment, indeed if $r = t$ and $x = y$ we do not get formal containment. We could get away with formal inclusion throughout, but for our purposes the somewhat blunt Lemma 3.8 below (as well as its relativised versions) will suffice. More specifically, we note that in the compact case containment is an arithmetic relation. We can also prove this for arbitrary open sets, not just basic open sets, for which we need the following definition.

Definition 3.7. Let U be an open set of a Polish space $X = \overline{M}$. A *name* of U is any collection of basic open balls $B_1, B_2, \ldots$ of X, with rational radii and centers in M, such that $U = B_1 \cup B_2 \cup \cdots$.

An open set U of a computably presented Polish space X is *computable* if it possesses a c.e. name. (Similarly, we say that U is A-computable, for $A \subseteq \omega$, if it has an A-c.e. name.)

Then we show:

Lemma 3.8. *Let M be a compact Polish space and U, V computable open sets. It is arithmetic ($0^{(3)}$-computable) to decide whether $U \subseteq V$. (If M is effectively compact, it is in fact $0^{(2)}$-computable.)*

Proof. Suppose that U is the union of a computable collection $\mathcal{U}$ of basic open sets and V is the union of a collection $\mathcal{V}$ of basic open sets. We claim that $U \subseteq V$ if and only if for every $B_\delta(x) \in \mathcal{U}$ and every $\delta' < \delta$, there are basic closed balls $C_1, \ldots, C_n$ and $V_1, \ldots, V_n \in \mathcal{V}$ such that each C_i is formally contained in V_i and $C_1, \ldots, C_n$ cover $\overline{B_{\delta'}(x)}$. This is $0^{(3)}$-computable because the property "$C_1, \dots, C_n$ is a cover of $\overline{B_{\delta'}(x)}$" is Π^0_1: $C_1, \dots, C_n$ fail to cover $\overline{B_{\delta'}(x)}$ if and only if there is a special point in $B_{\delta'}(x)$ which is not covered by the C_i.[3]

We have that $U \subseteq V$ if and only if every $B_\delta(x) \in \mathcal{U}$ is contained in V, and $B_\delta(x)$ is contained in V if and only if

$$\forall \delta' < \delta \qquad \overline{B_{\delta'}(x)} \subseteq V.$$

Finally, by compactness, $\overline{B_{\delta-\epsilon}(x)} \subseteq V$ if and only if there is a finite cover $C_1, \dots, C_n$ of $\overline{B_{\delta-\epsilon}(x)}$ by basic closed balls such that each C_i is formally contained in some basic open set making up V. (To see this, use the fact that if a basic open ball is formally contained in some other basic open ball, then the corresponding basic closed ball is also formally contained.) ☐

Using the same ideas we can check containment of closures.

Lemma 3.9. *Let M be a compact Polish space and U, V computable open sets. It is arithmetic ($0^{(2)}$-computable) to decide whether $\overline{U} \subseteq V$. (If M is effectively compact, it is in fact $0'$-computable.)*

Proof. Similarly to the previous lemma, $\overline{U} \subseteq V$ if and only if there is a finite cover $C_1, \dots, C_n$ of U (and hence of $\overline{U}$) by basic closed balls such that each C_i is formally contained in some basic open set making up V. If $C_1, \dots, C_n$ do not cover $\overline{U}$, there is a special point in $U - C_1, \dots, C_n$. ☐

Next we will prove some facts about connectedness. The main issue we will face is that basic open balls might not be connected. In this subsection we explain how to arithmetically replace basic open balls with the connected components of their centers. We show that given an open ball and a point x in that ball, we can construct the path-connected component of x in the ball. This path-connected component is itself an open set.

The setting here will be that of a locally path connected space, so that for open sets the notions of connectedness and path connectedness are synonymous. (This, however, fails for closed sets in general.) This includes any manifold.

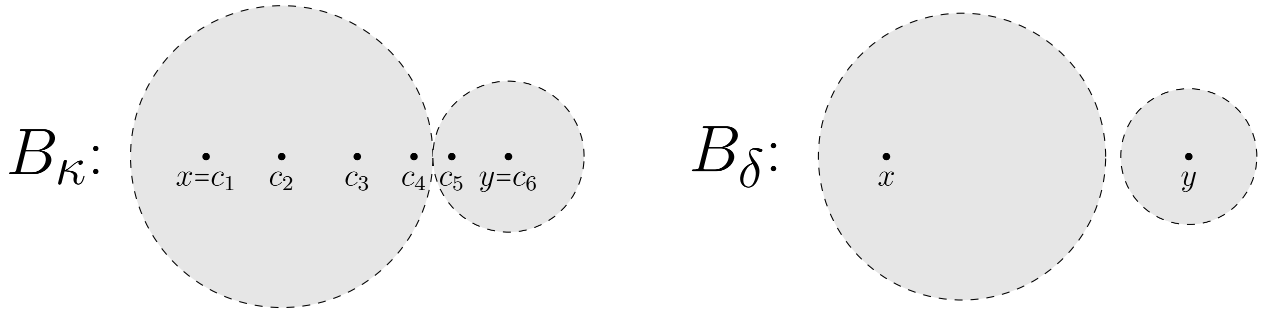
Lemma 3.10. *Given a point x of a computably metrised locally path connected Polish space $X = \overline{M}$ and a basic open ball $B \ni x$, we can arithmetically find an open set $U \ni x$, $U \subseteq B$, such that U is the path-connected component of x in B. Moreover, U has a $0^{(4)}$-computable name.*

Proof. For the proof, we use the notion of an *ϵ-path.* Given $a, b \in M$, an ϵ-path from a to b is a sequence of steps from a to b such that each step has distance at most ϵ: points $c_1 = a, c_2, \dots, c_{n-1}, c_n = b$ of M such that $d(c_i, c_{i+1}) < \epsilon$ for each i.

The first (naive) attempt would be to claim that, in a basic open ball, a and b are connected by a path if, and only if, there is an ϵ-path between these point, for every $\epsilon > 0$. However, there might be two path-connected components of a basic open $B_\kappa(c)$ which are arbitrarily

[3]Note that it is important here that we use the closure of the open ball, rather than the closed ball, as $C_1, \dots, C_n$ may fail to cover the closed ball $D_{\delta'}(x)$ without there being a special point of $B_\delta(x)$ witnessing this.

close to each other, as on the left of the diagram below.



By shrinking B_κ to $B_{\kappa-\delta}$ as on the right of the diagram, we separate these two components, and for small enough ϵ there is no longer an ϵ-path from x to y. This motivates the following definition.

Let $x \in B = B_\kappa(c)$. Say that $x \sim y$ if there is a $\delta > 0$, $\delta < \kappa$, such that $x, y \in B_\delta(c)$ and for all $\epsilon > 0$, there is an ϵ-path from x to y in $B_\delta(c)$.

Claim 3.10.1. *$x \sim y$ if and only if there is a path from x to y in $B = B_\kappa(c)$.*

Proof of claim. If there is a path from x to y in $B_\kappa(c)$, this path is a closed set in $B_\kappa(c)$. Consider the supremum of the distances between c and the points on this path; since the path is a compact set, this maximum distance is achieved by some point on the path, and the distance is $< \kappa$. Take $\delta < \kappa$ but greater than this maximal distance. The path will still be in $B_\delta(c)$. We can choose a finite sequence of special points near the path giving an ϵ-path from x to y in $B_\delta(c)$. (See Proposition 4.5 for a more complicated version of this argument.)

Suppose there is no path from x to y in $B_\kappa(c)$, but there exists a $\delta < \kappa$ such that $x, y \in B_\delta(c)$ and for all $\epsilon > 0$, there is an ϵ-path from x to y in $B_\delta(c)$. Let C_x and C_y be the path-connected components of x and y in $B_\kappa(c)$. These sets are closed in $B_\kappa(c)$ but need not be closed in the ambient space, however, C_x and $Y = B_\kappa(c) \setminus C_x \supseteq C_y$ are relatively clopen in $B_\kappa(c)$. Given closed subsets C and D the distance between C and D is defined by the formula $dist(C, D) = \inf_{c \in C, d \in D} d(c, d)$.

If $dist(C_x, Y) > \epsilon > 0$ then there is no $\epsilon/4$-path from x to y in $B_\kappa(c)$, let alone $B_\delta(c)$. If $dist(C_x, Y) = 0$, then consider the closures $\overline{C_x}$ and $\overline{Y}$ of C_x and Y in X. By compactness, it must be the case that $\overline{C_x} \cap \overline{Y} \neq \varnothing$, and indeed each $z \in \overline{C_x} \cap \overline{Y}$ must be at distance exactly κ from the center c of $B_\kappa(c)$, for otherwise we would have $z \in B_\kappa(c)$ contradicting that C_x and Y are relatively clopen in $B_\kappa(c)$. Repeat this argument for $B_\delta(c)$ to see that $dist(C_x \cap B_\delta(c), B_\delta(c) \setminus C_x)$ cannot be 0 (as long as both sets are non-empty) for any choice of $\delta < \kappa$. Indeed if it was then, in the notation above, we would be able to find a common limit point $z \in B_\kappa(c) \supseteq \overline{B_\delta(c)}$ of B_x and its compliment. This contradicts the fact that C_x and Y are relatively clopen in $B_\kappa(c)$. It therefore must be the case that for each $\delta < \kappa$ there is $\epsilon > 0$ such that $dist(C_x \cap B_\delta(c), B_\delta(c) \setminus C_x) > \epsilon > 0$, and so there is no $\epsilon/4$-path from x to y in $B_\delta(c)$. □

An open set V is contained in the connected component of x (in B) if and only if every special point y of V has $x \sim y$. So we can arithmetically define the connected component of x in B to be the collection of all basic open balls C formally contained in B such that $x \sim y$ for every special point $y \in C$. Looking at the definition of $\sim$, $0^{(4)}$ can list these basic open balls. □

In a locally path-connected space, the path-connected component of each point is open (since each point is contained in the component together with a small enough path-connected neighbourhood). As we already mentioned above, for open sets of such spaces, their path-connected components and their connected components coincide. In particular, each path-connected component of a basic open ball is open.

Definition 3.11. We write $B_r^{con}(x)$ for the (path-)connected component of x in the open ball $B_r(x)$. We call $B_r^{con}(x)$ the *basic connected open set centered at x with radius r.*

3.2. **Computability and manifolds.** Every topological manifold is a Polish space, so by a (computable) presentation of a manifold we just mean a (computable) presentation as a Polish space. We also define computable atlases and triangulations.

Definition 3.12. Let $X = \overline{M}$ be a computable presentation of a manifold. A *computable atlas* for X is a computable sequence $(U_i, f_i)_{i\in\omega}$ of (names for) computable open sets U_i covering X and computable homeomorphisms $f_i\colon U_i \to V_i \subseteq \mathbb{R}^n$. We call each such pair U_i, f_i a computable chart.

Definition 3.13. Let $X = \overline{M}$ be a compact manifold. A *computable triangulation* of X is a finite simplicial complex K, homeomorphic to X, together with a computable homeomorphism $h : |K| \to X$.

3.3. **Jordan curves.** In this paper we will rely heavily on the properties of Jordan curves. Recall that a *Jordan curve* is a continuous closed curve $\varphi\colon [0,1] \to \mathbb{R}^2$ with no self-intersections except $\varphi(0) = \varphi(1)$; and a Jordan arc is a continuous curve $[0,1] \to \mathbb{R}^2$ with no self-intersections. Of course the reader will recall the Jordan curve theorem:

Theorem 3.14 (Jordan curve theorem)**.** *Every Jordan curve in $\mathbb{R}^2$ separates the plane into two disconnected regions, exactly one of which is unbounded.*

We call the bounded region the *interior* of the Jordan curve, and the unbounded region the *exterior*. On a closed surface, we have a similar notion of Jordan curve and arc, but it is no longer true that every Jordan curve separates the surface into two disconnected regions. However we will generally work within a small Euclidean region. In this case, a Jordan curve (contained within this Euclidean region) separates the surface into two disconnected regions, but both are bounded. We will still talk about the interior and exterior of the curve and let context determine which is which.

Jordan curves can be very complex. For example, there are Jordan curves that are nowhere differentiable and have positive two-dimensional measure. See [Sag94, Chapter 8] for several such examples. Nevertheless, up to homeomorphism of the ambient space, all Jordan curves are essentially the same.

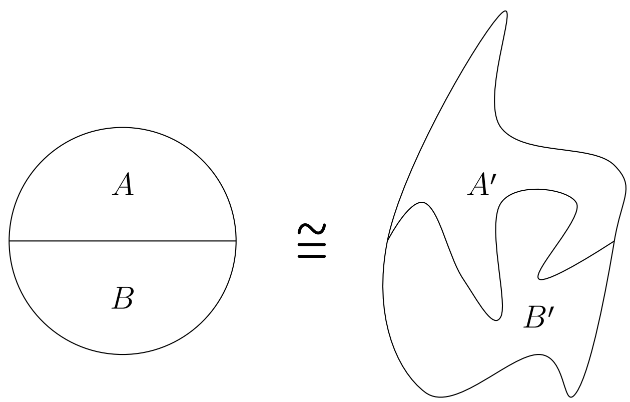
Theorem 3.15 (Schoenflies theorem)**.** *If $C \subset \mathbb{R}^2$ is a Jordan curve, then there is a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(C)$ is the unit circle in the plane.*

Thus conveniently, whenever we produce Jordan curves in our construction, we will always know what they "look like". In $\mathbb{R}^3$ there are counterexamples to the generalization such as Alexander's horned sphere, see, e.g., [Moi77]. As we have already mentioned in the introduction, this is one of the main barriers to generalising our results to 3-manifolds. (The Jordan curve theorem does generalize to three dimensions.)

The well-known radial extension theorem below, which can be viewed as a special case of Alexander's trick, will allow us to work with Jordan arcs and curves and not worry too much about the regions that they bound. For a reference, see Problem 3.2.10 of [Thu97].

Theorem 3.16. *Every homeomorphism of the boundary sphere S^{n-1} can be extended to a homeomorphism of the n-dimensional ball D^n.*

We use the combination of the above facts frequently in the following sort of situation. Suppose that we have a Jordan curve (which, by the Schoenflies theorem, we may assume is the unit circle S^1) and a Jordan arc from one point on the curve, through the interior, to another point on the curve. Then any two such figures are homeomorphic, regardless of which interior arc we choose. For example, there is a homeomorphism of the plane which maps the two figures below to each other, e.g., mapping the region A onto the region A' and the region B onto the region B'.



The same is true for any similarly constructed figures. We will use this fact implicitly throughout.

Now of course we want to bring computability in. We prove a few simple results about Jordan arcs.

Definition 3.17. Let $X = \overline{M}$ be a computably presented Polish space. A *computable Jordan arc* in X is a computable homeomorphic embedding $f\colon [0,1] \to X$.

Lemma 3.18. *Let $X = \overline{M}$ be a computably presented Polish space. We can compute, given $\epsilon > 0$ and a Jordan arc (or curve) J in X, a finite cover of J by basic open balls which is contained within the ϵ-neighbourhood of J.*

Proof. Search for finitely many pairs $(B_1, C_1), \dots, (B_n, C_n)$ in the name for $J : [0,1] \to X$, with each C_i of radius at most $\epsilon/2$, such that $B_1 \cup \dots \cup B_n$ covers $[0,1]$. Each B_i is a rational ball, so we can computably decide whether this happens. Then $C_1, \dots, C_n$ cover J. Since each C_i is of radius $< \epsilon/2$ and intersects J, they are contained in the ϵ-neighbourhood of J. □

Remark 3.19. We can then use Lemma 3.8 to test whether a given collection $U_1, \dots, U_n$ of open sets covers J: since J is compact, every open cover of J contains the ϵ-neighbourhood of J for some ϵ.

Lemma 3.20. *Let $X = \overline{M}$ be a computably presented Polish space. Uniformly using $0'$ we can compute, given two computable Jordan arcs J and K in X, whether J and K have an intersection, and if they do, we can find the first such intersection on J.*

Proof. If J and K do not intersect, then there is a some distance between the two, and so for some ϵ, their ϵ-neighbourhoods are disjoint. Thus, for sufficiently small ϵ, the open covers obtained by Lemma 3.18 will be disjoint. Of course, if J and K do intersect, then every open cover of J intersects every open cover of K.

We now argue that this can be checked with $0'$. Instead of checking if the covers are disjoint, search for covers that are formally disjoint. This means that, for any ball $B_r(c)$ of the first cover and every ball $B_q(d)$ of the second cover, we have $d(d,c) > r + q$. Since the arcs are compact sets, if they are disjoint then there is a non-zero distance between these arcs. Thus, it is not hard to see that such covers exist.

Now supposing that J and K do intersect, we want to find the least $x \in [0,1]$ such that $J(x)$ is on K. We can compute x by a sort of binary search. First ask $0'$ whether $J\restriction_{[0,\frac{1}{2}]}$ intersects K; if it does, then $x \in [0, \frac{1}{2}]$. Otherwise, $x \in [\frac{1}{2}, 1]$. Continuing in this way, we can compute x. □

A similar argument gives:

Lemma 3.21. *Let $X = \overline{M}$ be a computably presented Polish space. Uniformly using $0'$ we can compute, given a computable Jordan arc J in X and a computable point a, whether a is on J, and if it is, compute x such that $J(x) = a$.*

Now we show that we can find names for the interior and exterior of a Jordan curve (contained within some small Euclidean region). Note that the lemma also applies to Jordan curves in $\mathbb{R}^2$, since each such curve lies within some closed disk.

