Measuring the complexity of characterizing $[0,1]$, S^1, and $\mathbb{R}$ up to homeomorphism

Matthew Harrison-Trainor and Eissa Haydar*

April 12, 2025

Abstract

In analogy to the study of Scott rank/complexity of countable structures, we initiate the study of the Wadge degrees of the set of homeomorphic copies of topological spaces. One can view our results as saying that the classical characterizations of $[0,1]$ (e.g., as the unique continuum with exactly two non-cut points, and other similar characterizations), appropriated expressed, are the simplest possible characterizations of $[0,1]$. Formally, we show that the set of homeomorphic copies of $[0,1]$ is $\mathbf{\Pi}^0_4$-Wadge-complete. We also show that the set of homeomorphic copies of S^1 is $\mathbf{\Pi}^0_4$-Wadge-complete. On the other hand, we show that the set of homeomorphic copies of $\mathbb{R}$ is $\mathbf{\Pi}^1_1$-Wadge-complete. It is the local compactness that cannot be expressed in a Borel way; the set of homeomorphic copies of $\mathbb{R}$ is $\mathbf{\Pi}^0_4$-Wadge-complete within the locally compact spaces.

1 Introduction

A long-standing program in computable structure theory is the study of Scott rank and Scott complexity. The basic question is, given a countable structure $\mathcal{A}$, how hard is it to characterize $\mathcal{A}$ up to isomorphism among all countable structures? This began with Scott [Sco65] who showed that there is always a formula of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ characterizing $\mathcal{A}$ among countable structures. One can then look for the simplest such sentence, with its complexity called the *Scott rank* of $\mathcal{A}$.[1]

The particular perspective we will take in this paper is that of *Scott complexity* as introduced in [AGHTT21] by Alvir, Greenberg, Harrison-Trainor, and Turetsky. The idea is to consider, in the Polish space $\mathrm{Mod}(\mathcal{L})$ of $\mathcal{L}$-structures, the set

$$\mathrm{Copies}(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \cong \mathcal{A}\}$$

of all isomorphic presentations of $\mathcal{A}$. By a well-known theorem of Lopez-Escobar [LE65], $\mathcal{A}$ has a Σ_α Scott sentence if and only if $\mathrm{Copies}(\mathcal{A})$ is $\mathbf{\Sigma}^0_\alpha$, and similarly for other complexity

---

[1]There are numerous other definitions of Scott rank, most using some notion of back-and-forth relations, but for the purposes of this paper we can ignore them. See, e.g., [Mon].

classes such as $\Pi_\alpha/\mathbf{\Pi}_\alpha^0$ and the difference hierarchy. Thus, instead of considering sentences characterizing $\mathcal{A}$, we can instead study the Wadge degree of Copies($\mathcal{A}$). This Wadge degree is called the *Scott complexity* of $\mathcal{A}$, and measures the complexity of charactering $\mathcal{A}$ up to isomorphism.

In this paper, our goal is to, instead of characterising countable structures up to isomorphism, characterize certain topological spaces up to homeomorphism. We restrict our attention to Polish spaces, that is, separable completely metrizable spaces. In this case, there is no syntactic perspective, and so given a topological space X we study the Wadge degree of the homeomorphic copies of X,

$$\text{HCopies}(X) = \{Y : Y \cong_{\text{homeo}} X\}.$$

We call this the *(topological) Scott complexity* of X.[2] (Here, the copies Y are thought of as being presented as the completion of a countable metric space. The space of all such presentation is itself a Polish space. See Section 2.1 for details.) The Scott complexity of X measures the difficulty of characterizing X up to homeomorphism.

For a countable structure $\mathcal{A}$, Copies($\mathcal{A}$) is always Borel. This is not the case for HCopies(X) if X is not compact; indeed, HCopies($\mathbb{R}$) is not Borel, as shown later in this paper. However, if X is compact, HCopies(X) is Borel; indeed, the homeomorphism equivalence relation is bi-reducible to a complete orbit equivalence relation of a Borel action of a Polish group [Zie16]. Another perspective, which we do not use in this paper, is to use the effective Gelfand duality [BEF⁺] to reduce the problem to isometric isomorphism of unital abelian C^*-algebras. These are metric structures and thus subject to the metric Scott analysis [BYDNT17], but it is not yet clear how to fruitfully pull back the metric Scott analysis across the Gelfand duality.

In the case of discrete structures, there are many examples for which the Scott complexity has been computed. For example, the complexity of the groups free groups [CHK⁺12, MW12]: $\mathbb{F}_n$ has complexity $\mathbf{d}\text{-}\mathbf{\Sigma}_2^0$ for n finite and $\mathbb{F}_\infty$ has complexity $\mathbf{\Pi}_4^0$. Moreover, we know [AGHTT21] that the only possible Scott complexities are $\mathbf{\Sigma}_\alpha^0$, $\mathbf{\Pi}_\alpha^0$, and $\mathbf{d}\text{-}\mathbf{\Sigma}_\alpha^0$.

In the context of topological spaces not only do we not yet know any general results about the topological Scott complexity but to our knowledge there are as yet no non-trivial examples where we know the exact Wadge degree of HCopies(X). In this paper, we study some of the first 1-dimensional topological spaces to come to mind: the compact line $[0, 1]$, the circle S^1, and the real line $\mathbb{R}$. We exactly characterize their topological Scott complexity. For these spaces, there are classical characterizations coming from continuum theory, e.g., $[0, 1]$ is the unique continuum with exactly two non-cut points. Our setting allows us to analyse such characterizations and determine whether they are as simple as possible. We show that, appropriately expressed,[3] they are the simplest possible characterizations.