Lemma 3.22. *On a compact surface $X = \overline{M}$, suppose N is a Jordan region bounded by a $0^{(t)}$-computable Jordan curve C (contained in a small Euclidean region), where $t > 3$. Then N has a $0^{(t)}$-computable name as an open set.*

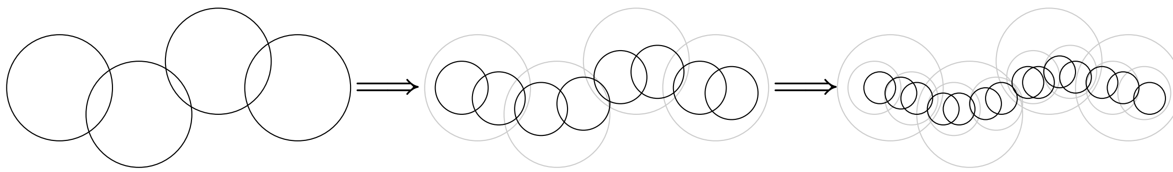
Proof. Begin by picking a connected basic open ball B_0 contained in N. (This differentiates between "inside" and "outside" C.)

At each stage s, using the uniform continuity of the Jordan curve as a continuous function $[0,1] \to C$, we can cover C by finitely many basic open balls of radius $1/s$, the centers of which are of distance at most $1/s$ from a point of C. (This requires $0^{(t)}$.) Let C_s be this open cover of C. If so far by stage s we have found connected basic open balls $B_0, \dots, B_k$ in N, put a basic open ball B into N if it does not intersect C_s, but intersects $B_0 \cup \dots \cup B_k$. This requires $0^{(4)}$ by Lemma 3.10; note that seeing whether two open sets intersect is c.e. in their names. At the end, we will have covered N by basic connected open balls. We need $0^{(4)}$ in order to write the basic connected open balls as unions of basic open balls. □

4. Arithmetic Atlases and the Proof of Theorem 1.1

Our goal in this section is to prove Theorem 1.1: for topological surfaces one can reconstruct the coordinate charts in an arithmetic way. We fix a compact computable metrized surface $X = \overline{M}$ throughout the section.

4.1. **Overview of the proof.** The main technical step in the proof of Theorem 1.1 is to show that one can approximate Jordan arcs by arithmetic Jordan arcs. This is done using sequences $U_1, \ldots, U_n$ of connected open sets, each with small diameter, such that each ball intersects the next. One can then refine such a sequence to a sequence of smaller diameter, and so on, yielding in the limit a Jordan arc.



There are a number of technical details involved in making sure that the limiting process works, and moreover to have the arithmetic Jordan arc begin and end at specific points, or to have it not cross some other arithmetic Jordan arc, and so on.

The next step is to define the domain of each chart in the atlas. The domain of each chart will be the open region bounded by an arithmetic Jordan curve. The fact that each Jordan curve can be approximated by an arithmetic Jordan curve will allow us to arithmetically cover the surface with such regions. Recall that a region in $\mathbb{R}^2$ bounded by a Jordan curve is homeomorphic to the open unit disk; and if X is indeed a compact surface then we can cover the surface X by finitely many such curves. Since such a covering exists, we will be able to search for one that works.

On the domain of each chart, we must define a homeomorphism with a subset of $\mathbb{R}^2$. The boundary of the domain is given by a Jordan curve, which we can think of as homeomorphic to the boundary of the unit square. The idea here is to cover the interior of the Jordan curve with "horizontal" and "vertical" gridlines, such that the horizontal gridlines do not intersect each other, the vertical gridlines do not intersect each other, and each pair of horizontal and vertical gridlines intersect exactly once. Moreover, we ensure that every pair of points is separated by some gridline. We can use such gridlines to build a homeomorphism. This is the most challenging step in the proof, because we must keep the procedure computable relative to some fixed $0^{(k)}$, but it seems that the more gridlines we introduce the more complex properties we will have to check (because we have to have the new gridlines have the right intersection properties with previous gridlines), so the complexity seems to go all the way up to $0^{(\omega)}$. In order to keep a bound on the complexity, we must be more clever—but we leave this until later. (See Section 4.3.3. We will keep the arcs locally more effective, but more complex towards the end-points.)

4.2. **Approximations of Jordan arcs.** Our goal is to show every Jordan arc can be approximated by an arithmetic Jordan arc. Unless otherwise specified, the setting for all of the results in this section is a compact connected computable Polish space $X = \overline{M}$ which is also locally connected (and hence locally path connected). In later sections, we restrict further to the case of computable closed surface $X = \overline{M}$ (or within a connected open ball with compact closure in an non-compact surface $X = \overline{M}$ without boundary).

The first step is to replace a Jordan arc by a sequence of open balls approximating it.

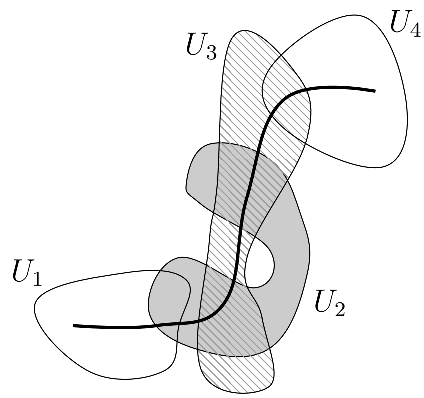
Definition 4.1. An ϵ-*arc* consists of open sets $U_1, \ldots, U_n$ such that:

(1) Each U_i is a basic connected open set $B^{con}_{\delta}(x)$, $\delta < \epsilon$ (see Def. 3.11);
(2) $U_i \cap U_{i+1} \neq \varnothing$;
(3) $U_i \cap U_j = \varnothing$ for $|i - j| > 1$.

Let a and b be special points in M. An ϵ-arc $U_1, \dots, U_n$ is *from a to b* if $a \in U_1$ and $b \in U_n$.

Remark 4.2. Given ϵ and basic connected open sets $U_1, \dots, U_n$, we can use $0^{(5)}$ to decide whether $U_1, \dots, U_n$ are an ϵ-arc (or an ϵ-arc from a to b). Recall that the U_i, as basic connected open sets, are $0^{(4)}$-computable, and of course two open sets intersect if and only if there is a special point that belongs to both sets.

If an ϵ-arc is supposed to approximate a Jordan arc, then it perhaps should at the very least contain a Jordan arc. This is indeed the case because each U_i is connected, $U_1 \cup \dots \cup U_n$ is connected by (2). One would also hope that such an arc can be chosen so that once it leaves U_i it will never goes back to U_i. This is not the case. Note that after the arc leaves U_i, it might have to return to U_i, as illustrated on the diagram below.

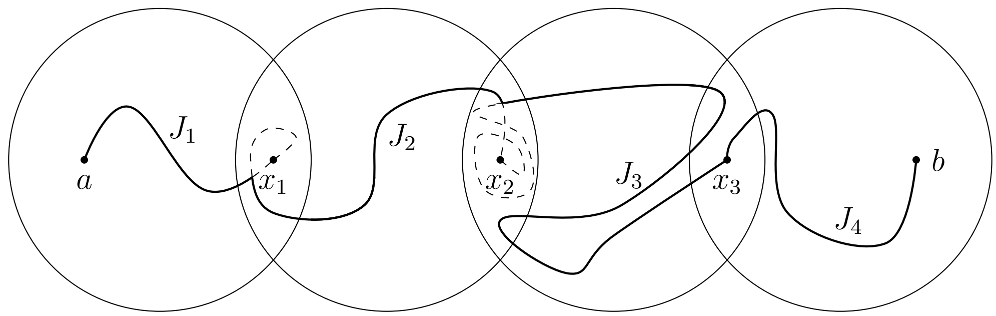


Every Jordan arc through this ϵ-arc must either leave U_2 and then later enter it again (as the arc shown does), or leave U_3 and then later enter it again. The best we can do is the proposition below: We can choose an arc so that once the arc leaves U_2 it never again returns to U_1, and once it leaves U_3 it never again returns to U_2, etc.

Proposition 4.3. *Let $U_1, \dots, U_n$ be an ϵ-arc from a to b. Then there is a Jordan arc from a to b contained in $U_1 \cup \dots \cup U_n$. Moreover, once the arc leaves U_{i+1}, we can choose it to be entirely in $U_{i+1}, U_{i+2}, U_{i+3}, \dots$, and so it never returns to U_i.*

Proof. Since U_i and U_{i+1} intersect for each i, and each U_i is connected, $U_1 \cup \dots \cup U_n$ is connected, hence path-connected. Since $a \in U_1$ and $b \in U_n$, there is an arc from a to b in $U_1 \cup \dots \cup U_n$. Indeed, we can choose $x_1, \dots, x_{n-1}$ in M such that $x_1 \in U_1 \cap U_2$, $x_2 \in U_2 \cap U_3$, and so on, and then choose Jordan arcs J_1 from a to x_1 in U_1, J_2 from x_1 to x_2 in U_2, and so on, up to J_n from x_n to b in U_n. Now the concatenation J of $J_1, J_2, \dots, J_n$ might not be a Jordan arc because it might be self-intersecting. It does, however, have the property that once the curve leaves U_{i+1}, it is contained in $U_{i+1}, U_{i+2}, U_{i+3}, \dots$. This is by (2) of Definition 4.1.

Now simply remove the intersections between the different J_i, e.g., as in the following example:



Each J_i can only intersect with J_{i+1} and J_{i-1} (by (3) of Definition 4.1), in U_{i+1} and U_{i-1} respectively. ☐

Our next step is to show that a Jordan arc can be approximated by an ϵ-arc. In order to say that the ϵ-arc is close to the Jordan arc, we ask that it be contained in some neighbourhood of the Jordan arc.

Definition 4.4. Let N be an open set. If $U_1, \ldots, U_n$ is an ϵ-arc, we say that it is *contained in* N, or that N *is a neighbourhood of the* ϵ*-arc*, if $U_1, \ldots, U_n$ are contained in N.

Of course, each Jordan arc, without any computability assumptions, has a computable open neighbourhood: by compactness, J is contained within finitely many basic open balls. Now, given a Jordan arc and a neighbourhood, we approximate the Jordan arc by an ϵ-arc in the neighbourhood.[4]

Proposition 4.5. *Let J be a Jordan arc from a to b contained in a neighbourhood N. Let $\epsilon > 0$. Then there is an ϵ-arc $U_1, \ldots, U_n$ from a to b contained in N. Moreover, if A and B are open sets containing a and b respectively, we may choose $U_1 \subseteq A$ and $U_n \subseteq B$.*

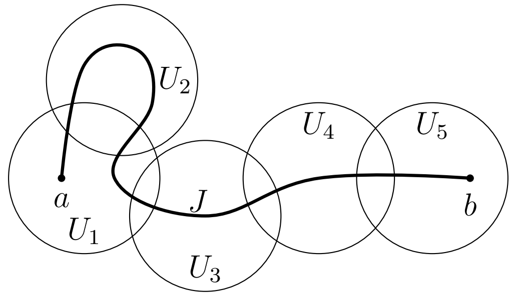
Proof. Fix a parameterization $f\colon [0,1] \to J$ of J. We may assume that $\epsilon < d(a,b)/4$.

Cover $J \cap N$ with basic connected sets $B_r^{con}(c) \subseteq N$, with each c a special point of M, and each radius a rational number $r < \epsilon$. Since J is compact, there are finitely many of these open sets $B_{r_1}^{con}(c_1), \ldots, B_{r_n}^{con}(c_n)$ that cover J. It is possible to ensure that $a \in B_{r_1}^{con}(c_1) \subset A$ and $b \in B_{r_n}^{con}(c_n) \subset B$. Let $U_i = B_{r_i}^{con}(c_i)$.

Before we proceed, we give an intuitive explanation of a potential issue and how to fix it. Note that $J \cap U_1$ does not have to be connected; however, U_1 is connected (and thus

[4]Later in the proof, we will also have to deal with the situation when we need to construct an arc from a to b where a, b are merely on the boundary of an open N. It seems to the authors that we cannot do this arithmetically for general N. Luckily, our neighbourhoods N will always be Jordan regions, which have the property that if a is on the boundary of N, and U is an open set containing a, then $U \cap N$ has exactly one connected component with a on its boundary. This component can be defined arithmetically.

path-connected). For example, the cover might look like this:



We are violating both (2) and (3) of Definition 4.1, and there is no way to use these basic connected open sets to give an ϵ-arc containing the Jordan arc J. Luckily, we are allowed to find an ϵ-arc that does not contain J. We can make a "shortcut" and skip the part of J which is in U_2, using U_1, U_3, U_4, U_5 as the ϵ-arc. Note that as $a \in U_1$ and $b \in U_n$, our ϵ-arc will start and end with these open sets.

More formally, we do the following. Let d_{i_0} be the rightmost limit point of U_1 along the arc J. More formally, let $t = \sup f^{-1}(U_1 \cap J)$. Since $\epsilon < d(a,b)/4$, $b \notin U_1$ and so $t < 1$, and $t \notin f^{-1}(U_1 \cap J)$. Let $d_{i_0} = f(t)$; $d_{i_0} \notin U_1$, but d_{i_0} is on the boundary of U_1. Similarly, let d_i be the rightmost limit point of U_i along J.

Since the U_i cover J, we can choose $i_1, \dots, i_k$ such that

(1) $d_{i_0} \in U_{i_1}$;
(2) $d_{i_j} \in U_{i_{j+1}}$ for $j < k$; and
(3) $d_{i_k} \in U_n$.

We can choose U_{i_1} because $d_{i_0} \notin U_1$. Given $U_{i_1}, \dots, U_{i_\ell}$, if $d_{i_\ell} \in U_n$ then we are done. Otherwise, there is some $U_{i_{\ell+1}} \ni d_{i_\ell}$. Note that $f^{-1}(d_{i_0}) < f^{-1}(d_{i_1}) < \cdots < f^{-1}(d_{i_k})$ and $i_0, \dots, i_k$ are distinct. Clearly, the process described above must end.

Let $V_j = U_{i_j}$. So for $V_0 = U_0, V_1, \dots, V_k, V_{k+1} = U_n$ we have that

(1) Each V_i is a basic connected set of radius $< \epsilon$ contained in N;
(2) $V_i \cap V_{i+1} \neq \varnothing$.

For (2), this is because $U_{i_{j+1}}$ is a neighbourhood of d_{i_j} and so $U_{i_{j+1}}$ intersects U_{i_j}.

We are missing the third part of being an ϵ-arc, namely that $V_i \cap V_j = \varnothing$ for $|i-j| > 1$. Define $W_0 = V_0$. Supposing that we have defined $W_0, \dots, W_t$, we define W_{t+1} to be V_i for the greatest i such that $W_t \cap V_i \neq \varnothing$. (Note that if $W_t = V_j$, then as $V_j \cap V_{j+1} \neq \varnothing$, $i > j$; so the sequences of W's is a subsequence of the sequence of V's.) Continue until we define $W_{t+1} = V_{k+1} = U_n$. The resulting sequence is an ϵ-arc from a to b contained in N. □

Recall that to construct an arithmetic Jordan arc, we want to build a sequence of more and more refined ϵ-arcs. We need to define what it means for an ϵ'-arc to refine an ϵ-arc. Essentially this should mean that $\epsilon' < \epsilon$ and that each open set of the ϵ-arc is split into many balls in the refining ϵ'-arc.

Definition 4.6. Let $0 < \epsilon' < \epsilon$. An ϵ'-arc $V_1, \dots, V_m$ *refines* an ϵ-arc $U_1, \dots, U_n$ if there are $1 = t_1 < \cdots < t_{m-1} < t_m = m$ such that for each i:

(1) $t_{i+1} - t_i \geq 2$,
(2) $\overline{V_{t_i}} \subseteq U_i$, and
(3) for each j with $t_i < j < t_{i+1}$ we have either $\overline{V_j} \subseteq U_i$ or $\overline{V_j} \subseteq U_{i+1}$.

We would really like to have that the first few V_i are contained in U_1, then the next few V_i are contained in U_2, and so on; but this might not be possible, due for example to the situation described just after the statement of Proposition 4.3.

Remark 4.7. Given an ϵ-arc $U_1, \ldots, U_n$ and an ϵ'-arc $V_1, \ldots, V_m$, we can use $0^{(6)}$ to decide whether $V_1, \ldots, V_m$ refines $U_1, \ldots, U_n$. This is because these are all $0^{(4)}$-computable open sets, and we check containment using Lemma 3.9.

For $\epsilon' < \epsilon$, each ϵ-arc can be refined to an ϵ'-arc.

Proposition 4.8. *Let $0 < \epsilon' < \epsilon$ and let $U_1, \ldots, U_n$ be an ϵ-arc from a to b. Let A and B be open sets containing a and b respectively. Then there is an ϵ'-arc $V_1, \ldots, V_k$ from a to b refining $U_1, \ldots, U_n$, with $V_1 \subseteq A$ and $V_k \subseteq B$.*

Proof. Let $J \subseteq U_1 \cup \cdots \cup U_n$ be a Jordan arc from a to b obtained from Proposition 4.3. Once J leaves U_{i+1}, it does not return to U_i. We can cut J up into non-trivial arcs $J_1, \ldots, J_{n-1}$ such that $J_1 \subseteq U_1 \cup U_2$, $J_2 \subseteq U_2 \cup U_3$, and so on, so that J is the concatenation $J_1 \to J_2 \to \cdots \to J_{n-1}$. Let $c_1 = a, c_1, \ldots, c_{n-1}, c_n = b$ be the endpoints of these arcs, so that J_1 is from c_1 to c_2, J_2 is from c_2 to c_3, and so on, and $c_i \in U_i$. By reducing ϵ', we may assume that $8\epsilon'$ is less than, for each i, the (positive) distance between the (disjoint) arcs $J_1 \to \cdots \to J_{i-1}$ and $J_{i+1} \to \cdots \to J_{n-1}$.

For each c_i, choose a basic connected open set $V_i^* \ni c_i$ of radius $< \epsilon'$, with $\overline{V_i^*} \subseteq U_i$, and with $V_1 \subseteq A$ and $V_n \subseteq B$. Now divide each J_i up into three arcs $J_i^*, J_i^{**}, J_i^{***}$ such that J_i^* starts at c_i and is contained entirely in V_i and J_i^{***} ends at c_{i+1} and is contained entirely in V_{i+1}. All three arcs should be non-trivial; let $d_i \in U_i$ be the right endpoint of J_i^* (and the left endpoint of J_i^{**}) and let $e_i \in U_{i+1}$ be the right endpoint of J_i^{**} (and the left endpoint of J_i^{***}). Let $0 < \epsilon'' < \epsilon'$ be such that ϵ'' is less than the distance between each J_i^{**} and J_j^{**}, $i \neq j$.

By Proposition 4.5, for each $i = 1, \ldots, n-1$, we can find an ϵ''-arc $W_1^i, \ldots, W_{k_i}^i$ from d_i to e_i within $U_i \cup U_{i+1}$ and also within the ϵ''-neighbourhood of J_i^{**}. We may also have $\overline{W_j^i} \subseteq U_i \cup U_{i+1}$ by choosing the ϵ''-arc within a sufficiently small neighbourhood of J_i^{**}.

Choose $1 \leq \ell_i < r_i \leq k_i$ such that ℓ_i is greatest such that V_i intersects $W_{\ell_i}^i$ and $r_i \geq \ell_i$ is the least such that V_{i+1} intersects $W_{r_i}^i$. Since $d_i \in V_i \cap W_1^i$ and $e_i \in V_{i+1} \cap W_{k_i}^i$, such ℓ_i and r_i exist. We have $\ell_i < r_i$ because c_i and c_{i+1} are of distance at least $8\epsilon'$ from each other, and each of $V_i, W_1^i, \ldots, W_{k_i}^i, V_{i+1}$ have diameter at most ϵ'.

Similarly, because $8\epsilon'' < 8\epsilon'$ is less than the distance between the J_i^{**} for different i, there are no intersections between the W_j^i for different i (and a similar argument also shows that there are no intersections between the W_j^i and the $V_{i'}$, for $i' \neq i, i+1$, between W_j^i and V_i for $j > \ell_i$, etc.). Then

$$V_1, W_{\ell_i}^1, \ldots, W_{r_i}^1, V_2, W_{\ell_2}^2, \ldots, W_{r_2}^2, V_3, \ldots, V_n$$

is the desired ϵ'-arc from a to b, with $t_1 = 1$, $t_2 = 2 + (1 + r_1 - \ell_1)$, $t_3 = 3 + (2 + r_1 + r_2 - \ell_1 - \ell_2)$, and so on, witnessing that it refines $U_1, \ldots, U_n$. (Here t_i is the index of V_i in this list.) $\square$

Finally, we put everything together to show that a Jordan arc can be approximated by an arithmetic Jordan arc. Essentially, we approximate the Jordan arc by more and more refined ϵ-arcs, and show that the limit of such ϵ-arcs is a Jordan arc.

Theorem 4.9. *Let J be a Jordan arc from arithmetic points a to b and N an arithmetic neighbourhood of J. Then there is an arithmetic Jordan arc from a to b contained in N. (If a and b are $0^{(t)}$-computable, then we get a $0^{(t+7)}$-computable Jordan arc.*[5]*)*

Proof. Let $a = \bigcap A_i$ be a $0^{(t)}$-computable name for a, and $b = \bigcap B_i$ be a $0^{(t)}$-computable name for B.

By Proposition 4.5 there is a 1-arc $U_1^1, \ldots, U_{k_1}^1$ from a to b contained in N, with $U_1^1 \subseteq A_1$ and $U_1^{k_1} \subseteq B_1$. We can find such a 1-arc arithmetically. Then, by Proposition 4.8, for each n we can find a $1/2^n$-arc $U_1^n, \ldots, U_{k_n}^n$ from a to b contained in N refining $U_1^{n-1}, \ldots, U_{k_{n-1}}^{n-1}$, with $U_n^1 \subseteq A_n$ and $U_n^{k_n} \subseteq B_n$. Again, we can find these arithmetically.

We can find $U_1^1, \ldots, U_{k_1}^1$ non-uniformly. Using $0^{(t)}$ we can compute the names for a and b. Note that the U_i^n are basic connected open sets, and hence are $0^{(4)}$-computable. We use Remarks 4.2 and 4.7 together with Lemmas 3.8 and 3.9 to check containments. The containments we check are of $0^{(4)}$-computable basic connected open sets in $0^{(t)}$-computable open sets, and so a bound one the complexity of doing this is $0^{(t+7)}$. So we can compute the sequence of ϵ-arcs using $0^{(t+7)}$.

We can transform these ϵ-arcs into a finitely branching tree, with the children ordered from left to right. Each node at level n on the tree will be an open set which is the union of two consecutive basic connected open sets from the $1/2^n$-arc. The children of the root node will be $U_1^1 \cup U_2^1,\ U_2^1 \cup U_3^1, \ldots, U_{k_1-1}^1 \cup U_{k_1}^1$, ordered from left to right in that order. At the nth level of the tree, the nodes will be $U_1^n \cup U_2^n,\ U_2^n \cup U_3^n, \ldots, U_{k_n-1}^n \cup U_{k_n}^n$, from left to right. Since $U_1^{n+1}, \ldots, U_{k_{n+1}}^{n+1}$ is a refinement of $U_1^n, \ldots, U_{k_n}^n$, there are $1 = t_1^n < \cdots < t_{k_n}^n = k_{n+1}$ as in Definition 4.6, i.e., such that $\overline{U_{t_i}^{n+1}} \subseteq U_i^n$ and for each $t_i^n < j < t_{i+1}^n$, $\overline{U_j^{n+1}} \subseteq U_i^n$ or $\overline{U_j^{n+1}} \subseteq U_{i+1}^n$. We put $U_{t_i^n}^{n+1} \cup U_{t_i^n+1}^{n+1}, \ldots, U_{t_{i+1}^n-2}^{n+1} \cup U_{t_{i+1}^n-1}^{n+1}, U_{t_{i+1}^n-1}^{n+1} \cup U_{t_{i+1}^n}^{n+1}$ as children of $U_i^n \cup U_{i+1}^n$, in that order from left to right. Since $t_{i+1}^n - t_i^n \geq 2$ this tree has no dead ends and each node has at least two children.

By Lemmas 3.8 and 3.9 we can build the tree using $0^{(t+7)}$.

Each path through this tree corresponds to a point, because: (a) the open sets at level n are contained within balls of radius at most $1/2^{n-1}$ (as the open sets at level n are the union of two intersecting open sets, each contained within a ball of radius $1/2^n$), and (b) if $U_j^{n+1} \cup U_{j+1}^{n+1}$ is a child of $U_i^n \cup U_{i+1}^n$, then

$$\overline{U_j^{n+1} \cup U_{j+1}^{n+1}} = \overline{U_j^{n+1}} \cup \overline{U_{j+1}^{n+1}} \subseteq U_i^n \cup U_{i+1}^n.$$

When $j = t_{i+1} - 1$ or $j + 1 = t_{i+1}$, we must use (2) of Definition 4.6 to see this; otherwise we use (3). The infinite paths through the tree are homeomorphic to a Cantor space $\mathcal{C}$. So we have a map $f\colon \mathcal{C} \to M$ mapping a path through the tree to the unique corresponding point of M. This map is easily seen to be continuous.

There is a surjective continuous map $\pi\colon \mathcal{C} \to [0,1]$. (Think of elements of $\mathcal{C}$ as binary or decimal (etc.) expansions, with each digit in a different base depending on the branching of the tree, and the base of the next digit depending on the previous digit.) We claim that if $\pi(x) = \pi(y)$, then $f(x) = f(y)$, so that f induces a continuous map $\tilde{f}\colon [0,1] \to M$. Indeed, $\pi(x) = \pi(y)$ if and only if there is some node σ with two children ρ_1 and ρ_2, with ρ_1 just to the left of ρ_2, such that (without loss of generality) x is the rightmost path through ρ_1 and y is the leftmost path through ρ_2. Then if x the path $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$, and y is the path $W_1 \supseteq W_2 \supseteq W_3 \supseteq \cdots$, then because for each n there is i such that $V_n = U_i^n \cup U_{i+1}^n$ and

[5] Note that the complexity of N does not affect the complexity of the arithmetic Jordan arc, though of course one needs a name for N in order to find the arc uniformly in the inputs.

$W_n = U^n_{i+1} \cup U^n_{i+2}$ we have $V_n \cap W_n \neq \varnothing$ for every n. This means that the unique element of $\bigcap V_n$ is the same as the unique element of $\bigcap W_n$, and so $f(x) = f(y)$.

It should be clear that $\tilde{f} \colon [0,1] \to M$ is computable from the tree T. We claim that $\tilde{f} \colon [0,1] \to M$ is a Jordan arc from a to b contained inside N.

Claim 4.9.1. *$\tilde{f}$ is injective.*

Proof. Suppose that $\tilde{f}(x) = \tilde{f}(y)$, with $x, y \in \mathcal{C}$, say $x < y$. Then if x corresponds to the path of open sets (V_n), and y corresponds to the path of open sets (W_n), it must be that $V_n \cap W_n \neq \varnothing$ for each n, as $\tilde{f}(x) \in \overline{V_{n+1}} \subseteq V_n$ and $\tilde{f}(y) \in \overline{W_{n+1}} \subseteq W_n$. This can only happen if for every n there is i such that $V_n = U^n_i \cup U^n_{i+1}$ and either $W_n = U^n_{i+1} \cup U^n_{i+2}$ or $W_n = U^n_{i+2} \cup U^n_{i+3}$ (recall $x < y$). We will argue that because this is true for all n, it must be that $W_n = U^n_{i+1} \cup U^n_{i+2}$.

Indeed, suppose that $W_n = U^n_{i+2} \cup U^n_{i+3}$. Then $V_{n+1} = U^{n+1}_j \cup U^{n+1}_{j+1}$ for some j with $t^n_i \leq j < t^n_{i+1}$ and $W_{n+1} = U^{n+1}_k \cup U^{n+1}_{k+1}$ for some k with $t^n_{i+2} \leq k < t^n_{i+3}$. Since $t_{i+2} - t_{i+1} \geq 2$, $k - (j+1) \geq 2$. Thus $V_{n+1} = U^{n+1}_j \cup U^{n+1}_{j+1}$ and $W_{n+1} = U^{n+1}_k \cup U^{n+1}_{k+1}$ are disjoint, giving a contradiction. We conclude that $W_n = U^n_{i+1} \cup U^n_{i+2}$.

Since for all n, V_n and W_n are adjacent nodes at the nth level of the tree, x and y are adjacent in the Cantor space $\mathcal{C}$, so their images $\pi(x) = \pi(y)$ in $[0,1]$ are equal. Thus $\tilde{f}$ is injective. ☐

Claim 4.9.2. *$\tilde{f}(0) = a$ and $\tilde{f}(1) = b$.*

Proof. We argue that $\tilde{f}(0) = a$; $\tilde{f}(1) = b$ for similar reasons. Note that 0 corresponds to the leftmost path in $\mathcal{C}$, so that $\tilde{f}(0)$ is the unique element of $\bigcap U^n_1$. We have $U^n_1 \subseteq A_n$ for each n, so that $\bigcap U^n_1 = \{a\}$. ☐

Claim 4.9.3. *For $x \in [0,1]$, $\tilde{f}(x) \in N$.*

Proof. We have that $\tilde{f}(x) \in U^1_1 \cup \cdots \cup U^1_{k_1} \subseteq N$. ☐

Thus $\tilde{f} \colon [0,1] \to M$ is a Jordan arc from a to b inside N. ☐

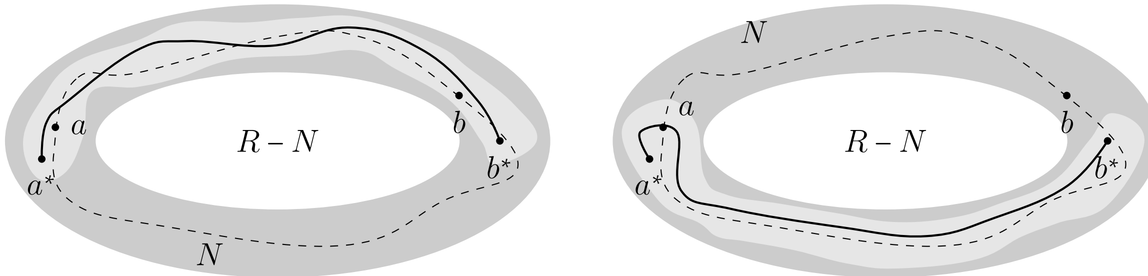
We can also approximate a Jordan curve by an arithmetic curve. The proof is to split up a curve into two arcs, and to approximate those arcs.

Theorem 4.10. *Let J be a Jordan curve, and let N be a neighbourhood around J. Then there is an arithmetic Jordan curve contained in N. Furthermore, if M is a surface and if J bounds a Jordan region R, then the arithmetic Jordan curve also bounds a region containing $R - N$. J can be chosen to be $0^{(8)}$-computable.*

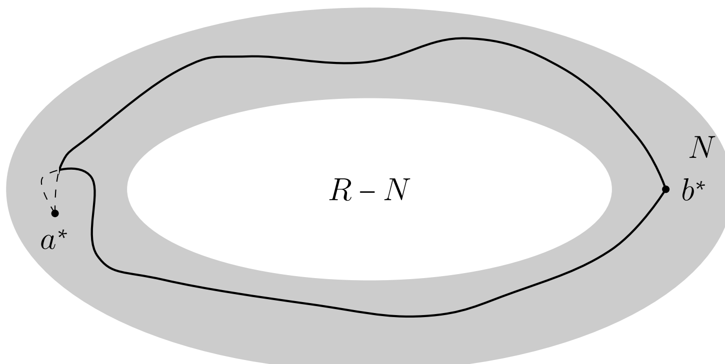
This is again not uniform, and the neighbourhood N can be of any complexity.

Proof. Shrinking N to a smaller neighbourhood of J, we may assume that $R - N$ is non-empty. First, pick two (not necessarily arithmetical) points a and b on J and (non-arithmetically) split J into two Jordan arcs J_1 and J_2 with common endpoints a and b. Let N_1 and N_2 be sufficiently small *connected* neighbourhoods of J_1 and J_2, contained in N, so that every point of N_1 is within ϵ of J_1, and similarly for J_2, for some sufficiently small ϵ. Fix special points a^* and b^* within N ϵ-close to a and b, respectively. Then by Theorem 4.9 there are

$J_1^* \subseteq N_1$ and $J_2^* \subseteq N_2$ which are arithmetic Jordan arcs from a^* to b^*.



Now these might intersect near a^* and b^*, but by cutting off an initial and end segment of each arc, we can put them together to form a Jordan curve. This Jordan curve bounds a region containing $R - N$. To compute the intersection, we must use compactness; see Lemma 3.20. (Note the new curve does not have to go though a and b, it it stays in the neighbourhood N.)



The two initial Jordan arcs can be chosen to be $0^{(7)}$-computable by Theorem 4.9. By Lemma 3.20, $0^{(8)}$ can compute the intersection and the modified arcs. □

4.3. **Gridlines.** We now assume that $X = \overline{M}$ is a computable closed surface (or just a computable surface without boundary, and we work locally inside of a connected open ball with compact closure).

Using Theorem 4.10 we will be able to search for $0^{(8)}$-computable Jordan curves so that the respective interior regions cover the whole surface. But to initiate such a search, we must also be able to arithmetically tell whether each of these $0^{(8)}$-computable potential Jordan regions are homeomorphic to the unit disc. Thus, our next goal is to prove the following theorem:

Theorem 4.11. *Let $X = \overline{M}$ be a computable closed surface,*[6] *and let J be a $0^{(8)}$-computable Jordan curve in X such that the interior of J is homeomorphic to the interior of the disk. Then there is an arithmetic ($0^{(16)}$-computable) homeomorphism between J with its interior and the unit disk with its boundary.*

We begin by fixing some notation. Let I be the open region bounded by J. Let $B = [0,1] \times [0,1]$ be the unit square, and ∂B its boundary. We may assume that J is given by a map $J\colon \partial B \to X$; fix this notation for the remainder of this section.

[6]Or a computable surface without boundary, and J is contained inside of a connected basic open ball with compact closure.

The idea is to "draw", in the interior of J, a number of (arithmetic) "horizontal" and "vertical" gridlines, and then to use these gridlines to define the homeomorphism between B and J together with its interior.

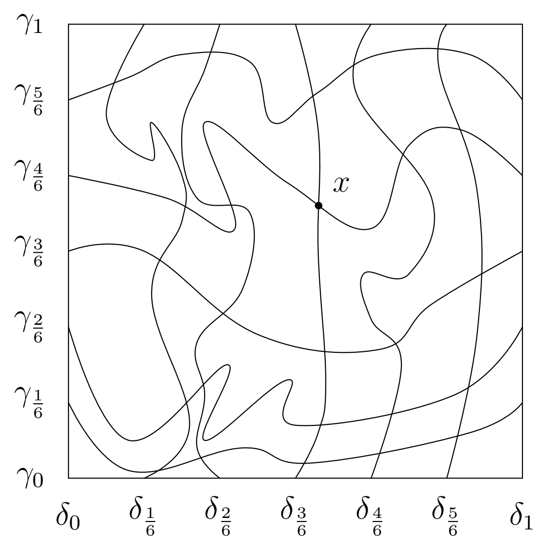
Notation 4.12. Denote by $[a,b]_{\mathbb{Q}}$ the rational numbers in the interval $[a,b]$, i.e., $[a,b]_{\mathbb{Q}} = [a,b] \cap \mathbb{Q}$.