Theorem 1.1. *The unit interval $[0, 1]$ has topological Scott complexity $\mathbf{\Pi}_4^0$, i.e., the set*

$$\text{HCopies}([0, 1]) = \{Y : Y \cong_{\text{homeo}} [0, 1]\}$$

---

[2]We also note that there is an analogous program for characterizing metric spaces (or indeed, metric structures) up to isometry, by considering $\{Y : Y \cong_{isometric} X\}$. While we are not aware of any work on this problem specifically, there is existing work along these lines, e.g., [MN13, CMS19, BM20]. The isometry classes are known to be Borel by [BYDNT17] or [EFP⁺13].

[3]For example, to express the notion of a cut point it is helpful to first be in a locally connected space.

is $\mathbf{\Pi}_4^0$-*Wadge-complete.*

Theorem 1.2. *The circle S^1 has topological Scott complexity $\mathbf{\Pi}_4^0$, i.e., the set*

$$\text{HCopies}(S^1) = \{Y : Y \cong_{\text{homeo}} S^1\}$$

is $\mathbf{\Pi}_4^0$-Wadge-complete.

All of the results in this paper are effective, i.e., these sets are actually lightface Π_4^0 and the Wadge reductions are not just continuous but computable. In particular we could also state theorems for index sets, e.g., the index set

$$\{i \mid \text{the } i\text{th computable Polish space is homeomorphic to } [0,1]\}$$

is Π_4^0 m-complete.

For the real line $\mathbb{R}$, because it is not compact, the set of homeomorphic copies is not necessarily Borel; in fact, we show that it is $\mathbf{\Pi}_1^1$-Wadge-complete. (We conjecture that this might always be the case for any locally compact non-compact space.) However we also show that the set of homeomorphic copies of $\mathbb{R}$ is the intersection of the set of locally compact spaces and a Borel set. This suggests that for working with non-compact locally compact spaces one should work within the locally compact spaces.

Theorem 1.3. *The set*

$$\text{HCopies}(\mathbb{R}) = \{Y : Y \cong_{\text{homeo}} \mathbb{R}\}$$

is $\mathbf{\Pi}_1^1$-Wadge-complete, and $\mathbf{\Pi}_4^0$-Wadge-complete within the class of locally compact spaces.

This paper is a proof-of-concept for a program of analysing the characterizations of topological spaces up to homeomorphism. $[0,1]$, S^1, and $\mathbb{R}$ are the most basic 1-dimensional topological spaces. A natural next class of examples to consider are two-dimensional topological spaces such as S^2 and $\mathbb{R}^2$. In [HTM24], Melnikov and the first author studied closed surfaces, and one of their results is that for X a closed surface, $\text{HCopies}(X)$ is always arithmetic. Thus $\text{HCopies}(S^2)$ is arithmetic, but the upper bound obtained is around Σ_{27}^0 which is almost certainly not the best possible. There are no known non-trivial known lower bounds.

2 Background

2.1 Presentations of separable metric spaces

The traditional approach to studying the class of $\mathcal{L}$-structures in computable structure theory is to consider the space of *representations* of $\mathcal{L}$-structures with domain $\mathbb{N}$. Such a representation can be thought of as an encoding of the structure in machine-readable form, and the space of representations forms a Polish space $\text{Mod}(\mathcal{L})$.

In this work, we want to consider representations of topological (and more specifically Polish) spaces. Since Polish spaces typically have uncountably many points, we cannot represent each point by a natural number. Turing [Tur36, Tur37] had already realized this long ago, and instead suggested representing a real number by a Cauchy sequence of rational numbers. Moreover, in order to perform any computations, we need to know how fast the Cauchy sequence converges.

Definition 2.1. A Cauchy sequence $(x_i)_{i \in \mathbb{N}}$ in a metric space (M, d) is *fast* if $d(x_i, x_{i+1}) < 2^{-i}$.

One thinks of the reals $\mathbb{R}$ presented as a completion of the rationals $\mathbb{Q}$, with each real number r represented by the Cauchy sequences converging to r. We extend this to arbitrary Polish spaces, using any countable dense subset of the space in place of $\mathbb{Q}$.

Definition 2.2. A *presentation* of a Polish space X is a countable set M of points (identified with $\mathbb{N}$) and a metric d such that X is homeomorphic to the completion $\overline{M}$ of the metric space (M, d). We think of representing the metric d by giving, for every two elements x, y of M, a fast Cauchy sequence converging to $d(x, y)$. Thus, a presentation of X can be identified with an infinite binary string.

Given a presentation $X = \overline{M}$ of a Polish space, we refer to the points of M as *special points*. We represent the non-special points of X using fast Cauchy sequences from M. The presentation $\mathbb{R} = \overline{\mathbb{Q}}$ is the natural presentation of $\mathbb{R}$ using $\mathbb{Q}$ as the special points, but there are, of course, other possible presentations of $\mathbb{R}$.

When we talk about the topology on X, we will generally use the open balls centered on special points with rational radii. We call these the *basic open balls* and denote them by $B_r(x)$. We note that the basic open balls might not even be connected, even when the space is, e.g., connected and locally path connected. We denote the basic closed balls by $\overline{B}_r(x)$, noting that this is different from the closure of the basic open ball.

We will frequently make use of ϵ-*paths*. An ϵ-path from x to y is a sequence of special points $x = u_1, \ldots, u_n = y$ such that $d(u_i, u_{i+1}) < \epsilon$. The beginning and ending points x and y need not be special points, but for convenience we require that all the points in between them must be special. The ϵ-path is within a set U if all of the points are within U. If U is open, then there is an ϵ-path (using special points) from x to y within U if and only there is an ϵ-path (not neccesarily consisting of special points) from x to y within U.

The space of all presentations of Polish spaces is a Π^0_2 subset of $2^{<\omega}$ (under the identification given above) and so is itself a Polish space in its own right. We call this space TOP. Note that TOP is homeomorphic to Baire space ω^ω, and so as described in the following subsection we can thus talk about Wadge reducibility and the Wadge degrees of subsets of TOP.