Definition 4.13. Let $\Gamma = \{\gamma_p\}_{p \in [0,1]_{\mathbb{Q}}}$ and $\Delta = \{\delta_q\}_{q \in [0,1]_{\mathbb{Q}}}$ be collections of Jordan arcs $[0,1] \to \overline{I} \subseteq X$. We say that Γ and Δ are *gridlines*, and that Γ are *horizontal gridlines* and Δ are *vertical gridlines*, if:

(1) $\gamma_0(\cdot) = J(0,\cdot)$ and $\gamma_1(\cdot) = J(1,\cdot)$;
(2) for each p, $\gamma_p(0) = J(p,0)$ and $\gamma_p(1) = J(p,1)$;
(3) for $0 < p < 1$, other than its endpoints, γ_p is in I, the open region bounded by J;
(4) no two γ_p intersect;
(5) $\delta_0(\cdot) = J(\cdot,0)$ and $\delta_1(\cdot) = J(\cdot,1)$;
(6) for each q, $\delta_q(0) = J(0,q)$ and $\delta_q(1) = J(1,q)$;
(7) for $0 < q < 1$, other than its endpoints, δ_q is in I, the open region bounded by J;
(8) no two δ_q intersect; and
(9) each γ_p and δ_q intersect at only one point, $\gamma_p(q) = \delta_q(p)$.

We say that γ_p is the *(horizontal) p-gridline* and that δ_q is the *(vertical) q-gridline*.

The idea is that the gridlines look something like this:



When we have a finite sets of arcs $\Gamma = \{\gamma_{p_1}, \ldots, \gamma_{p_m}\}$ with $p_1 = 0 < p_2 < \cdots < p_m = 1$ and $\Delta = \{\delta_{q_1}, \ldots, \delta_{q_n}\}$ with $q_1 = 0 < q_2 < \cdots < q_n = 1$, we sometimes say that they are gridlines if they satisfy the above conditions.

We will define a homeomorphism with the unit box so that points on the intersections of two gridlines are mapped to the coordinates corresponding to the two gridlines; for example, the point x shown on the intersection of $\gamma_{\frac{4}{6}}$ and $\delta_{\frac{3}{6}}$ would be mapped to $(\frac{3}{6}, \frac{4}{6}) \in [0,1] \times [0,1]$. To extend the homeomorphism to points not between two gridlines, we need the following additional property:

Definition 4.14. We say that Γ and Δ are a *complete set of gridlines* if

(†) every pair of distinct points on the interior of J is separated by a gridline.

Each vertical gridline divides I into two "halves". Because no two vertical gridlines intersect, each other vertical gridline is contained in one half or the other. We shall adopt this self-explanatory terminology. More specifically, we say that the q-gridline divides I into a *left half* and a *right half*, where (because of where the gridlines meet the boundary) the left half contains all of the q'-gridlines, $q' < q$, and the right half contains all of the q''-gridlines, $q'' > q$. We also say that points are *to the left of* δ_q or *to the right of* δ_q. Similarly each horizontal gridline divides I into a *top half* and a *bottom half*. We say that two points are *separated* by a gridline if one point is on one side of the gridline, and the other point is on the other side. (A point on a gridline is not separated from another point off the gridline by that gridline.)

We are ready to begin the proof of Theorem 4.11. The proof is relatively long and is divided into a number of subsections.

4.3.1. *Using gridlines to define a homeomorphism.* We begin by showing that we can use a complete set of gridlines to build a homeomorphism from $B = [0,1] \times [0,1]$ to J and its interior I. Suppose that $\Gamma = \{\gamma_p\}_{p \in [0,1]_{\mathbb{Q}}}$ and $\Delta = \{\delta_q\}_{q \in [0,1]_{\mathbb{Q}}}$ are a complete set of gridlines; see (†). Define $f\colon [0,1]_{\mathbb{Q}} \times [0,1]_{\mathbb{Q}} \to \overline{I} \subseteq X$ as follows. Given (p,q), let $f(p,q)$ be the unique intersection point of γ_p and δ_q.

Claim 4.14.1. *Given $(x,y) \in [0,1] \times [0,1]$ and a rational sequence $(x_n, y_n)_{n \in \mathbb{N}} \in [0,1]_{\mathbb{Q}} \times [0,1]_{\mathbb{Q}}$ which converges to (x,y), the limit*

$$\lim_{n \to \infty} f(x_n, y_n)$$

exists and is independent of the choice of sequence.

Thus f extends to a continuous map $\tilde{f}\colon [0,1] \times [0,1] \to \overline{I} \subseteq X$ on the completion of its domain, by defining

$$f(x,y) = \lim_{n \to \infty} f(x_n, y_n)$$

for any rational sequence $x_n \to x$ and $y_n \to n$. (The claim implies that f is continuous on $[0,1]_{\mathbb{Q}} \times [0,1]_{\mathbb{Q}}$, but this alone is not enough to know that it extends.)

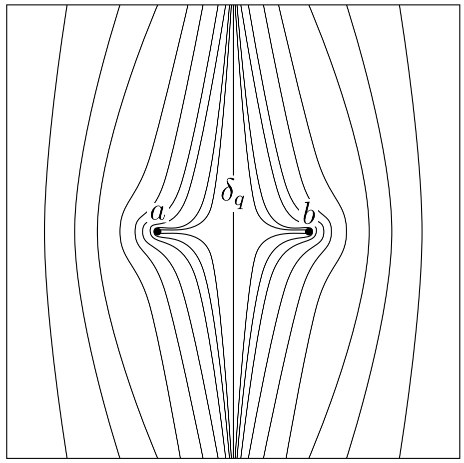
Proof of Claim 4.14.1. First, suppose that we have two rational sequences (x'_n, y'_n) and (x''_n, y''_n) with the same limit (x,y), but such that

$$\lim_{n \to \infty} f(x'_n, y'_n) = a \neq b = \lim_{n \to \infty} f(x''_n, y''_n).$$

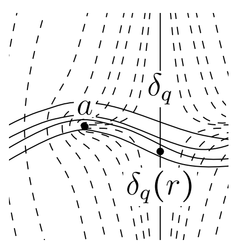
Now there is a gridline, say without loss of generality a vertical gridline δ_q, separating a and b. Suppose that a is on the left half of δ_q, and b is on the right half.

Now $f(x'_n, y'_n)$ lies on the vertical y'_n-gridline, and so for large enough n, we must have $y'_n < q$. (Since $\lim_{n \to \infty} f(x'_n, y'_n) = a$ is on the left half of δ_q, $f(x'_n, y'_n)$ must be on the left half for sufficiently large n.) Similarly, $y''_n > q$ for sufficiently large n. Then, since y is the common limit of both y'_n and y''_n, we must have $y = q$.

For each $q' < q$, a must be on the right half of $\delta_{q'}$. This is because for sufficiently large n, $y'_n > q'$, and so $f(x'_n, y'_n)$ is on the right half of $\delta_{q'}$. Similarly, for each $q'' > q$, b must be on the left half of $\delta_{q'}$. So the situation must be as follows:



The key observation is that vertical gridlines fail to separate a from any point of δ_q. We claim that, indeed, there is a point on δ_q which is not separated from a by horizontal gridlines either. Some of the horizontal gridlines must be above a, and some must be below a, and at most one horizontal gridline goes through a. There is a real number r such that for each rational $p < r$, γ_p is below a, and for $p > r$, γ_p is above a. Recall that (9) of Definition 4.13 says that each γ_p and δ_q intersect at only one point, $\gamma_p(q) = \delta_q(p)$. Thus, for each rational $p < r$, γ_p is below $\delta_q(r)$, and for each $p > r$, γ_p is above $\delta_q(r)$. Then a is not separated from $\delta_q(r)$ by any gridline, horizontal or vertical.



This gives a contradiction. We conclude that for any two rational sequences (x'_n, y'_n) and (x''_n, y''_n) with the same limit (x, y), we have

$$\lim_{n\to\infty} f(x'_n, y'_n) = \lim_{n\to\infty} f(x''_n, y''_n).$$

Second, suppose that (x_n, y_n) is a sequence of rationals pairs converging to (x, y). If $f(x_n, y_n)$ does not converge, and since the codomain $\overline{I}$ is compact, there are two subsequences (x'_n, y'_n) and (x''_n, y''_n) such that $f(x'_n, y'_n)$ converges to a point a and $f(x''_n, y''_n)$ converges to a different point b. By the above argument, this cannot happen. So $\lim_{n\to\infty} f(x_n, y_n)$ exists. □

Claim 4.14.2. *f is injective.*

Proof. Suppose that $(x, y) \neq (x', y')$, say, without loss of generality, $x < x'$. Choose rationals q, q' with $x < q < q' < x'$. Then $f(x, y)$ is the limit of points to the left of δ_q, and $f(x', y')$ is the limit of points to the right of $\delta_{q'}$. Thus $f(x, y) \neq f(x', y')$. □

Claim 4.14.3. *f is surjective onto $\overline{I}$.*

Proof. Fix $a \in R$. It suffices to show that for each $\epsilon > 0$, there are $(x, y) \in [0,1]_{\mathbb{Q}} \times [0,1]_{\mathbb{Q}}$ with $f(x, y)$ within ϵ of a.

Consider the connected component of a in the basic open ball of radius ϵ around a, and choose another point b within this connected component. (We need to consider the connected component since otherwise, the points might be at close proximity but be separated by a gridline which does not even intersect the basic ball, i.e., "far-far away".) Since a and b are separated by a gridline, there must be a gridline, say δ_q, passing within ϵ of a. So there must be some rational p with $d(\gamma_q(p), a) < \epsilon$. Thus $f(p, q) = \gamma_q(p)$ is within ϵ of a. □

Claim 4.14.4. *f is a homeomorphism.*

Proof. Since f is continuous and injective with a compact domain, it is a homeomorphism onto its image $\overline{I}$. □

Claim 4.14.5. *f is computable from*

$$\left(\bigoplus_{p \in [0,1]_{\mathbb{Q}}} \gamma_p \;\oplus\; \bigoplus_{q \in [0,1]_{\mathbb{Q}}} \delta_q \right)'.$$

Proof. We can compute $f \colon [0,1]_{\mathbb{Q}} \times [0,1]_{\mathbb{Q}} \to \overline{I} \subseteq X$ from the oracle $\bigoplus_{p \in [0,1]_{\mathbb{Q}}} \gamma_p \;\oplus\; \bigoplus_{q \in [0,1]_{\mathbb{Q}}} \delta_q$. Indeed, $f(p, q) = \gamma_p(q) = \delta_q(p)$. Then using an extra jump we can also compute a name for the extension $\hat{f} \colon [0,1] \times [0,1] \to X$ using the uniform continuity of $\hat{f}$.[7] □

Thus if we can find a complete set of gridlines, we can construct the desired homeomorphism. We return to the problem of constructing such a set of gridlines.

4.3.2. *An overview and an obstacle.* Our construction of a complete set of gridlines will be stage-by-stage, at each stage adding finitely many new gridlines. As they are already predetermined, we can start with $\gamma_0(\cdot) = J(0, \cdot)$, $\gamma_1(\cdot) = J(1, \cdot)$, $\delta_0(\cdot) = J(\cdot, 0)$, and $\delta_1(\cdot) = J(\cdot, 1)$. Fixing some listing $(q_i)_{i \in \mathbb{N}}$ of the remaining rational numbers in $(0,1)_{\mathbb{Q}}$, at stage $3k+1$ and $3k+2$ we will define the q_k-gridlines γ_{q_k} and δ_{q_k} if they have not already been defined at a previous stage. The remaining conditions for a set of gridlines are all things to avoid; e.g., when defining a new horizontal gridline, we must keep it disjoint from all other horizontal gridlines, and it must intersect the vertical gridlines at the correct points.

To make our set of gridlines complete, at stages $3k$ we must work to meet condition $(*)$:

$(*)$ every pair of distinct points in I is separated by a gridline.

There are uncountably many instances of the second condition to be a complete set of gridlines. Thus we cannot hope to meet these requirements one-by-one, satisfying only finitely many of them at every stage of the construction. Luckily, in Lemma 4.15 we will show that $(*)$ is equivalent to the following condition $(**)$ which has only countably many instances:

[7] Given a positive rational ϵ, we have to compute a rational δ such that $|(p, q) - (v, w)| < \delta$ implies $|f(p, q) - f(v, w)| \le \epsilon$. For each such fixed potential δ corresponding to the given ϵ, the collection of points for which it works is a closed set. In particular, if it fails then it fails on special points, which is $\Sigma^{0,f}_1$. Thus it takes one extra jump over the complexity of f to check if δ works for a given ϵ. We can search for the first found δ that works for ϵ, and by compactness such a δ must exist.

($**$) for every pair of disjoint basic closed balls C_1 and C_2 contained in I with rational radii and centered at special points, there are finite covers by basic connected open balls $B_1, \ldots, B_n$ of C_1 and $B_1^*, \ldots, B_m^*$ of C_2 in I such that each B_i is separated from each B_j^* by a gridline.

In the condition above, a gridline separates two basic connected open sets if one open set is one one side of the gridline, the other open set is on the other side, and *the gridline is disjoint from the closures of the open sets.* Note that we ask the open sets to be connected so that they are contained entirely on one side or the other of the gridline.

Lemma 4.15. *Let $\Gamma = \{\gamma_p\}_{p \in [0,1]_{\mathbb{Q}}}$ and $\Delta = \{\delta_q\}_{q \in [0,1]_{\mathbb{Q}}}$ be a set of gridlines. Then they are a complete set of gridlines if and only if they satisfy ($**$).*

Proof. It is important for the proof that $\overline{I}$ is compact. It is easy to see that ($**$) implies ($*$): given any two points, we can find disjoint basic closed balls C_1 and C_2 as in ($**$) containing them. Then, given any finite cover by open balls $B_1, \ldots, B_n$ of C_1 and $B_1^*, \ldots, B_m^*$ of C_2, there are balls B_i and B_j^* containing the two points. These two balls, and hence the two points, are separated from each other by a gridline.

Now we must show that ($*$) implies ($**$). Fix C_1 and C_2 as in ($**$). For each $x \in C_1$, $y \in C_2$ there is either a horizontal or vertical gridline separating the two. Call this gridline $G_{x,y}$ and choose open balls $U_{x,y} \ni x$ and $V_{y,x} \ni y$ that are separated from each other by the gridline. For each x, there are finitely many $y_1, \ldots, y_n$ such that the $V_{y_i,x}$ cover C_2, and take $U_x = \bigcap_{y_i} U_{x,y}$. Then each point of C_2 is separated from each point of U_x by a gridline G_{x,y_i}. Now there are finitely many $x_1, \ldots, x_m$ such that the U_{x_i} cover C_1. Take the gridlines G_{x_i,y_i}; every point of C_1 is separated from a point of C_2 by one of these gridlines. □

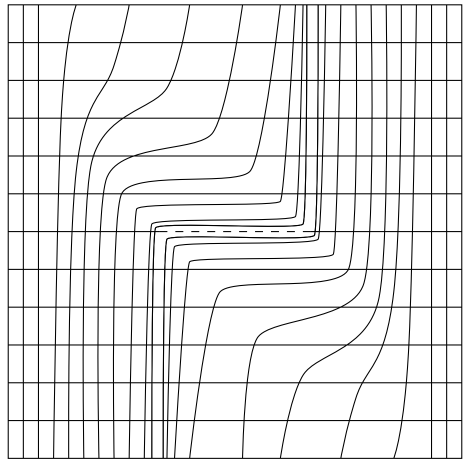
Note that because ($*$) holds in $[0,1] \times [0,1]$ with the standard rational gridlines, ($**$) is also satisfied.

Can we simplify ($**$) to a simpler condition which would not involve separating closed sets? One might consider a third condition ($***$) which says that the gridlines are very close to each other. For example:

($***$) For every horizontal p-gridline and $\epsilon > 0$, there is $\delta > 0$ such that every horizontal p'-gridline, with $|p - p'| < \delta$, has every point of $\gamma_{p'}$ within ϵ of a point of γ_p; and similarly for vertical gridlines.

It would be much easier to construct gridlines satisfying ($***$). Unfortunately, ($***$) is not equivalent to ($*$) and ($**$). We will explain why so as to show something that we must be careful to avoid when we construct our gridlines.

In the following set of gridlines, each horizontal gridline intersects each vertical gridline exactly once. They also satisfy ($***$) (though of course we cannot draw all the gridlines, but one can imagine that there are more gridlines in between the ones shown). And yet none of the points along the dashed line are separated from each other by any horizontal or vertical gridlines. Using a set of gridlines satisfying ($***$), one could still define a continuous map from J and its interior to $[0,1] \times [0,1]$, but it would not be injective.



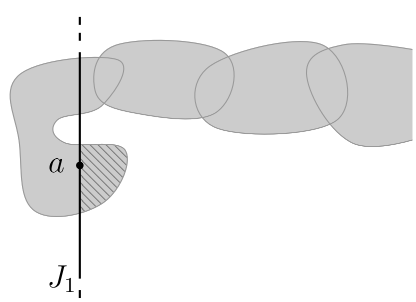
This example also shows that it is not sufficient to ask that every two special points be separated; the points on the dashed line are not separated, and it could be that there are no special points on that line.

4.3.3. *Effectiveness.* We have to make sure that our construction of the gridlines is arithmetic, so that by Claim 4.14.5, the resulting homeomorphism is arithmetic. We will use the results of Section 4.2, though there are two new issues that we did not deal with there, and so we will need to strengthen these results. The general setup will be that we want to make a Jordan arc from a to b, with a on a Jordan arc J_1 and b on a Jordan arc J_2, such that the new Jordan arc does not intersect J_1 and J_2 except at a and b respectively.

The two issues are:

- There is an open neighbourhood N between J_1 and J_2 in which we want our new arc J to be. But the end-points of the new arc do not belong to the neighbourhood N, since we merely have $a, b \in \partial\overline{N}$. But in Theorem 4.9 the assumption is that $J \subset N$.
- We need to have an arithmetic bound on all of the gridlines. If we build an arc from a to b naively, it will have greater complexity than J_1 and J_2 because it needs to ask questions about J_1 and J_2. Then there will not be an arithmetic bound on the gridlines (but rather $0^{(\omega)}$ will be a bound).

The first issue will mostly require taking greater care when constructing the series of ϵ-arcs for smaller and smaller ϵ's. If we follow the strategy of Theorem 4.9, we might end up choosing our first ϵ-arc like this:



Then any Jordan curve starting at a inside the ϵ-arc will intersect the vertical curve J_1 at some point other than a. By being more careful, we will make sure that the curve, near a, is contained within the hatched region. This will require that second ball of the ϵ-arc should intersect this hatched region.

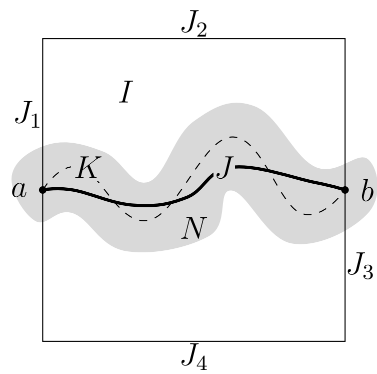
The second issue will require a finer notion of complexity for Jordan arcs. The key is to note that a and b are in the middle of their arcs, while the new arc J that we are constructing attaches to a and b at its endpoints. We will be able to arrange that the arcs are easier to compute in the middle, and only harder to compute near their endpoints; and then since we attach the endpoints of new arcs to the middles of existing arcs, we can maintain a bound on the complexity of the arcs. We make the following definition capturing this idea.

Definition 4.16. A Jordan arc $J\colon[0,1]\to M$ is globally $0^{(n)}$/locally $0^{(m)}$-computable if:

- J is $0^{(n)}$-computable, and
- given rationals x, y with $0 < x < y < 1$, $J\restriction_{[x,y]}$ is $0^{(m)}$-computable, and uniformly in x and y we can use $0^{(n)}$ to compute a $0^{(m)}$-index for $J\restriction_{[x,y]}$.

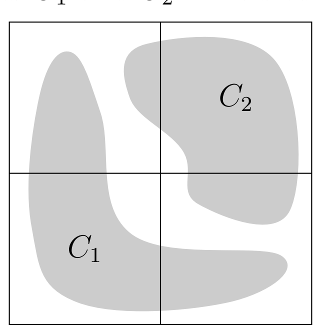
The following theorem is the required refinement of Theorem 4.9 (though it uses Theorem 4.9 in its proof).

Theorem 4.17. *Let J_1, J_2, J_3, J_4 be four globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arcs, such that $J_1 \to J_2 \to J_3 \to J_4$ is a Jordan curve. Let I be the interior of this curve, and suppose that the curve and its interior are homeomorphic to the unit disk and its boundary. Let a be a point on J_1, and let b be a point on J_3, both $0^{(9)}$-computable. Let K be a Jordan arc from a to b contained in I, and let N be an open neighbourhood of K. Then there is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc from a to b contained in $N \cap I$.*

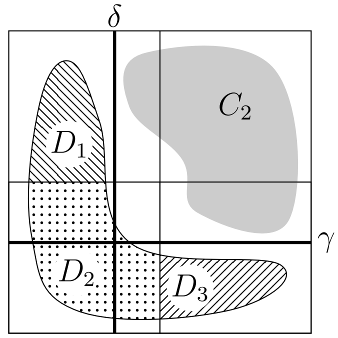


We delay the proof until Section 4.6.

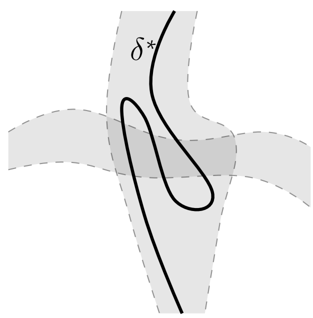
4.4. **Discussion of technical issues.** In Section 4.3.2 we described condition $(**)$ that we must meet. Suppose that at some point in the construction, we have already constructed finitely many horizontal and vertical gridlines. We now want to add finitely many new gridlines, separating two closed sets C_1 and C_2. The situation might look like this:



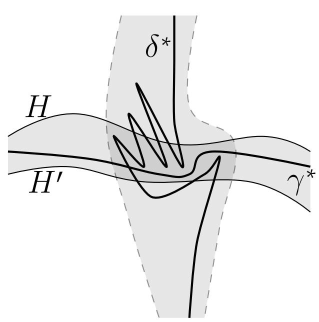
To separate C_1 and C_2, we would want to divide C_1 up into D_1, D_2, and D_3 and use gridlines δ and γ like the following:



Now we need to replace δ and γ by arithmetic gridlines δ^* and γ^*, such that δ^* still separates D_1 and C_2, and γ^* still separates D_3 and C_2. (D_2 is already separated from C_2 by the original gridlines.) Any close enough approximation to δ will still separate D_1 and C_2, in the sense that there is a neighbourhood of δ such that any other Jordan arc in the neighbourhood also separates D_1 and C_2. We already know—by Theorem 4.9—how to draw arithmetic approximations of each of δ and γ separately. But how do we make sure that they intersect at a single point? The following picture shows why this might be difficult; suppose that we use Theorem 4.9 to find an arithmetic approximation δ^* of δ in some neighbourhood of δ, and now we want to find an arithmetic approximation γ^* of γ through a neighbourhood of γ. If the δ^* we obtain is as shown below, we cannot choose γ^* without having δ^* and γ^* intersect at multiple points.

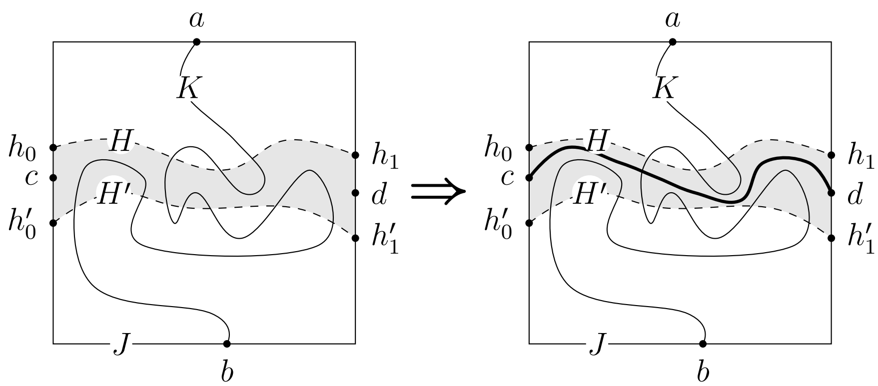


The solution is as follows. We can find two Jordan arcs, H and H', one on each side of γ, such that any Jordan arc between H and H' separates the same closed sets that γ was chosen to separate. Now we want to make sure that δ^* has the property that once it reaches H', it never again intersects H. Intuitively, one should think of δ^* as looking as follows:

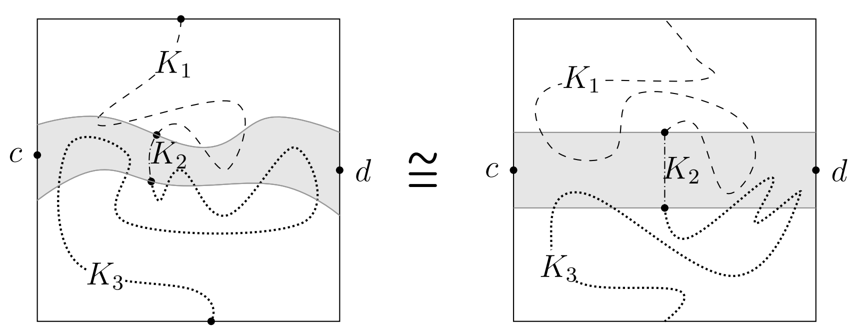


In this picture, we can easily see how can could draw a γ^*, contained between H and H', which intersects δ^* exactly once. In general, the following lemma says that in this situation, one can always find such a γ^*. (The lemma does not give an arithmetic such γ^*, but once we know that there is *some* such Jordan arc, Theorem 4.17 allows us to construct one arithmetically.)

Lemma 4.18. *Let J be a Jordan curve that bounds a Jordan region. Let h_0, h_1, h_0', h_1' be four points on J. Suppose that H and H' are Jordan arcs, with H going from a point h_0 to h_1, and H' going from h_0' to h_1'. Suppose that H and H' do not intersect, so that, in order around the arc J, the points are ordered h_0, h_0', h_1', h_1. Let K be a Jordan arc from a point a on J between h_0 and h_1 to a point b on J between h_0' and h_1'. Suppose that once K intersects H', it never again intersects H. Let c be a point on J between h_0 and h_0', and let d be a point on J between h_1 and h_1'. Then there is a Jordan arc H^* from c to d such that H^* does not intersect H or H', and such that H^* intersects K exactly once.*



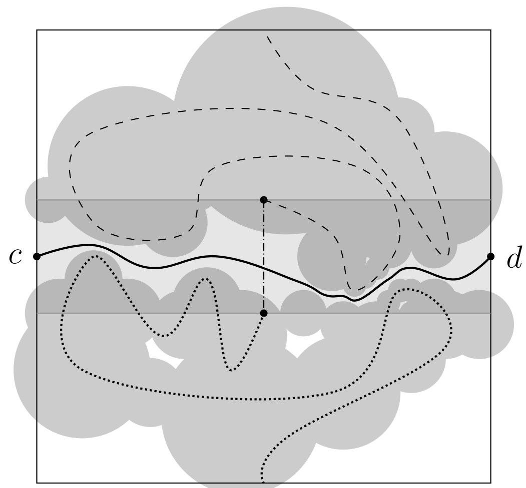
Proof. Up to homeomorphism, we can assume that J is a unit square, and that H, H', and K are straight lines. There is some last intersection of K with H, and first intersection of K with H'. The segment of K between these two intersections lies between H and H'. So divide K up into three segments, K_1, K_2, K_3, with K_2 between H and H', K_1 all above H', and K_3 all below H.



It is intuitively clear that there is a Jordan arc from c to d, between H and H', crossing K_2 exactly once, and disjoint from K_1 and K_3. One way to argue formally is as follows. For the next paragraph we work with the standard metric, so that the basic open balls are in fact circles, and their boundaries are given by a Jordan curve.

First, put small open balls B and B' around the common points of intersection of K_1, K_2, and H, and of K_2, K_3, and H'. Now the following closed sets are all disjoint: $K_1 \cup H - B$, $K_3 \cup H' - B$, $K_2 - B$, the point c, and the point d. Thus we can cover $K_1 \cup H - B$ by finitely many open balls $B_1, \dots, B_\ell$ disjoint from the other closed sets, and so that $B \cup B_1 \cup \dots \cup B_\ell$

is connected. Similarly we can cover $K_3 \cup H' - B'$ by finitely many open balls $B'_1, \dots, B'_m$. Now the exterior boundaries of $B \cup B_1 \cup \dots \cup B_\ell$ and of $B' \cup B'_1 \cup \dots \cup B'_m$ are given by Jordan curves (which are made up of finitely many circular arcs). These two curves meet K_2 at only one point each, on the boundary of B and B'. So we can essentially think of these boundary curves as a new H and H' which are closer together than the originals. One can then easily draw a Jordan arc from c to d between them, crossing K_2 exactly once.



□

So we need a modification of Theorem 4.17 in which we can also obtain this nice behaviour with pairs of Jordan arcs in the other direction. We may need to have more than one pair of gridlines H, H'.

Theorem 4.19. *In addition to the hypotheses of Theorem 4.17, suppose that*

$$H_1, H'_1, H_2, H'_2, \dots, H_\ell, H'_\ell$$

are Jordan arcs, all non-intersecting, and each from a point on J_2 to a point on J_4, neither point being on the ends of J_2 or J_4. Suppose that $H_1, H'_1, H_2, H'_2, \dots, H_\ell, H'_\ell$ are listed in that order from closest to J_1 to closest to J_3. Then we can find a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc as in Theorem 4.17 with the additional property that once it crosses H'_i, it never again crosses H_i.

We also leave the proof of this fact to Section 4.6.

4.4.1. *Construction of a complete set of gridlines.* Recall that we have a Jordan curve $J\colon \partial B \to M$, where B is the unit box $[0,1] \times [0,1]$. We want to construct a complete set of gridlines covering the region I bounded by J.

We construct the gridlines using $0^{(15)}$. Each gridline itself will be composed of finitely many pieces, each of which is globally $0^{(13)}$/locally $0^{(9)}$-computable. Since $0^{(15)}$ can compute all of the gridlines, by the results in Section 4.3.1, $0^{(16)}$ will be able to compute a homeomorphism between J together with its interior I and $[0,1] \times [0,1]$. Modulo the proofs of Theorems 4.17 and 4.19 in Section 4.6, this will give a proof of Theorem 4.11.

The construction will be a stage-by-stage construction, where at each stage s we have defined finitely many gridlines Γ_s and Δ_s. Suppose that at stage s we have defined $\Gamma_s = \{\gamma_{p_1}, \dots, \gamma_{p_m}\}$ with $p_1 = 0 < p_2 < \dots < p_m = 1$ and $\Delta_s = \{\delta_{q_1}, \dots, \delta_{q_n}\}$ with $q_1 = 0 < q_2 <$

$\cdots < q_n = 1$. Then for each i and j, $\delta_{q_i} \restriction_{[p_j, p_{j+1}]}$ and $\gamma_{p_j} \restriction_{[q_i, q_{i+1}]}$ will be globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arcs.

Let r_k be a computable listing of $[0,1]_{\mathbb{Q}}$ and let (C_1^k, C_2^k) be a listing of pairs of disjoint basic closed balls with rational radii and centered at special points. We have three sets of requirements to meet:

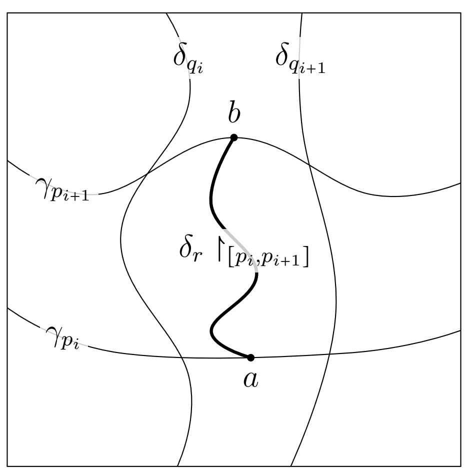
(1) Define δ_{r_k}.
(2) Define γ_{r_k}.
(3) Satisfy an instance of $(**)$: If C_1^k, C_2^k are in I, there are finite covers by basic connected open balls $B_1, \ldots, B_n$ of C_1 and $B_1^*, \ldots, B_m^*$ of C_2 such that each B_i is separated from each B_j^* by a gridline.

At each stage, we add finitely many gridlines to meet one of these requirements.

Stage $s = 0$: We begin at stage $s = 0$ with $\Gamma_s = \{\gamma_0, \gamma_1\}$ and $\Delta_s = \{\delta_0, \delta_1\}$ where $\gamma_0(\cdot) = J(0,\cdot)$, $\gamma_1(\cdot) = J(1,\cdot)$, $\delta_0(\cdot) = J(\cdot, 0)$, and $\delta_1(\cdot) = J(\cdot, 1)$. These are $0^{(8)}$-computable, hence globally $0^{(13)}$/locally $0^{(9)}$-computable.

Stage $s + 1 = 3k$: We meet the kth requirement of type (1), defining δ_{r_k}. For simplicity, we write $r = r_k$ at this stage. Suppose that at the previous stage we defined $\Gamma_s = \{\gamma_{p_1}, \ldots, \gamma_{p_m}\}$, with $p_1 = 0 < p_2 < \cdots < p_m = 1$, and $\Delta_s = \{\delta_{q_1}, \ldots, \delta_{q_n}\}$, with $q_1 = 0 < q_2 < \cdots < q_n = 1$. If δ_r has already been defined in Δ_s, then we do not have to do anything at this stage. So suppose that it has not been defined, and let i be such that $q_i < r < q_{i+1}$.

For each $j = 1, \ldots, m-1$, consider the region bounded by $\delta_{q_i} \restriction_{[p_j, p_{j+1}]}$, $\delta_{q_{i+1}} \restriction_{[p_j, p_{j+1}]}$, $\gamma_{p_j} \restriction_{[q_i, q_{i+1}]}$, and $\gamma_{p_{j+1}} \restriction_{[q_i, q_{i+1}]}$. (Recall (9) of Definition 4.13 which gives the intersection points of these gridlines.) Each of these is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc. Consider also the $0^{(9)}$-computable points $a = \gamma_{p_j}(r)$ and $b = \gamma_{p_{j+1}}(r)$ on the boundary of this region.



By Theorem 4.17 there is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc from a to b contained in this region. Denote this Jordan arc by δ_r^j and assume that it has domain $[p_j, p_{j+1}]$.

Now we can define δ_r to be the Jordan arc with domain $[0,1]$ which is the concatenation of each of the Jordan arcs δ_r^j.

Stage $s + 1 = 3k + 1$: We meet the kth requirement of type (2), defining γ_{r_k}. This is the same construction as meeting a requirement of type (1), except horizontal instead of vertical.

Stage $s+1 = 3k+2$: We meet the kth requirement of type (3), separating $C_1 = C_1^k$ and $C_2 = C_2^k$. Suppose that at the previous stage we defined $\Gamma_s = \{\gamma_{p_1}, \dots, \gamma_{p_m}\}$, with $p_1 = 0 < p_2 < \dots < p_m = 1$, and $\Delta_s = \{\delta_{q_1}, \dots, \delta_{q_n}\}$, with $q_1 = 0 < q_2 < \dots < q_n = 1$.

Using $0^{(8)}$, ask whether C_1^s and C_2^s are in the region I bounded by J.