Remark 2.3. There is also an alternate perspective which is more traditional in descriptive set theory. In this perspective we view the space of compact Polish spaces as the space of closed subsets of the Hilbert cube $[0, 1]^{\mathbb{N}}$ under the Hausdorff metric (which makes the Hilbert cube into a Polish space in its own right). As far as we are aware there are as yet no formal results connecting the complexity of a class of topological spaces in the first perspective and in the second. But it is clear that, by effectivizing the universality of the Hilbert cube, there are low-level Borel maps taking presentations of a compact space to a closed subset of the Hilbert cube, and vice versa. Thus, while the exact complexity of a class of topological spaces might change depending on whether we think of presentations or of closed subsets of $[0, 1]^{\mathbb{N}}$, morally speaking the two approaches are the same.

2.2 Wadge degrees and reductions

Definition 2.4 (Wadge). Let A and B be subsets of Baire space $\omega^\omega \cong \mathsf{TOP}$. We say that A is *Wadge reducible* to B, and write $A \leq_W B$, if there is a continuous function f with

$A = f^{-1}[B]$, i.e.

$$x \in A \Longleftrightarrow f(x) \in B.$$

The equivalence classes under this pre-order are called the Wadge degrees. The Wadge hierarchy is the set of Wadge degrees under continuous reductions. With enough determinacy, the Wadge hierarchy is very well-behaved; it is well-founded (as proved by Martin and Monk) and almost totally ordered (in the sense that any anti-chain has size at most two) [Wad83].

Pointclasses such as $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Pi}^1_\alpha$, and $\mathbf{\Sigma}^1_\alpha$ are all closed downwards in the Wadge degrees. For Γ a pointclass, and A a set, we say that A is Γ-Wadge-complete (or simply Γ-complete) if it is Γ and also Γ-hard, which means that for every $B \in \Gamma$, $B \leq_W A$.

The reader may be a computability theorist used to many-one reductions of index sets. All of our Wadge reductions will actually be computable reductions, and so the same completeness theorems hold for many-one reductions. But building a Wadge reduction is more general (e.g., there are computable structures such that the set of indices of computable copies is Σ^0_3, but the set of indices of $0'$-computable copies is d-Σ^0_2 relative to $0'$). In this paper every upper bound on the complexity of a set will be a lightface upper bound.

Sometimes the complexity of a particular set is dominated by one aspect. In computable structure theory, one such example is finitely generated groups. The class of all finitely generated groups is Σ^0_3, and there are particular finitely generated groups G with Copies(G) being $\mathbf{\Sigma}^0_3$-complete [HTH18]. On the other hand for any finitely generated group G Copies(G) is the intersection of a Π^0_3 set and the set of all finitely generated groups. Thus the complexity of saying that G is finitely generated "drowns out" the complexity of saying which finitely generated group it is. Another example from [CHK$^+$12, MW12] is that, with F_∞ the free group on countably infinitely many generators, Copies(F_∞) is $\mathbf{\Pi}^0_4$-complete, but $\mathbf{\Pi}^0_3$-complete within the class of free groups. The standard way to separate these is to measure the complexity of Copies(G) within the set of finitely generated groups/free groups, as follows.

Definition 2.5. Let A and C be sets in ω^ω.

(1) We say that A is Γ within C if there is a set $A^* \in \Gamma$ with $A = A^* \cap C$.

(2) We say that A is Γ-Wadge-complete (or simply Γ-complete) within C if for every $B \in \Gamma$, there is a continuous function f such that for all x, $f(x) \in C$, and $x \in B \Longleftrightarrow f(x) \in A$.

The main example of this phenomenon in our paper is in Theorem 1.3 where when characterizing $\mathbb{R}$ up to homeomorphism, the fact that it is locally compact (which is Π^1_1) drowns out everything else. We show that HCopies$(\mathbb{R})$ is $\mathbf{\Pi}^0_4$-complete within the class of locally compact spaces.

The second example is that HCopies$([0,1])$ is $\mathbf{\Pi}^0_4$-complete, but the most difficult part is in expressing local connectedness. In Theorem 4.1 we show that HCopies$([0,1])$ is actually $\mathbf{\Pi}^0_3$-complete within the compact, connected, locally connected spaces.

3 A Π^0_4 characterization of $[0,1]$

In this section we prove the first part of Theorem 1.1, that HCopies$([0,1])$ is Π^0_4. We prove the second half of the theorem, that HCopies$([0,1])$ is $\mathbf{\Pi}^0_4$-hard, in the next section.

We recall the classical characterization of $[0,1]$ as the unique continuum (non-empty compact connected metric space) with exactly two non-cut points [Nad92, Theorem 6.17]. It is not obvious how to express the existence of non-cut points as we are limited to quantifying over special points. Instead, we will express an idea of "betweenness." In the following subsections, we will express various properties of a topological space in a Π_4^0 (or often simpler) way, and then show that they characterize $[0,1]$.

3.1 Compactness

Following [MN13], compactness can be expressed in a Π_3^0 way in terms of total boundedness: We say that a countable metric space satisfies CPCT if and only if for every $\epsilon > 0$, there is $n \in \mathbb{N}$ and n special points points $x_1, \ldots, x_n$ such that for all special points y, y is within ϵ of some x_i. We take a second here to note that $d(y, x_i) < \epsilon$ is a Σ_1^0 condition, because in a presentation we are only given a fast Cauchy sequence converging to $d(y, x_i)$. To make CPCT Π_3^0 we must instead interpret "y is within ϵ of some x_i" as that "$d(y, x_i) \leq \epsilon$" which is a Π_1^0 condition. As a general principal, these types of conditions are really about *sufficient smallness* and so asking for strict or non-strict inequalities are equivalent.