> *Since J is $0^{(8)}$-computable, by Lemma 3.22 I is a $0^{(8)}$-computable open set, and so by Lemma 3.8 we can test containment using $0^{(8)}$.*

If C_1 and C_2 are not in the region I, we do not have to do anything. If they are, then look for:

- new rational numbers $p'_1, \dots, p'_{m'}$ and $q'_1, \dots, q'_{n'}$ (write $q^*_1, \dots, q^*_{n+n'}$ for $q_1, \dots, q_n, q'_1, \dots, q'_{n'}$ in increasing order),
- additional vertical gridlines $\delta_{q'_1}, \dots, \delta_{q'_{n'}}$, such that for each $j = 1, \dots, n'$ and $i = 1, \dots, m-1$, $\delta_{q'_j} \restriction_{[p_i, p_{i+1}]}$ is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc;
- additional horizontal gridlines $\gamma_{p'_1}, \dots, \gamma_{p'_{m'}}$, such that for each $i = 1, \dots, m'$ and $j = 1, \dots, n+n'-1$, $\gamma_{p'_i} \restriction_{[q^*_j, q^*_{j+1}]}$ is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc;
- a finite cover $\mathcal{B}_1$ of C_1 by basic connected open balls; and
- a finite cover $\mathcal{B}_2$ of C_2 by basic connected open balls

such that each $B \in \mathcal{B}_1$ is separated from each $B' \in \mathcal{B}_2$ by one of the gridlines

$$\gamma_{p_1}, \dots, \gamma_{p_m}, \gamma_{p'_1}, \dots, \gamma_{p'_{m'}}, \delta_{q_1}, \dots, \delta_{q_n}, \delta_{q'_1}, \dots, \delta_{q'_{n'}}.$$

Assuming they exist, we can find such gridlines using $0^{(15)}$. Let $\Delta_{s+1} = \{\delta_{q_1}, \dots, \delta_{q_n}, \delta_{q'_1}, \dots, \delta_{q'_{m'}}\}$ and let $\Gamma_{s+1} = \{\gamma_{p_1}, \dots, \gamma_{p_m}, \gamma_{p'_1}, \dots, \gamma_{p'_{n'}}\}$.

> *Since these arcs are all $0^{(13)}$-computable, by Lemma 3.20 we can check for the right kind of intersections using $0^{(14)}$. To check that an arc is contained in $\overline{I}$, we check that for every $\epsilon > 0$, there is an open cover of the arc by ϵ-balls such that every ϵ ball intersects I. We can decide this using $0^{(15)}$.*

4.4.2. *Verification.* For the verification, it is sufficient to argue that such gridlines exist. First we will argue that we can find such gridlines non-effectively, and then we will show how to find them arithmetically. So first we argue non-effectively that there are

- new rational numbers $p'_q, \dots, p'_{m'}$ and $q'_1, \dots, q'_{n'}$ (write $q^*_1, \dots, q^*_{n+n'}$ for $q_1, \dots, q_n, q'_1, \dots, q'_{n'}$ in increasing order),
- additional vertical gridlines $\delta^*_{q'_1}, \dots, \delta^*_{q'_{n'}}$;
- additional horizontal gridlines $\gamma^*_{p'_1}, \dots, \gamma^*_{p'_{m'}}$;
- a finite cover $\mathcal{B}_1$ of C_1 by basic connected open balls; and
- a finite cover $\mathcal{B}_2$ of C_2 by basic connected open balls

such that each $B \in \mathcal{B}_1$ is separated from each $B' \in \mathcal{B}_2$ by a one of the gridlines

$$\gamma_{p_1}, \dots, \gamma_{p_m}, \gamma^*_{p'_1}, \dots, \gamma^*_{p'_{m'}}, \delta_{q_1}, \dots, \delta_{q_n}, \delta^*_{q'_1}, \dots, \delta^*_{q'_{n'}}.$$

There is a homeomorphism θ taking J and its interior to $[0,1] \times [0,1]$ which takes each of the original horizontal gridlines γ_{p_i} to the horizontal gridline with equation $y = p_i$ in $[0,1] \times [0,1]$, and each original vertical gridline δ_{q_j} to the vertical gridline with equation $x = q_j$ in $[0,1] \times [0,1]$. Moreover, we can choose such a homeomorphism to map $\gamma_{p_i}(r)$ to the point (r, p_i) in $[0,1] \times [0,1]$ and $\delta_{q_j}(r)$ to (q_j, r). (See Section 3.3.) Now $[0,1] \times [0,1]$ satisfies the following variant of $(**)$ for arbitrary closed sets, the standard rational gridlines, and any fixed basis of connected open sets:

$(****)$ Let $\mathcal{B}$ be an open basis for $[0,1]\times[0,1]$ consisting entirely of connected sets. For every pair of disjoint closed sets C_1 and C_2 in the interior of $[0,1]\times[0,1]$, there are finite covers $\mathcal{B}_1$ of C_1 and $\mathcal{B}_2$ of C_2, where each of these open sets is from $\mathcal{B}$, such that each $B\in\mathcal{B}_1$ is separated from each $B'\in\mathcal{B}_2$ by a rational gridline of $[0,1]\times[0,1]$.

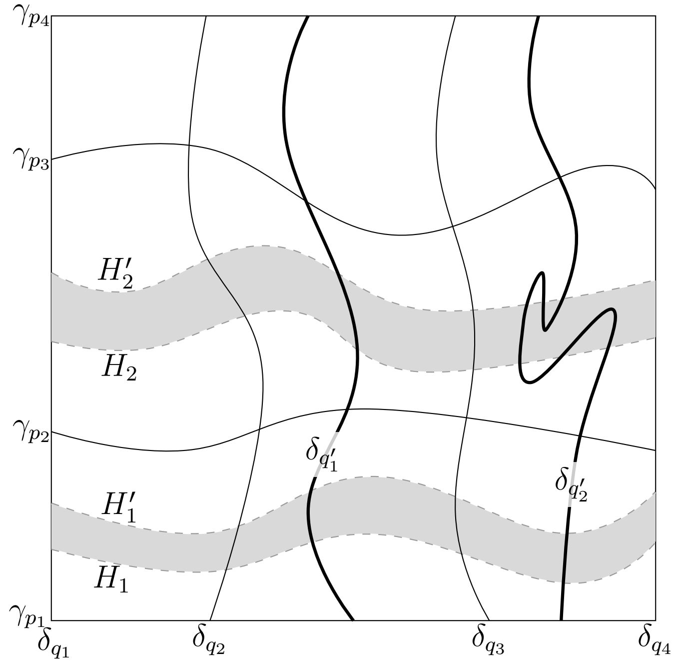
The argument that $[0,1]\times[0,1]$ satisfies $(****)$ is exactly the same as that $(*)$ implies $(**)$. Now taking as $\mathcal{B}$ the homeomorphic image under θ of the basic connected open balls, using $(****)$ in $[0,1]\times[0,1]$, and pulling back through θ, we get the desired gridlines $\gamma^*_{p'_1},\dots,\gamma^*_{p'_{m'}},\delta^*_{q'_1},\dots,\delta^*_{q'_{n'}}$ and covers $\mathcal{B}_1,\mathcal{B}_2$.

Now we must argue that there are arithmetic approximations of these gridlines. (We will use the same rationals $p'_1,\dots,p'_{m'}$ and $q'_1,\dots,q'_{n'}$ and open covers $\mathcal{B}_1$ and $\mathcal{B}_2$.) For each of the new (non-effective) horizontal gridlines $\gamma^*_{p'_i}$, choose two gridlines H_i and H'_i, the first below $\gamma^*_{p'_i}$, the second above, and both close enough to $\gamma^*_{p'_i}$ that any other horizontal gridline between them still separates the basic connected open sets from $\mathcal{B}_1$ and $\mathcal{B}_2$ separated by $\gamma^*_{p'_i}$. Moreover, make sure that of all the gridlines considered so far, H_i is the one just below $\gamma^*_{p'_i}$ and H'_i is the one just above (i.e., none of the other gridlines are between H_i and H'_i other than $\gamma^*_{p'_i}$). We do not need any effectiveness bound on the H_i and H'_i.

For each q'_j, choose a neighbourhood of $\delta^*_{q'_j}$ which is a finite union of basic open balls such that any Jordan arc through that neighbourhood still separates any basic connected open balls from $\mathcal{B}_1$ and $\mathcal{B}_2$ that were separated by the original arc. Moreover, we want that any two of these neighbourhoods, for different $\delta^*_{q'_j}$, are disjoint, and that they do not intersect any of the original vertical gridlines $\delta_{q_1},\dots,\delta_{q_n}$.

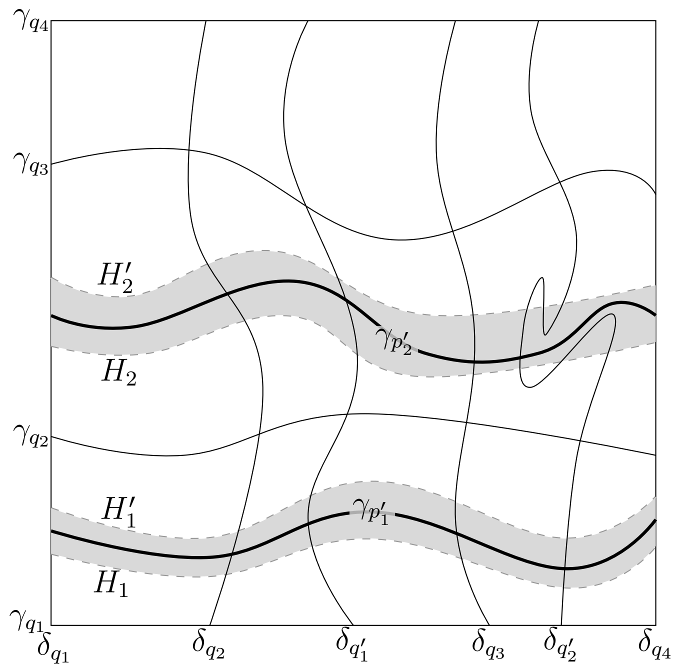
Now we are ready to make our arithmetic approximations. Fix $j\in\{1,\dots,n'\}$. By Lemma 4.19, for each $i=1,\dots,m$, there is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc from $\gamma_{p_i}(q'_j)$ to $\gamma_{p_{i+1}}(q'_j)$ within the neighbourhood around $\gamma^*_{q'_j}$, and with the property that once it crosses any H'_{i^*}, $i^*\in\{1,\dots,k\}$, it never again crosses H_{i^*}. Piecing these together, we get a vertical gridline $\delta_{q'_j}$. These vertical gridlines $\delta_{q'_i}$ still separate all of the basic connected open sets that were separated by the non-effective $\delta^*_{q'_i}$. (Note that, because each H_i and H'_i is a gridline, the original vertical gridlines $\delta_{q_1},\dots,\delta_{q_n}$ only intersect each H_i and H'_i once.)

Step One: Construct Vertical Gridlines



Recall that we write $q_1^*, \ldots, q_{n+n'}^*$ for $q_1, \ldots, q_n, q_1', \ldots, q_{n'}'$ in increasing order, so that the vertical gridlines, from left to right, are $\delta_{q_1^*}, \ldots, \delta_{q_{n+n'}^*}$. For each $i = 1, \ldots, m'$ and $j = 1, \ldots, n + n' - 1$, we can use Lemma 4.18 to construct a Jordan arc from $\gamma_{q_j^*}(p_i')$ to $\gamma_{q_{j+1}^*}(p_i')$, between H_i and H_i'. Piecing these together, we get a new horizontal gridline $\gamma_{p_i'}$.

Step Two: Construct Horizontal Gridlines



4.5. **Proof of Theorem 1.1.** Recall that Theorem 1.1 says that every computable surface without boundary has an arithmetic atlas.

Proof of Theorem 1.1. Make a $0^{(10)}$-computable list (x_i, δ_i, J_i) consisting of special points $x_i \in M$, rationals δ_i, and $0^{(8)}$-computable Jordan curves J_i formally contained within $B_{\delta_i}(x_i)$ (these exist by Theorem 4.10, and we need two more jumps to check that they are Jordan curves). By Lemma 3.22, for a given i, the open region bounded by J_i is $0^{(8)}$-computable, and using Lemma 3.8, $0^{(11)}$ can decide whether the regions bounded by finitely many Jordan curves $J_{i_1}, \dots, J_{i_\ell}$ covers X. We want to find a finite cover by such regions which are homeomorphic to open disks in $\mathbb{R}^2$, and to compute the homeomorphisms. By compactness and Theorem 4.10, there is a finite cover of X consisting of such regions. Also, $0^{(20)}$ can find such a cover: If J_i bounds a region containing x_i and homeomorphic to the interior of the disk, then by Theorem 4.11 there is a $0^{(16)}$-computable such homeomorphism. By Fact 3.6, for each i we can ask $0^{(20)}$ whether such a homeomorphism exists, and if it exists, find it. □

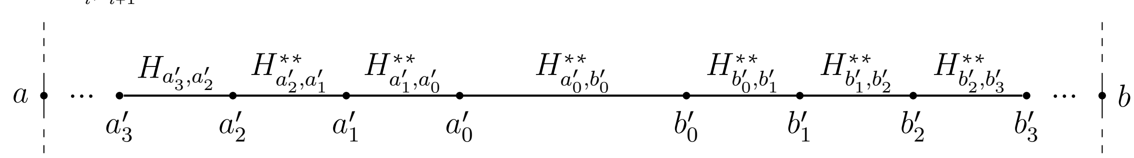
4.6. **Constructing globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arcs.** In this section we will prove Theorem 4.17. Recall the statement.

Theorem 4.17. *Let J_1, J_2, J_3, J_4 be four globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arcs, such that $J_1 \to J_2 \to J_3 \to J_4$ is a Jordan curve. Let I be the interior of the region bounded by of this curve. Let a be a point on J_1, and let b be a point on J_3, both $0^{(9)}$-computable. Let K be a Jordan arc from a to b contained in I, and let N be an open neighbourhood of K which is a finite union of basic open balls. Then there is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc from a to b contained in $N \cap I$.*

Uniformity is not required of this theorem; if the reader recalls where the theorem is used in the construction of the gridlines, it is purely for a proof of the *existence* of a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc satisfying certain properties; we then separately search for *some* Jordan arc satisfying those properties (but the Jordan arc we find may not, e.g., be contained in N unless we specifically want to keep it in N).

Proof. We will define a Jordan arc H which is a $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc from a to b contained in $N \cap R$. We will define H by piecing together infinitely many smaller arcs. We will define special points $a'_0, a'_1, a'_2, \dots$ which limit to a, and $b'_0, b'_1, b'_2, \dots$ which limit to b. These will be the endpoints of the constituent arcs. These constituent arcs will be:

- $H^{**}_{a'_0,b'_0}$ from a'_0 to b'_0;
- $H^{**}_{a'_{i+1},a'_i}$ from a'_{i+1} to a'_i; and
- $H^{**}_{b'_i,b'_{i+1}}$ from b'_i to b'_{i+1}.



Then we define H as follows:

- on the interval $[\frac{1}{4}, \frac{3}{4}]$, H is the arc $H^{**}_{a'_0,b'_0}$ from a'_0 to b'_0;
- on the interval $[\frac{1}{4+i+1}, \frac{1}{4+i}]$, H is the arc $H^{**}_{a'_{i+1},a'_i}$ from a'_{i+1} to a'_i;
- on the interval $[1 - \frac{1}{4+i}, 1 - \frac{1}{4+i+1}]$, H is the arc $H^{**}_{b'_i,b'_{i+1}}$ from b'_i to b'_{i+1}.

We also define $H(0) = a$ and $H(1) = b$. To know that the resulting H is continuous, we will need to have not just that the a'_i converge to a, but that the entire arcs $H^{**}_{a'_{i+1},a'_i}$ are contained in smaller and smaller open neighbourhoods of a, and similarly the arcs $H^{**}_{b'_i,b'_{i+1}}$ are contained in smaller and smaller open neighbourhoods of b. These arcs also should not intersect each other, except at their common endpoints. Each of the constituent arcs will individually be $0^{(9)}$-computable, while the complete set of arcs will be $0^{(13)}$-computable. Thus H will be globally $0^{(13)}$/locally $0^{(9)}$-computable.

Initial attempt to define the arc segments. Our first attempt to define the arc segments described above will result in special points $a_0, a_1, a_2, \ldots$ which limit to a, and $b_0, b_1, b_2, \ldots$ which limit to b, as well as arc segments:

- H_{a_0,b_0} from a_0 to b_0;
- H_{a_{i+1},a_i} from a_{i+1} to a_i; and
- $H_{b_i,b_{i+1}}$ from b_i to b_{i+1}.

These are almost the arc segments we want, except that there might be too many intersections between the different segments. We begin by giving the construction of these segments, and following this we will explain how to remove the intersections. We subdivide the description into several phases. These arcs will also be $0^{(7)}$-computable, but calculating their indices will require $0^{(13)}$.

The plan is as follows. In Phases 1-3 we define H_{a_0,b_0} which is our first attempt to build the "middle" segment of the arc between a and b, and calculate an estimate of the complexity of H_{a_0,b_0}. In Phases 4-5 we give a detailed description of H_{a_1,a_0}. The definitions of H_{a_{i+1},a_i} (and $H_{b_i,b_{i+1}}$) are essentially the same as the definitions of H_{a_0,a_1} (and H_{b_0,b_1}), up to a change of notation. After Phases 1-5, we define new subarcs denoted by H^{**} which remove the intersections.

Phase 1. Search for a $\delta > 0$, finite covers of each of $J_1, \ldots, J_4$ by basic open δ-balls, and basic open balls $A \ni a$ and $B \ni b$ of radius at most δ such that:

- $A, B \subseteq N$;
- the centers of A and B are at a distance at least 4δ from each other;
- A is disjoint from the covers of J_2, J_3, and J_4;
- the center of A is of distance greater than 3δ from the endpoints $J_1(0)$ and $J_1(1)$;
- B is disjoint from the covers of J_1, J_2, and J_4; and
- the center of B is of distance greater than 3δ from the endpoints $J_3(0)$ and $J_3(1)$.

Note that A and B are disjoint.