3.2 Connectedness

Connectedness for arbitrary spaces is not Borel, even in the case of finite dimensional Euclidean spaces, as shown in [DSR20]. However, connectedness in a compact space can be expressed much more simply.

We say that $X = \overline{(M, d)}$ satisfies CONN if for every $\epsilon > 0$ and special points $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$, if $U = \overline{B}_\epsilon(x_1) \cup \cdots \cup \overline{B}_\epsilon(x_m)$ and $V = \overline{B}_\epsilon(y_1) \cup \cdots \cup \overline{B}_\epsilon(y_m)$ cover the special points (i.e., for all special points z, there is some x_i with $d(z, x_i) \leq \epsilon$ or some y_i with $d(z, y_i) \leq \epsilon$) the sets $U^* = B_{2\epsilon}(x_1) \cup \cdots \cup B_{2\epsilon}(x_m)$ and $V^* = B_{2\epsilon}(y_1) \cup \cdots \cup B_{2\epsilon}(y_m)$ are not disjoint (i.e., there is a special point z such that for some x_i and y_j, $d(z, x_i) < 2\epsilon$ and $d(z, y_i) < 2\epsilon$). Note that CONN is a Π_2^0 property.[4] We remark again that to make this Π_2^0 we were careful to use the closed condition in the antecedent of the implication and the open condition in the consequence. Going forward, we will no longer explicitly comment on such issues.

Lemma 3.1. *Let $X = \overline{(M, d)}$ be compact. Then X is connected if and only if (M, d) satisfies* CONN.

Proof. We argue that a compact space X is connected if and only if it satisfies CONN. Suppose that X is not connected, with $X = U \sqcup V$ being a partition. Let d be the distance between U and V, and choose $\epsilon > 0$ such that $2\epsilon < d$. Let $U = \overline{B}_\epsilon(x_1) \cup \cdots \cup \overline{B}_\epsilon(x_m)$ and $V = \overline{B}_\epsilon(y_1) \cup \cdots \cup \overline{B}_\epsilon(y_m)$ be ϵ-covers of U and V by closed balls centered at special points. Note that $U^* = B_{2\epsilon}(x_1) \cup \cdots \cup B_{2\epsilon}(x_m)$ and $V^* = B_{2\epsilon}(y_1) \cup \cdots \cup B_{2\epsilon}(y_m)$ are still disjoint by choice of ϵ. Thus we witness the negation of CONN.

On the other hand, suppose that X fails to satisfy CONN. Take a witness with $U^* = B_{2\epsilon}(x_1) \cup \cdots \cup B_{2\epsilon}(x_m)$ and $V^* = B_{2\epsilon}(y_1) \cup \cdots \cup B_{2\epsilon}(y_m)$. Then U^* and V^* are disjoint, since

---

[4]It is easy to see that the set of connected spaces is also Π_2^0-complete within the compact spaces; one builds a copy of the space $[0, 1 - \frac{1}{n}] \cup [1, 2]$, and lets $n \to \infty$ in the infinitary outcome.

if they intersected then there would be a special point z in their intersection, as they are open. They also cover X, because $U = \overline{B}_\epsilon(x_1) \cup \cdots \cup \overline{B}_\epsilon(x_m)$ and $V = \overline{B}_\epsilon(y_1) \cup \cdots \cup \overline{B}_\epsilon(y_m)$ cover the special points of X, and every point of X is within ϵ of a special point of X. Thus $X = U^* \sqcup V^*$ witness that X is not connected. $\qquad\square$

3.3 Local connectedness

We expect that, in general, local connectedness might be quite hard to express (like local compactness, which is not Borel [NS15]). However, we do not have to find a condition which is both necessary and sufficient for local connectedness, but just a condition equivalent to local connectedness within compact spaces. Our condition is as follows.

LC: For all basic open sets $A = B_r(c)$ and $B = B_{2r}(c)$ there are special points $x_1, \ldots, x_n \in A$ and $\epsilon > 0$ such that:

   (1) for all special points $y \in A$, there is some i such that there is an ϵ-path from y to x_i in B.

   (2) for distinct i, j there is no ϵ-path from x_i to x_j in B.

   (3) for all i and for all special points $y \in A$ and $\epsilon' > 0$ if there is an ϵ-path from y to x_i in B then there is an ϵ'-path from y to x_i in B.

This is Π^0_4. First, we show that our condition LC implies local connectedness via weak local connectedness. We recall the difference between locally connected and weakly locally connected spaces. Let X be a topological space. X is locally connected if every point has a neighbourhood base consisting of open connected sets. X is weakly locally connected if every point has a neighbourhood base consisting of connected sets, which are not necessarily open. However these are equivalent; see, e.g., Theorem 27.16 of [Wil70].

Lemma 3.2. *If $X = \overline{(M, d)}$ is compact and satisfies LC, then X is weakly locally connected and hence locally connected.*

Proof. Let x be any point, and U an open set containing x. We can choose basic open balls $x \in B_r(c) \subseteq \overline{B_r(c)} \subseteq B_{2r}(c) \subseteq \overline{B_{2r}(c)} \subseteq U$. Let $A = B_r(c)$ and $B = B_{2r}(c)$. We will find a connected neighbourhood D of x contained in B. To witness that D is a neighbourhood, we will have $D \cap A$ open.

By LC, there are $x_1, \ldots, x_n \in A$ and ϵ satisfying (1), (2), and (3). Define

$$D_i = \{ y \in \overline{B} : \text{for all } \epsilon' > 0 \text{ there is an } \epsilon'\text{-path from } y \text{ to } x_i \text{ in } B \}.$$

Clearly $D_i \subseteq \overline{B} \subseteq U$. We prove the following five claims.