Given $\delta > 0$, we can find a cover of each of $J_1, \ldots, J_4$ by basic open balls of radius at most δ, each contained in the δ-neighbourhood of the arc, using $0^{(13)}$. This is because each of $J_1, \ldots, J_4$ is $0^{(13)}$-computable; see Lemma 3.18. Each of these covers is finite so checking, for example, that A is disjoint from the cover of J_2 can be done using $0'$ as if they are not disjoint then they contain a special point in common. Thus we can find δ, A, and B using $0^{(13)}$. The points $J_1(0)$, $J_1(1)$, $J_3(0)$, and $J_3(1)$ are $0^{(13)}$-computable, so we can recognize in a c.e. in $0^{(13)}$ way that the distances from the centers of A and B to these points are sufficiently large.

Now using $0^{(13)}$ find $p, q \in [0,1]_{\mathbb{Q}}$, $p < q$, such that $J_1\restriction_{[0,p]}$ and $J_1\restriction_{[q,1]}$ are disjoint from A.

We can choose p using $0^{(13)}$ as follows. Using the name for J_1, search for a basic open ball C containing $J_1(0)$, with the radius of C being less than δ, and a rational p such that $J_1^{-1}([0,p)) \subseteq C$. Then since A has radius at most δ and the center of A is of distance greater than 3δ from $J_1(0)$, A and C are disjoint. Choose q similarly.

Thus the portion of J_1 that passes through A is entirely contained in $J_1 \restriction_{[p,q]}$ which is $0^{(9)}$-computable (with parameters p, q). For simplicity, we write $\tilde{J}_1$ for $J_1 \restriction_{[p,q]}$. So $\tilde{J}_1$ is $0^{(9)}$-computable, and we can find a $0^{(9)}$-computable index using $0^{(13)}$. We can do the same thing with B and J_3, writing $\tilde{J}_3$ for the subarc of J_3.

Let $\tilde{A}$ be the open set which is the connected component of $A \cap I$ with a on its boundary. (Recall the example in Section 4.3.3 which shows why we need to consider this set.) Define $\tilde{B}$ similarly.

Claim 4.19.1. *The set $\tilde{A}$ has a $0^{(9)}$-computable name, and we can compute an index for this name using $0^{(13)}$. The same holds for $\tilde{B}$.*

Proof. Note that we could compute a name for $\tilde{A}$ using Lemma 3.22 to find a name for I and $A \cap I$, and then, using Lemma 3.10, calculate its connected component with a on the boundary. However, the complexity of this name would be too high.

We compute the name for $\tilde{A}$ in the following way. Suppose that $A = B_r(q)$. Fix a special point $p \in I$ using $0^{(13)}$ (see Lemma 3.22). We will use this point to distinguish between the inside and the outside of the region.

Search for $\rho > 0$, a ρ-cover of J, and a special point $a' \in A$ such that:

(1) p and a' are not in the ρ-cover of J;
(2) for every $\epsilon > 0$, there is an ϵ-path from p to a' disjoint from the ρ-cover of J;
(3) for every $\epsilon > 0$, there is a point $a'' \in A$ ϵ-close to a and a $\rho' > 0$ such that for every $\epsilon' > 0$ there is an ϵ'-path from a' to a'' within $B_{r-\rho'}(q)$ and avoiding a finite ρ'-cover of $\tilde{J}_1$.

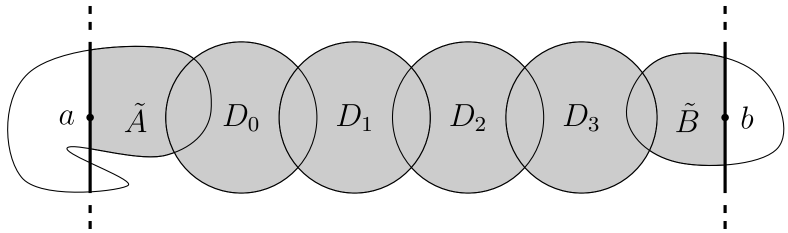
Conditions (1) and (2) say that a' is inside the Jordan region I bounded by J. Condition (3) relates a' with a. Indeed, we have $a' \in \tilde{A}$; the proof of this is almost literally the same as the proof of Lemma 3.10.

We can find a' using $0^{(13)}$. Given $\rho > 0$, $0^{(13)}$ can find a finite ρ-cover of J. Using $0^{(9)}$ we can find a finite ρ'-cover of $\tilde{J}_1$. Checking the second item uses $0^{(3)}$ and the third uses four jumps over the complexity of a and $\tilde{J}_1$, both of which are $0^{(9)}$.

First, fix a listing of all basic connected sets contained in A; this can be done using $0^{(7)}$ (by Lemma 3.10 and Lemma 3.8). Now, to list $\tilde{A}$, iterate the following procedure. Put a basic connected open set $C \subseteq A$ in $\tilde{A}$ if there is a $\rho > 0$ such that C is disjoint from a finite ρ-cover of $\tilde{J}_1$, and either $a' \in C$ or C intersects a basic connected open set which we have already determined is in $\tilde{A}$. This is a $0^{(9)}$-computable name; we used $0^{(13)}$ to find (finite) parameters. ☐

Phase 2. Now find an $\epsilon > 0$, $\delta' > 0$, an ϵ-arc $D_0, \dots, D_\ell$ contained in the open neighbourhood N of K, $\ell \geq 3$, and a finite δ'-cover of $J_1, \dots, J_4$, such that:

- $D_0, \dots, D_\ell$ are disjoint from the δ'-cover of $J_1, \dots, J_4$;
- $D_0 \cap \tilde{A} \neq \varnothing$;
- $D_\ell \cap \tilde{B} \neq \varnothing$;
- $D_i \cap \tilde{A} = \varnothing$ for all $i > 0$;
- $D_i \cap \tilde{B} = \varnothing$ for all $i < \ell$.



Claim 4.19.2. *Such an ϵ-arc exists and can be found using* $0^{(13)}$.

Proof of Claim. Since $K - \tilde{A} - \tilde{B}$ is closed, and so has some distance from $J_1, \ldots, J_4$, we can find a $\delta' > 0$, $\delta' < \delta$ and an open neighbourhood N' of $K - \tilde{A} - \tilde{B}$ which is a finite union of basic open sets, with $N' \subseteq N$, such that N' is disjoint from a finite δ'-cover of $J_1, \ldots, J_4$. Since N' is open and contains $K - \tilde{A} - \tilde{B}$, and $K \subseteq N' \cup \tilde{A} \cup \tilde{B} \cup \{a\} \cup \{b\}$, $K \cap N'$ must intersect $\tilde{A}$ and $\tilde{B}$, say at special points a^* and b^*. Find, using Proposition 4.5, an ϵ-arc $D_0, \ldots, D_\ell$ from a^* to b^* contained in N', with $\epsilon < \delta$. Now $a^* \in D_0 \cap \tilde{A}$ so the intersection is non-empty, and similarly $b^* \in D_\ell \cap \tilde{B}$. Let i be greatest such that $D_i \cap \tilde{A}$ is non-empty, and let $j > i$ be least such that $D_j \cap \tilde{B}$ is non-empty. Since $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$, A and B are basic open balls of radius less than δ, the centers of A and B are of distance at least 4δ from each other, and $\epsilon < \delta$, we must have that $j - i \geq 3$. Then $D_i, \ldots, D_j$ is the desired ϵ-arc.

We claim that it takes at most $0^{(13)}$ to find such an ϵ-arc by Lemma 4.19.1. The oracle $0^{(13)}$ is used to find the finite δ'-cover of $J_1, \ldots, J_4$ and a special point inside I, but after fixing these parameters the names of $\tilde{A}$ and $\tilde{B}$ are computable in $0^{(9)}$. The rest requires much less powerful oracles than $0^{(13)}$. We use, e.g., $0^{(4)}$ to recognize an ϵ-arc as in Remark 4.2. We use $0^{(10)}$ to check the intersections with $\tilde{A}$ and $\tilde{B}$. We also need to check containment of $D_0, \ldots, D_\ell$ in N, which we can do with $0^{(7)}$ by Lemma 3.8. Using $0^{(13)}$ we search through all possible ϵ-arcs until we find one. ☐

Phase 3. Choose special points $a_0 \in D_0 \cap \tilde{A}$ and $b_0 \in D_\ell \cap \tilde{B}$. Since $D_0 \cup \cdots \cup D_\ell$ is a finite union of basic connected open sets, it is a $0^{(4)}$-computable open neighbourhood. Using Theorem 4.9, there is a $0^{(7)}$-computable Jordan arc H_{a_0,b_0} from a_0 to b_0 within $D_0 \cup \cdots \cup D_\ell$. We can use $0^{(13)}$ to find it.

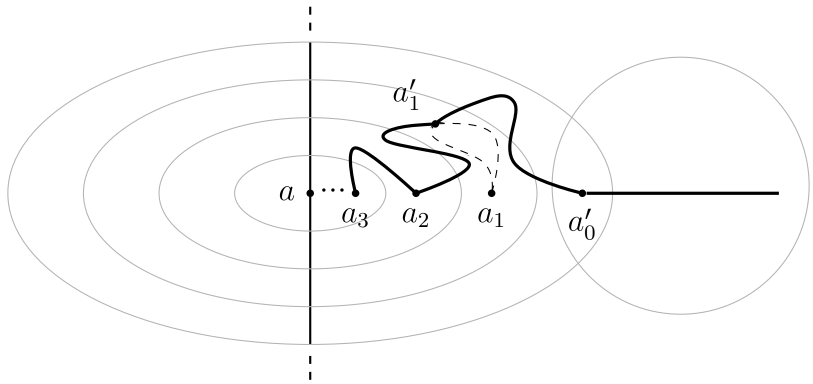
Phase 4. Now choose a basic open set $A_1 \ni a$, $A_1 \subseteq A$, disjoint from $D_0 \cup \cdots \cup D_\ell$, and of radius $< \delta$ and < 1. Define $\tilde{A}_1$ to be the connected component of $A_1 \cap R$ with a on its boundary. The connected open set $\tilde{A}_1$ has a $0^{(9)}$-computable name, and we can compute an index for this name using $0^{(13)}$; this is the same as the computation of names for $\tilde{A}$ and $\tilde{B}$ earlier. (Note that the curve H_{a_0,b_0} is disjoint from $\tilde{A}_1$.)

Phase 5. Choose a special point $a_1 \in \tilde{A}_1$. Find an $\epsilon_1 > 0$, an ϵ_1-arc from a_0 to a_1 in $\tilde{A}_0$, a $\delta_1 > 0$, and a finite δ_1-cover of $\tilde{J}_1$, such that the ϵ_1-arc is disjoint from the δ_1-cover. Such an ϵ_1-arc exists because $\tilde{A}_0$ is connected, and so there is a Jordan arc from a_0 to a_1 within $\tilde{A}_0$. Using Theorem 4.9 choose a $0^{(7)}$-computable Jordan arc H_{a_0,a_1} from a_0 to a_1 within this ϵ_1-arc, and so within $\tilde{A}_0$ and disjoint from the finite δ_1-cover of $\tilde{J}_1$.

As before with H_{a_0,b_0}, even though the arc is $0^{(7)}$-computable searching for the suitable finite parameters used in its definition requires the more powerful oracle $0^{(13)}$.

It should be clear how this process can be iterated to define H_{a_{i+1},a_i} (and $H_{b_i,b_{i+1}}$). For that, define $\tilde{A}_i \ni a_i$ (and $\tilde{B}_i \ni b_i$) of radius $< \frac{1}{i}$ and keep them disjoint from (finite covers of) the arcs already constructed, and also use δ_i-covers of $\tilde{J}_1$ to keep the new arcs away from the boundary. Each of these arcs is $0^{(7)}$-computable, however, computing their indices requires $0^{(13)}$. We are using here, among other things, the fact that a and b have $0^{(9)}$-computable names.

Refining the arcs. The issue is that H_{a_1,a_0} might intersect the Jordan arc H_{a_0,b_0} at some point other than a_0, and similarly H_{a_2,a_1} and H_{a_1,a_2} can intersect at a point other than a_1, etc.



Any intersection of H_{a_0,b_0} must be in $\tilde{A}_0$, since the Jordan arc H_{a_0,a_1} is contained in $\tilde{A}_0$; and the intersection cannot be in $\tilde{A}_1$ as the Jordan arc H_{a_0,b_0}, being contained in $D_0 \cup \cdots \cup D_\ell$, is disjoint from this. So we can choose some common point $a_0' \in \tilde{A}_0 - \tilde{A}_1$ of the two arcs (possibly with $a_0' = a_0$ if they did not intersect anywhere else; see the diagram) and, by Lemma 3.20,

- a $0^{(8)}$-computable Jordan arc $H^*_{a_0',b_0}$ from a_0' to b_0 which is a subarc of H_{a_0,b_0}, and
- a $0^{(8)}$-computable Jordan arc $H^*_{a_1,a_0'}$ from a_1 to a_0' which is a subarc of H_{a_1,a_0}

such that $H^*_{a_0',b_0}$ and $H^*_{a_1,a_0'}$ meet only at their common endpoint a_0'.

Now H_{a_2,a_1} might intersect $H^*_{a_1,a_0'}$, but it cannot intersect $H^*_{a_0',b_0}$ (because H_{a_2,a_1} is in $\tilde{A}_1$ but $H^*_{a_0',b_0}$ is disjoint from $\tilde{A}_1$). As before, we claim that we can find common point $a_1' \in \tilde{A}_1 - \tilde{A}_2$ of the two arcs H_{a_2,a_1} and $H^*_{a_1,a_0'}$ and

- a $0^{(8)}$-computable Jordan arc $H^{**}_{a_1',a_0'}$ from a_1' to a_0' which is a subarc of $H^*_{a_1,a_0'}$, and
- a $0^{(8)}$-computable Jordan arc $H^*_{a_2,a_1'}$ from a_2 to a_1' which is a subarc of H_{a_2,a_1}

such that $H^{**}_{a_1',a_0'}$ and $H^*_{a_2,a_1'}$ meet only at their common endpoint a_1'. It seems that, however, there is an obstacle here since $H^*_{a_1,a_0'}$ is already a $0^{(8)}$-computable arc; if we found a $0^{(9)}$-computable intersection point a_1', then the arc $H_{a_2,a_1'}$ would be $0^{(9)}$-computable, and we would keep increasing the complexity.

To circumvent this issue, use the fact that $H^*_{a_1,a_0'}$ was just the restriction of a $0^{(7)}$-computable arc to a $0^{(8)}$-computable interval. Then a_1' is on the restriction of $H^*_{a_1,a_0'}$ to some rational interval, and this restriction is a $0^{(7)}$-computable arc (though its index requires a more powerful oracle to be computed). So again by Lemma 3.20 this intersection point is (non-uniformly) $0^{(8)}$-computable. But of course, we can use $0^{(13)}$ to compute a $0^{(8)}$-computable name for the intersection point a_1'. Thus, using $0^{(13)}$ we can compute indices for these as $0^{(8)}$-computable arcs.

Continue in this way to define points $a_i' \in \tilde{A}_i$ and Jordan arcs $H^{**}_{a'_{i+1},a'_i}$ such that $H^{**}_{a'_{i+1},a'_i}$ is contained entirely in $\tilde{A}_i$. Similarly, on the other side, we can define points $b_i' \in \tilde{B}_i$ and Jordan arcs $H^{**}_{b'_{i+1},b'_i}$ such that $H^{**}_{b'_{i+1},b'_i}$ is contained entirely in $\tilde{B}_i$. Moreover, these arcs intersect each other only at their common endpoints. Thus we can piece them together to get the desired Jordan arc H from a to b. (The fact that H is continuous at the left- and right-hand endpoints is because $\tilde{A}_i$ and $\tilde{B}_i$ are contained in open balls of radius at most $1/i$.) Note that H is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc. $\square$

We also had a strengthening of this theorem:

Theorem 4.19. *In addition to the hypotheses of Theorem 4.17, suppose that*

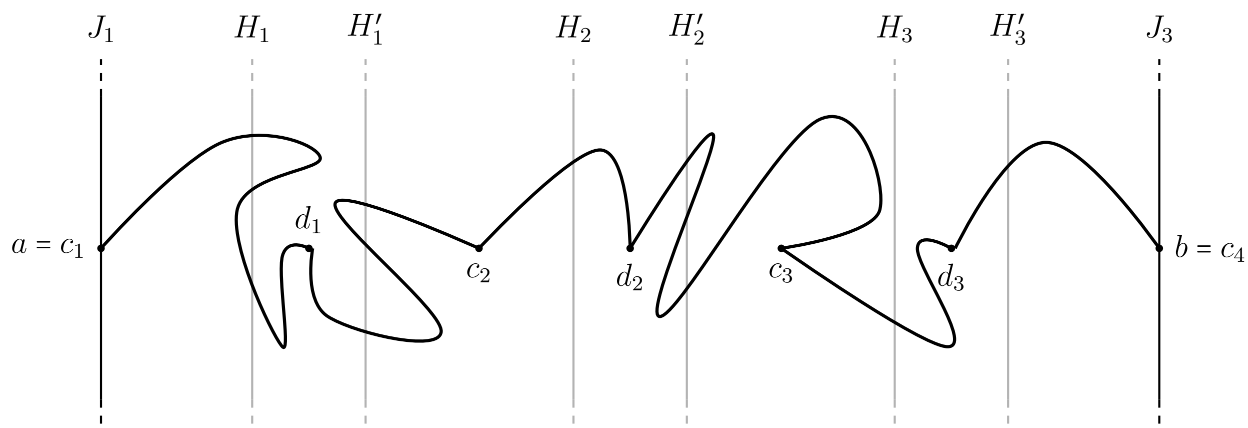
$$H_1, H_1', H_2, H_2', \dots, H_\ell, H_\ell'$$

are Jordan arcs, all non-intersecting, and each from a point on J_2 to a point on J_4, neither point being on the ends of J_2 or J_4. Suppose that $H_1, H_1', H_2, H_2', \dots, H_\ell, H_\ell'$ are listed in that order from closest to J_1 to closest to J_3. Then there is a globally $0^{(13)}$/locally $0^{(9)}$-computable Jordan arc as in Theorem 4.17 with the additional property that once it crosses H_i', it never again crosses H_i.

Like in Theorem 4.17 we are not saying anything about the uniformity of finding such an arc; indeed, there is no effectivity assumption on the H_i and H_i'.

Proof. The proof is essentially just a refinement of the proof of Theorem 4.17 where we are more careful about what happens between a_0 and b_0. To begin, choose A and B so that they do not intersect any of the H_i. As before, we define an arc H_{a_0,b_0} from a_0 to b_0, but this time we will do it more carefully. After this, the rest of the proof is the same as for Theorem 4.17.

Choose points $c_1 = a_0, d_1, c_2, d_2, \dots, c_\ell, d_\ell, c_{\ell+1} = b_0$ such that c_i is between H_i' and H_{i+1} and d_i is between H_i and H_i'. Now by Theorem 4.9 there are $0^{(7)}$-computable arcs K_{c_i,d_i} from c_i to d_i and $K_{d_i,c_{i+1}}$ from d_i to c_{i+1} such that K_{c_i,d_i} is contained entirely between H_{i-1}' and H_i' and $K_{d_i,c_{i+1}}$ is contained entirely between H_i and H_{i+1}. (K_{c_1,d_1} is contained entirely between J_1 and H_1, and $K_{d_\ell,c_{\ell+1}}$ is contained entirely between H_ℓ' and J_3.) Even though the H_i and H_i' are not necessarily even arithmetic, we can non-effectively choose a neighbourhood consisting of finitely many open sets around such an arc, and disjoint from the H_i and H_i', and then choose our $0^{(7)}$-computable arc within this neighbourhood.



As in Theorem 4.17, some of these arcs might intersect others. Once again, we can remove these intersections, replacing $c_1, d_1, c_2, \dots, d_\ell, c_{\ell+1}$ with new points $c_1', d_1', c_2', \dots, d_\ell' c_{\ell+1}'$

and finding subarcs $K^*_{c_1,d'_1}, K^*_{d'_1,c'_2}, \dots, K^*_{c'_\ell,d'_\ell}, K^*_{d'_\ell,c_{\ell+1}}$ of the original arcs, so that the arcs only intersect at their common endpoints. (We keep the leftmost point $c'_1 = c_1 = a_0$ and the rightmost point $c'_{\ell+1} = c_{\ell+1} = b_0$ the same.) The new arcs are $0^{(8)}$-computable arcs, and $0^{(13)}$ can find indices for them. Then let H_{a_0,b_0} be the concatenation of $K^*_{c_1,d'_1}, K^*_{d'_1,c'_2}, \dots, K^*_{c'_\ell,d'_\ell}, K^*_{d'_\ell,c_{\ell+1}}$. This is an arc from a_0 to b_0.

Of the constituent arcs of H_{a_0,b_0}, the only one that crosses H_i is $K^*_{c'_i,d'_i}$, and the only one that crosses H'_i is $K^*_{d'_i,c'_{i+1}}$. Thus H_{a_0,b_0} has the property that once it crosses H'_i, it never again crosses H_i.

Now that we have defined H_{a_0,b_0}, the rest of the proof is the same as for Theorem 4.17. Since A was disjoint from H_1, and B was disjoint from $H_{\ell'}$, none of the other arcs making up H can cross any H_i or H'_i. □

5. Triangulation and categoricity. Proof of Theorem 1.3

In this subsection we prove two important consequences of Theorem 1.1.

Theorem 5.1. *Every computable closed surface $X = \overline{M}$ has an arithmetic ($0^{(22)}$-computable) triangulation. We can find such a triangulation in an arithmetically ($0^{(25)}$-computably) uniform way.*

Proof. By Theorem 1.1, X has an arithmetic atlas. The theorem will then follow from the relativization of the following lemma.

Lemma 5.2. *A computable closed surface $X = \overline{M}$ with a computable atlas (with computable inverse functions) has a ($0'$-computable) triangulation, and we can find this triangulation uniformly using $0'$.*

Proof. We adapt very slightly the proof that every surface has a triangulation given by Thomassen [Tho92b]. Let $\varphi_1 : D_1 \to X, \dots, \varphi_n : D_n \to X$ be computable coordinate charts forming an atlas for X, with the inverse of the φ_i computable as well. Thomassen's argument shows that there are finitely many quadrangles $Q_1, \dots, Q_m$, with each Q_i contained in some coordinate chart D_{ℓ_i}, such that the images $\varphi_i(\mathrm{int}(Q_i))$ in X cover X.[8] Moreover, Thomassen shows that we can choose these quadrangles such that any two of them have only finitely many points of intersection. During Thomassen's argument, we can also choose the vertices of each Q_i to be special points in D_i. The images under the coordinate maps φ_i of the edges of the Q_j are computable Jordan arcs in X, so we can compute their intersections using $0'$ by Lemma 3.20. Then the union $\Gamma = \bigcup Q_i$ can be thought of as a graph drawn on X in a $0'$-computable way. One can easily extend this to obtain a triangulation. □

Combined with Theorem 1.1, this lemma shows that every computable surface $X = \overline{M}$ has an arithmetic triangulation. Moreover, we get a bound of $0^{(22)}$ on the complexity of the triangulation. To finish the proof of Theorem 5.1, we must argue that we can also arithmetically find such a triangulation; we can do this with a few more jumps—$0^{(25)}$—simply by searching for one and checking that it works. □

[8] One must make a slight adaptation to the compactness argument at the beginning of the proof, since we are working with a fixed coordinate chart $D_1, \dots, D_n$. We can make the slight modification to Thomassen's argument by choosing, for each point p, the initial quadrangles $Q_1(p)$ and $Q_2(p)$ to have diameter smaller than the Lebesgue number of the covering of X by $\varphi_1(D_1), \dots, \varphi_n(D_n)$.

Remark 5.3. We suspect that the upper bound $0'$ on the complexity of a triangulation provided by the proof of Lemma 5.2 is not optimal. Indeed, it seems that one can completely avoid using Lemma 3.20 by further modifying the arcs locally, but this looks a bit tedious (if possible). *We leave this as an open problem.* (In a slightly different set-up, this problem was first raised in [AC17].)

We say that a space is Δ^0_{n+1}-categorical if any two computable Polish presentations of the space are $0^{(n)}$-homeomorphic.

Remark 5.4. We do not specify here whether the inverses of homeomorphisms have to be $0^{(n)}$-computable as well; by Claim 3.6.1 there is actually no ambiguity here. Also, note that being arithmetically categorical is likely stronger than just saying that every pair of presentations are $0^{(n)}$-isomorphic for some n. For countable structures under isomorphism, such an example can be found in [DIM18]. For Polish spaces up to isometry, a similar example can be found because there is a computable functor from countable structures to metric spaces (see [GMKT18]) that, in particular, preserves this sort of categoricity. It seems that none of these results implies that the notions are also different for Polish spaces up to homeomorphism; we leave this as an open problem.

Theorem 1.3 follows from the theorem below and the well-known classification of compact surfaces.

Theorem 5.5. *Every computable closed surface is Δ^0_{26}-categorical.*

We first discuss why Theorem 1.3 follows from this theorem, and then we prove the theorem. Indeed, there is a uniformly computable list $(L_i)_{i\in\mathbb{N}}$ of all homeomorphism types of compact surfaces represented as simplices in $\mathbb{R}^n$ for a suitable n, and without repetition. To see if X is a compact surface, ask if there is a homeomorphism between X and one of the L_i; it is an arithmetic question. Also, to see whether two surfaces are homeomorphic, arithmetically find the respective L_i and L_j and check if $i = j$.

Proof of Theorem 5.5. One way to see why Theorem 5.5 holds uses Theorem 5.1 combined with the following deep result in combinatorial topology. Let K and L be simplicial complexes, and f a map $|K| \to |L|$. We say that f is piecewise linear if there is a subdivision K' of K such that for each $\sigma \in K'$, the restriction $f|\sigma$ of f to σ maps σ linearly into a simplex of L (equivalently in either ambient coordinates or barycentric coordinates).

Fact 5.6 (The Hauptvermutung; e.g., Theorems 8.5 and 36.2 of [Moi77])**.** *Let K_1 and K_2 be triangulated 2-manifolds or 3-manifolds. If there is a homeomorphism $|K_1| \to |K_2|$, then there is a piecewise linear homeomorphism $|K_1| \to |K_2|$. Hence K_1 and K_2 are combinatorially equivalent.*

Using Theorem 5.2, find $0^{(25)}$-computable triangulations of the manifolds. By the Hauptvermutung, if they are homeomorphic, then there is a homeomorphism between them that is piecewise linear, hence computable relative to the triangulations. ☐

6. Embeddings in Euclidean space. Proof of Theorem 1.2

The goal of this section is to prove that *every computable compact surface is homeomorphic to an arithmetic closed subspace of a finite-dimensional Euclidean space.* Combined with Fact 6.1 below, Theorem 5.1, and Theorem 1.1, it will give Theorem 1.2.

A closed subset C of a computable Polish space M is Π^0_1 if we can computably list basic open balls making up its complement. A closed set C is Σ^0_1 if we can additionally list all basic open balls that intersect C. A closed set is *computable* if it is both Π^0_1 and Σ^0_1. It is well-known that C is Σ^0_1 if, and only if, it is *computably overt* which means that it contains a computable dense sequence of points. We obtain:

Fact 6.1. *A computable closed set has a computable Polish presentation.*

We will use:

Corollary 6.2 (Folklore)**.** *Let M be a compact Polish space and C an arithmetic closed set. Then C is arithmetically overt.*

Proof. Let $M - C = V$. Given an open set U, $U \cap C$ is empty if and only if $U \subseteq V$. This is arithmetic by Lemma 3.8. ☐

We will give two different proofs of this fact, both are of course much more general; one covers arbitrary Polish spaces of finite dimensions and the other works for arbitrary manifolds. Both proofs are essentially based on a careful analysis of classical proofs of similar results, but some care must be taken. As far as we know, the computable versions of the former result is new, while the second result can essentially be found in [AC17] in a significantly different notation and terminology. We give its short proof for the sake of completeness of exposition.

Definition 6.3. The dimension of M the least $n \in \mathbb{N} \cup \{\infty\}$ such that every open cover of M has a refinement of order $n+1$, i.e., each point belongs to at most $n+1$ sets. Here by a refinement we mean that each open set in the refinement is contained in an open set in the original cover.

Theorem 6.4. *Let M be a compact computable Polish space of dimension n. Then there is a $0'$-computable homeomorphic embedding of M into $\mathbb{R}^{2n+1}$. (If M is effectively compact, we there is a computable homeomorphic embedding of M into $\mathbb{R}^{2n+1}$.)*

Proof. This argument is based on the classical proof as given in e.g. Section 50 of [Mun00]. Say that a continuous function f is an ϵ-homeomorphism if $f^{-1}(x)$ has diameter at most ϵ for every x in the range. The standard proof is to show that:

Fact 6.5. *The set of ϵ-homeomorphisms form a dense open set in $C[M, \mathbb{R}^{2n+1}]$.*

From this, it follows by the Baire category theorem that the intersection, over all $\epsilon = 1/k$, of the ϵ-homeomorphisms is non-empty. Any element f of the intersection is injective, since for each $x \in \mathbb{R}^{2n+1}$, $f^{-1}(x)$ has diameter less than ϵ for every ϵ. Since M is compact and f is continuous, the inverse of f is also continuous and so f is a homeomorphism.

The fact that the ϵ-homeomorphisms are dense proceeds by the following argument. Recall that points $u_0, \ldots, u_k$ in $\mathbb{R}^n$ are affinely independent if they generate a k-dimensional plane, or equivalently if $u_0 - u_1, u_0 - u_2, \ldots, u_0 - u_k$ are linearly independent. A set of points A in $\mathbb{R}^n$ is in *general position* if any $N+1$ of them are affinely independent. The following fact is simple to prove:

Fact 6.6. *Given a finite set of points $x_1, \ldots, x_k$ of $\mathbb{R}^n$ and $\delta > 0$, there is a set of rational points $y_1, \ldots, y_k$ in general position in $\mathbb{R}^n$ such that $|x_i - y_i| < \delta$.*

The density of the ϵ-homeomorphisms is witnessed by following construction. Given a continuous $f : M \to \mathbb{R}^{2n+1}$ and $\delta > 0$, define an ϵ-homeomorphism g with $|f - g| < \delta$ as follows:

(1) Cover M by finitely many open sets $U_1, \dots, U_k$ such that
   (a) $\mathrm{diam}(U_i) < \frac{\epsilon}{2}$;
   (b) $\mathrm{diam}(f(U_i)) < \frac{\delta}{2}$;
   (c) $\{U_1, \dots, U_k\}$ has order at most $n+1$.

(2) For each i, choose a special point $x_i \in U_i$ and choose a rational point $z_i \in \mathbb{R}^{2n+1}$ such that $|z_i - f(x_i)| < \delta$ and such that $z_1, \dots, z_k$ are in general position. (This is possible by Fact 6.6.)

(3) Choose a partition of unity $\{\phi_i\}$ supported by U_i, i.e. such that the support of ϕ_i is contained within U_i. We can do this computably by Theorem 4.4.64 of [Bra98].

(4) Define
$$g(x) = \sum_i \phi_i(x) z_i.$$

Then, as argued in [Mun00], g is an ϵ-homeomorphism and $|f-g| < \delta$:

- To see that $|f-g| < \delta$,
$$f(x) - g(x) = \sum \phi_i(x)(z_i - f(x)) = \sum \phi_i(x)(z_i - f(x_i)) + \sum \phi_i(x)(f(x_i) - f(x)).$$
We have $|z_i - f(x_i)| < \delta/2$ and, if $\phi_i(x) > 0$ then $x \in U_i$ and so $|f(x_i) - f(x)| < \delta/2$. Since $\sum \phi_i(x) = 1$, we have $|f(x) - g(x)| < \delta$.
- Now to show that g is an ϵ-homeomorphism, we show that if $g(x) = g(y)$ then for some i, $x, y \in U_i$. If $g(x) = g(y)$ then
$$\sum (\phi_i(x) - \phi_i(y)) z_i = 0.$$
Because the covering $\{U_i\}$ has order at most $n+1$, at most $n+1$ of the $\phi_i(x)$ are non-zero, and at most $n+1$ of the $\phi_i(y)$ are non-zero. So there are at most $2n+2$ non-zero terms in the equation above. We have
$$\sum (\phi_i(x) - \phi_i(y)) = 1 - 1 = 0$$
and so since the $\{z_i\}$ are in general position in $\mathbb{R}^{2n+1}$ (and $(2n+1)+1 = 2n+2$) each of the coefficients $\phi_i(x) - \phi_i(y)$ must be zero. Now $\phi_i(x) > 0$ for some i so that $x \in U_i$, and so $\phi_i(y) > 0$ and $y \in U_i$.

Moreover, given f and $\delta \in \mathbb{Q}$, we can find such a g using f'. (2) and (3) can be done computably in f. For (1), we need to use $0'$ to recognize when finitely many open sets cover M. (If M is effectively compact, then (1) is in fact c.e.) We can look for finitely many closed sets that cover M, and then expand their radius a tiny bit. Seeing that $\mathrm{diam}(f(U_i)) < \delta/2$ can be done by f'. While it takes f' to find such a g, the resulting g is computable since the points z_i are rational and the partition of unity $\{\phi_i\}$ is computable.

To construct a $0'$-computable homeomorphic embedding of M into $\mathbb{R}^{2n+1}$, we need to effectivise the Baire category argument. We can begin by fixing a computable 1-homeomorphism f_1; this can be obtained (non-uniformly) by using the process given above applied to any elements of $C[M, \mathbb{R}^{2n+1}]$. Now, repeating this process, we can, for each n in turn, find a computable $1/n$-homeomorphism f_n such that $\|f_n - f_{n-1}\| < 2^{-n}$. It takes $0'$ to run this construction. The limit of the f_n's exists and is $0'$ computable. It gives an injective homeomorphic embedding of M into R^{2n+1}, thus a homeomorphic embedding. Because the data from one stage to the next is finite, this can all be done using $0'$. We get closer and closer at each stage so the limiting function is indeed computable by $0'$. □

Theorem 6.7 ([AC17]). *Let M be a computable compact topological manifold with computable atlas. Then there is a computable embedding of M into $\mathbb{R}^n$ for some n.*

Proof. This is again an effectivisation of a classical argument. Let $U_i, \ldots, U_k$ be the open cover of M given by the computable atlas, and let ϕ_i be a computable partition of unity supported by U_i. Let $g_i: U_i \to \mathbb{R}^m$ be the computable embedding of U_i in $\mathbb{R}^m$. Then define $h: M \to \mathbb{R}^m$ by

$$h_i(x) = \begin{cases} \phi_i(x) g_i(x) & x \in U_i \\ 0 & x \notin U_i \end{cases}.$$

Define

$$F: M \to \mathbb{R}^{k+km}$$

by

$$F(x) = (\phi_1(x), \ldots, \phi_k(x), h_1(x), \ldots, h_k(x)).$$

Then F is clearly continuous, and it is injective because if $F(x) = F(y)$, then $\phi_i(x) = \phi_i(y)$ for each i. For some i, $\phi_i(x) = \phi_i(y) > 0$, and so $x, y \in U_i$ and $h_i(x) = h_i(y)$. This implies that $g_i(x) = g_i(y)$, and since g_i is an embedding, $x = y$. Since M is compact, the inverse of F is also continuous, and so F is a homeomorphism. Moreover, F is clearly continuous. $\square$

In the case of compact surfaces, combine the above theorem with Theorem 1.1 to get an arithmetic embedding.

The arithmetic continuous image of a compact space is compact and thus closed. It is also arithmetically overt since the computable dense set will be mapped to an arithmetic dense set of the image. Therefore, we conclude that in both theorems above the space is homeomorphic to an arithmetically overt subspace of a suitable power of $\mathbb{R}$.

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