Claim 1. *Each D_i is closed.*

Proof. Given $y \in \overline{B}$, suppose that y is a limit point of D_i. We argue that $y \in D_i$. Given $\epsilon' > 0$, there is some point y' of D_i with $d(y, y') < \epsilon'/2$. Then there is an $\epsilon'/2$-path from y' to x_i in B, which, by replacing y' by y, is an ϵ'-path from y to x_i. $\qquad\square$

Claim 2. *Each D_i is connected.*

Proof. Suppose that D_i is not connected. Since D_i is closed and compact, there are disjoint relatively clopen $U, V \subseteq D_i$ which partition D_i and points $u \in U$ and $v \in V$. There is some distance ϵ' between U and V. Then there is no $\epsilon'/2$-path from u to v in B. $\qquad \square$

Claim 3. *The D_i are pairwise disjoint.*

Proof. Suppose that there is some $y \in D_i \cap D_j$. Then there is an $\epsilon/2$-path from y to x_i and an $\epsilon/2$-path from y to x_j. Putting them together, and deleting y, we get an ϵ-path from x_i to x_j. This contradicts (2). $\qquad \square$

Claim 4. *Each $D_i \cap A$ is open. Indeed, given $y \in D_i \cap A$ and $z \in A$ with $d(y, z) < \epsilon/2$, $z \in D_i$.*

Proof. Choose some ϵ'. We wish to find an ϵ'-path from z to x_i in B, which would place $z \in D_i$. Since $y \in D_i$, we have an ϵ-path from y to x_i in B. Since the special points are dense, we may choose a path among them, other than the y, which may be non-special. Call the path-elements a_i, where $a_1 = y$ and $a_n = x_i$. We have that y is in $B_{\epsilon/2}(z)$ and $B_\epsilon(a_2)$, and since both of these are open, there is some ball around y in both of these as well. Choose in particular a special $y' \in B$ within these two and also within ϵ' of y. Choose also a $z' \in A$ with $d(y, z') < \epsilon/2$ and $d(z, z') < \epsilon'$. Then we have an ϵ-path $z', y', a_2, \ldots, a_n = x_i$ among the special points, which by (3) gives us an ϵ'-path from z' to x_i in B, which by our choice of z' gives us an ϵ'-path from z to x_i in B. Thus $z \in D_i$.

Together with the fact that A is open, this shows that $D_i \cap A$ is open. $\qquad \square$

Claim 5. *The D_i cover A.*

Proof. For any $y \in A$, choose some special y' with $d(y, y') < \epsilon/2$. By the "Indeed" statement in the above claim, we have that $y' \in D_i \Rightarrow y \in D_i$ for any i. Since y' is special, (1) and (3) give us that $y \in D_i$ for some i. $\qquad \square$

Now since the D_i cover A, $x \in D_i$ for some i. Then D_i is a connected neighbourhood of x contained in U. This completes the proof of the lemma. $\qquad \square$

Second, we will show that any compact and locally connected space satisfies the condition LC.

Lemma 3.3. *If $X = \overline{(M, d)}$ is compact and locally connected then it satisfies LC.*

Proof. Firstly, take basic open sets with $A = B_r(c)$ and $B = B_{2r}(c)$ for some c and r. We want to show that there is a finite set of x_i together with an ϵ that satisfy (1)-(3) of LC.

Let $C_1, C_2, \ldots$ be the connected components of B, which are open sets because B is locally connected. Define an equivalence relation $\sim$ on the C_i saying that $C_i \sim C_j$ if for every $\epsilon' > 0$ and $x \in C_i$ and $y \in C_j$, there is an ϵ'-path from x to y in B. (Note that this is true of any two points x, y in the same C_i.) For each equivalence class, define an open set D which is the union of the C_i in that equivalence class. Let $D_1, D_2, \ldots$ be these sets. Then:

(1) For each i, $\overline{D_i}$ is connected. If not, then there is a clopen splitting of $\overline{D_i}$ with points x and y on different halves. Though the sets in this splitting are only clopen within $\overline{D_i}$, they are closed in X. Since there is some distance between these two sets, there is some ϵ' smaller than this distance such that there is no ϵ'-path from x to y in B.

8

(2) Each pair of distinct $\overline{D_i}$ and $\overline{D_j}$ are disjoint. Otherwise, they would have shared a limit point and would then have been of a distance less than any ϵ, contradicting that they come from different equivalence classes.

(3) The D_i are an open cover of B, and hence of the compact set $\overline{A}$. So there are finitely many of them, say $D_1, \ldots, D_n$, which cover $\overline{A}$.

For each i, j, choose $\epsilon_{i,j} > 0$ such that there is no $\epsilon_{i,j}$-path from any point $x \in D_i$ to any point $y \in D_j$. Let $\epsilon = \min_{i,j} \epsilon_{i,j}$, and choose a special point x_i in each D_i.

Claim 1. ϵ and $x_1, \ldots, x_n$ satisfy (1).

Proof. Since the D_i cover $\overline{A}$, they cover A, and so since $y \in A$, $y \in D_i$ for some i. By the construction of the D_i and since $x_i \in D_i$, this means that there is an ϵ'-path from y to x_i for all $\epsilon' > 0$, including ϵ. $\qquad\square$

Claim 2. ϵ and $x_1, \ldots, x_n$ satisfy (2).

Proof. Suppose that the converse of (2) was true. Then for some $i \neq j$, there would be an ϵ-path from x_i to x_j in B. But this contradicts the choice of ϵ in the construction above. $\qquad\square$

Claim 3. ϵ and $x_1, \ldots, x_n$ satisfy (3).

Proof. This follows from the construction of the D_i as the union of connected components which have ϵ-paths between any two elements for all ϵ. By the choice of ϵ in the above construction, y having an ϵ-path to some x_i places y in D_i, which then, for any ϵ', has an ϵ'-path to any other point within D_i, including x_i. $\qquad\square$

The claims prove the lemma. $\qquad\square$

If X is locally connected, compact, and connected, then it is path-connected and locally path-connected [Eng89, 6.3.11].

Corollary 3.4. *If $X = \overline{(M, d)}$ satisfies* CPCT, CONN, *and* LC *then it is path-connected and locally path-connected.*

3.4 Betweenness

In a standard proof of the classification of $[0, 1]$ as the unique continuum with exactly two non-cut points, the separation ordering plays an important role. We will deal not with the ordering, but with the corresponding notion of betweenness where one point is between two others if removing it separates the two points.

Definition 3.5. Define a ternary betweenness relation $\mathsf{BTW}(x, y, z)$ by setting $\mathsf{BTW}(x, y, z)$ if x, y, z are all distinct and for all $\delta > 0$, there are $\delta', \epsilon < \delta$ such that for all ϵ-paths $x = a_1, \ldots, a_n = z$ from x to z, there is some i such that $d(y, a_i) \leq \delta'$.

If $\mathsf{BTW}(x, y, z)$ holds, then we say that y *is between x and z*. It is easy to see that there is a symmetry between the first and third arguments, i.e., $\mathsf{BTW}(x, y, z)$ if and only if $\mathsf{BTW}(z, y, x)$. Note that the betweenness relation is Π_3^0.

Definition 3.6. Let ORD express that for all distinct special points x, y, z, either $\mathsf{BTW}(x, y, z)$, $\mathsf{BTW}(y, x, z)$, or $\mathsf{BTW}(z, x, y)$.

This expresses totality of the betweennes relation, and is Π_3^0.

Lemma 3.7. *If $X = \overline{(M, d)}$ is homeomorphic to $[0, 1]$ then X satisfies* ORD.

Proof. Take distinct $x, y, z \in X$. Since $\overline{X} \cong [0, 1]$, we can relabel x, y, and z so y is between x and z. We wish to show that for all rational δ, there exists $\delta' < \delta$ and $\epsilon < \delta'$ such that any ϵ-path from x to z goes within δ' of y. Take any $\delta' < \delta$ such that $x, z \notin B_{\delta'}(y)$. Then in $X - B_{\delta'}(y)$, the points x and z are in disjoint closed sets. Let $\epsilon < \delta$ be smaller than the distance between these two closed sets. Then there is no ϵ-path from x to z in $X - B_{\delta'}(y)$, hence any ϵ-path from x to z goes within δ' of y. $\qquad\square$

3.5 Putting everything together: arcs and cutpoints

So far, none of the properties we have introduced stipulate that a space must be non-trivial, so we introduce one final property. Recall that a continuum is said to be non-degenerate if it has more than one point. We say that $X = \overline{(M, d)}$ satisfies NDEGEN if it has at least two distinct points. This is Σ_2^0.

Combining various previous lemmas, we have the following.

Lemma 3.8. *If $X = \overline{(M, d)}$ is homeomorphic to $[0, 1]$, then it satisfies* NDEGEN, CPCT, CONN, LC, *and* ORD.

We must now show the converse.

Lemma 3.9. *If $X = \overline{(M, d)}$ satisfies* NDEGEN, CPCT, CONN, LC, *and* ORD, *then it is homeomorphic to $[0, 1]$.*

Proof. Since $X = \overline{(M, d)}$ satisfies CPCT, CONN, and LC we know that it is compact, connected, locally connected, path connected, and locally path connected. In particular, X is a continuum. We will show that X has exactly two non-cut points, which implies that is is homeomorphic to $[0, 1]$. (See [Nad92, Theorem 6.17].)

Claim 1. *Suppose that there is an arc from x to z that does not pass through y. Then $\neg\mathsf{BTW}(x, y, z)$*

Proof. Let f be the arc from x to z and suppose that y is not on this arc. There is some distance d between y and f. For sufficiently small ϵ, an ϵ-path from x to z approximating f avoids $B_{d/2}(y)$. Thus $\neg\mathsf{BTW}(x, y, z)$. $\qquad\square$

Claim 2. *For $x, y \in M$, there is a unique arc from x to y.*

Proof. Much of the technical difficulty with proving this claim is the fact that ORD only tells us about special points, but to prove the claim we need to argue for all points x, y not necessarily special.

The existence of such an arc follows from path-connectedness (which is equivalent to arc-connectedness in Hausdorff spaces). To prove uniqueness, suppose we have two distinct

arcs, f and g, from x to y. Take a point z on f that is not on g. Take a path-connected set around z that is disjoint from g, say U, and take a special point $z' \in U$. Then there is an arc h from z to z' within U, and thus disjoint from g. We may assume by moving z' and shortening h that z is the only point on both h and f. Break up f at the point z into two segments f_1 from x to z and f_2 from z to y. Choose also special points x' and y' close to x and y, respectively, with arcs from x to x' near x and from y to y' near y. The picture is something like this:
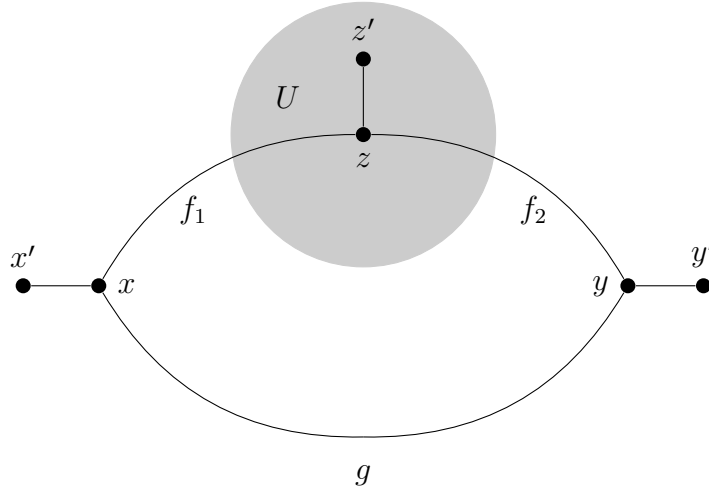
Thus we have arcs containing x' and y' but not z', y' and z' but not x', and x' and z' but not y'. By Claim 1 we have that $\neg\mathsf{BTW}(y', z', x')$, $\neg\mathsf{BTW}(x', y', z')$, and $\neg\mathsf{BTW}(z', x', y')$. This contradicts ORD and proves the lemma. $\qquad\square$

Claim 3. *There are at most two non-cut points.*

Proof. Suppose that there were three non-cut points a, b, c. Without loss of generality, we may assume that the arc f from a to b does not contain c. (If it did, then take the shorter arc from a to c and swap b and c.) We may also assume that the arc g from b to c does not contain a, as if it did, then we could take the arc from a to c and swap a and b.

Now we argue that b is in fact a cut point. Suppose not; then $X - \{b\}$ is connected, and since it is still locally path connected, it is path connected. Thus there is an arc from a to c avoiding b, a contradiction. $\qquad\square$

It is a theorem of Moore that any non-degenerate continuum must have at least two non-cut points (see [Nad92, Theorem 6.6]). Thus we have exactly two non-cut points. This proves the lemma. $\qquad\square$

Putting together all of the lemmas, we have one half of Theorem 1.1.

Theorem 3.10. $X = \overline{(M, d)}$ *satisfies* NDEGEN, CPCT, CONN, LC, *and* ORD *if and only if* X *is homeomorphic to* $[0, 1]$. *In particular,* $\mathrm{HCopies}([0, 1])$ *is* Π_4^0.

Note that of all of these conditions, it is only LC which is Π_4^0. Thus, in some sense, the hard part of saying that $X = \overline{(M, d)}$ is homeomorphic to $\mathrm{HCopies}([0, 1])$ is in saying that X is locally connected.

Corollary 3.11. HCopies($[0, 1]$) *is Π_3^0 within the locally connected spaces.*

In Theorem 4.1 to follow, we will show that HCopies($[0, 1]$) is Π_3^0-complete within the locally connected spaces.

4   A warmup: Π_3^0-hardness within the compact, connected, locally connected spaces

We will prove the following theorem as a warmup.

Theorem 4.1. HCopies($[0, 1]$) *is $\mathbf{\Pi_3^0}$-Wadge-complete within the compact, connected, locally connected spaces.*

Proof. It follows from Corollary 3.11 that HCopies($[0, 1]$) is Π_3^0 within the compact locally connected spaces. So it remains to prove a hardness result. Consider the $\mathbf{\Pi_3^0}$-Wadge-complete set $S = \{W \subseteq \omega \times \omega \ | \ \forall n \ |W_n| < \infty\}$ where $W_n = \{(n, m) \ | \ (n, m) \in W\}$ is the nth column of W. Given W we must build a compact, connected, locally connected space $X = X_W$ such that $W \in S$ if and only if $X_W \cong [0, 1]$. Fix W for which we must define $X = X_W$. We will work somewhat dynamically by letting $W_n[i] = W_n \restriction i = \{m \in W_n \ | \ m < i\}$. Note that $W_n[0] = \varnothing$.

We construct X inside the unit square $[0, 1] \times [0, 1]$. We will always include in X the line $[0, 1] \times \{0\} = \{(x, 0) : x \in [0, 1]\}$. We will also include in X the graph Γ_f of a continuous function $f : [0, 1] \to [0, 1]$. Thus

$$X = \Gamma_f \ \cup \ [0, 1] \times \{0\}.$$

The function f will have the property that $f(0) = 0$ and $f(1) = 1$. Thus X will always be connected, locally connected, and compact. X will be homeomorphic to $[0, 1]$ if and only if $f(x) > 0$ for all $x > 0$.

It just remains to define f. Pick some sequence of elements, alternating between a_i and b_i,

$$0 < \cdots < b_2 < a_2 < b_1 < a_1 < b_0 < a_0 = 1$$

converging to 0. In each interval $[a_{i+1}, a_i]$ choose an increasing sequence

$$a_{i+1} = \ell_i^0 < \ell_i^1 < \ell_i^2 < \cdots < b_i$$

converging to b_i, and a decreasing sequence

$$b_i < \cdots < r_i^2 < r_i^1 < r_i^0 = a_i$$

converging to b_i. Thus we have divided the interval $[0, 1]$ into intervals $[a_{i+1}, a_i]$, and we have divided each of these intervals into a left interval $[a_{i+1}, b_i]$ and a right interval $[b_i, a_i]$. The left and right intervals are in turn divided into intervals $[\ell_i^j, \ell_i^{j+1}]$ on the left and $[r_i^{j+1}, r_i^j]$ on the right:

$$a_{i+1} = \ell_i^0 < \ell_i^1 < \ell_i^2 < \cdots < b_i < \cdots < r_i^2 < r_i^1 < r_i^0 = a_i.$$

Every $t \in [0,1]$ is included in exactly one interval $[\ell_i^j, \ell_i^{j+1}]$ or $[r_i^{j+1}, r_i^j]$, except the endpoints which are included in two of these intervals, and the points 0 and b_i are not included in any of these intervals.

The idea is that X will look somewhat like the following:
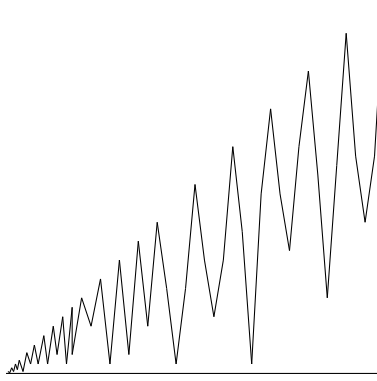
The straight horizontal line is $[0,1] \times \{0\}$ and the squiggly line is the graph of f. Each of the downward spikes in the graph of f happens between some a_i and a_{i+1}, with the points at b_i. The larger $|W_i|$, the lower the spike gets, so that if $|W_i| = \infty$, then the spike touches the horizontal line. This makes the space X not homeomorphic to $[0,1]$. If no spike touches the horizontal line, then X will be homeomorphic to $[0,1]$.

Given $t \in [0,1]$, define $f(t)$ as follows:

(1) If $t = 0$, then $f(t) = f(0) = 0$.

(2) If $t = b_i$, then $f(t) = f(b_i) = t2^{-|W_i|} = b_i 2^{-|W_i|}$ (which is equal to 0 if W_i is infinite).

(3) If $t \in [\ell_i^j, \ell_i^{j+1}]$ then

$$ f(t) = t \left[\frac{t - \ell_i^j}{\ell_i^{j+1} - \ell_i^j} \left(2^{-|W_i[j+1]|} - 2^{-|W_i[j]|} \right) + 2^{-|W_i[j]|} \right]. $$

So $f(\ell_i^j) = t2^{-|W_i[j]|} = \ell_i^j 2^{-|W_i[j]|}$ and $f(\ell_i^{j+1}) = t2^{-|W_i[j+1]|} = \ell_i^{j+1} 2^{-|W_i[j+1]|}$. In between, $f(t)$ is t times a linear interpolation.

(4) If $t \in [r_i^{j+1}, r_i^j]$ then

$$ f(t) = t \left[\frac{t - r_i^{j+1}}{r_i^j - r_i^{j+1}} \left(2^{-|W_i[j]|} - 2^{-|W_i[j+1]|} \right) + 2^{-|W_i[j+1]|} \right]. $$

This is essentially (3) in reverse. We have $f(r_i^j) = t2^{-|W_i[j]|} = r_i^j 2^{-|W_i[j]|}$ and $f(r_i^{j+1}) = t2^{-|W_i[j+1]|} = r_i^{j+1} 2^{-|W_i[j+1]|}$.

We must check that $f(t)$ is well-defined; that is, for any point which is in two of these intervals, the definitions of f agree on those common points. This is easy to see for the points ℓ_i^j with $j > 0$ which are parts of the intervals $[\ell_i^{j-1}, \ell_i^j]$ and $[\ell_i^j, \ell_i^{j+1}]$. Similarly, it is easy to see for the points r_i^j with $j > 0$. The non-trivial case is the points $a_i = \ell_i^0 = r_{i-1}^0$. But in this case, $|W_i[0]| = |W_{i-1}[0]| = 0$ and so $f(a_i) = a_i$.

We also need to check that the function is continuous, which means checking that (1) and (2) agree with the limiting behaviour of f. For (1), we have $f(t) \leq t$ and so $\lim_{t \to 0} f(t) = 0$. For (2), for $t \in [\ell_i^j, \ell_i^{j+1}]$ or $t \in [r_i^{j+1}, r_i^j]$ we have $t2^{-|W_i[j+1]|} \leq f(t) \leq t2^{-|W_i[j]|}$. Thus $f(b_i) = b_i 2^{-|W_i|}$ is the limit as $t \to b_i$.

We also need to be able to produce $X = X_W$ computably and uniformly from W. Note that we cannot compute $|W_i|$ uniformly from W, so we cannot compute $f(b_i)$. But we can compute f on all other inputs, and this gives a dense set to produce a presentation of X_W.

Finally, note that if each W_i is finite, then $f(t) > 0$ when $t > 0$, but if W_i is infinite, then $f(b_i) = 0$. As discussed before, X_W is homeomorphic to $[0,1]$ if and only if $f(t) > 0$ for all $t > 0$. This completes the argument. $\qquad\square$

5 $\mathbf{\Pi}_4^0$-hardness for $[0,1]$

We now turn to proving that HCopies($[0,1]$) is $\mathbf{\Pi}_4^0$-complete.

5.1 The main idea

We begin by showing that HCopies($[0,1]$) is $\mathbf{\Sigma}_3^0$-complete. By attaching infinitely many instances of this construction end-to-end, we will get a $\mathbf{\Pi}_4^0$ completeness result.

Looking at the characterization above, we see that the $\mathbf{\Pi}_4^0$ complexity comes from the local connectedness, and so we should try to make a line in the $\mathbf{\Sigma}_3^0$ outcome, and a non-locally-connected space in the $\mathbf{\Pi}_3^0$ outcome. The non-locally-connected spaces we build will be based off of the following:
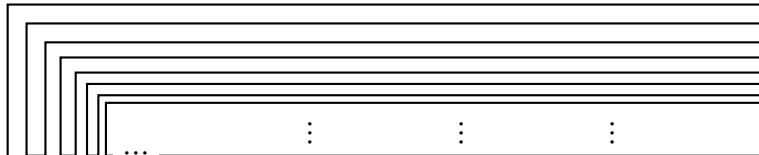
The space is built up from infinitely many *tendrils*. Above we drew each tendril with right-angled turns, like the following:
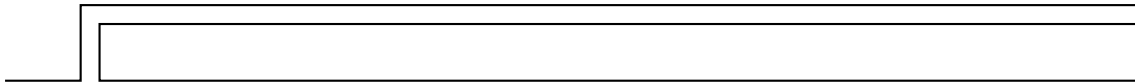

In the actual construction, we will have the tendrils consist of two straight lines, as follows:
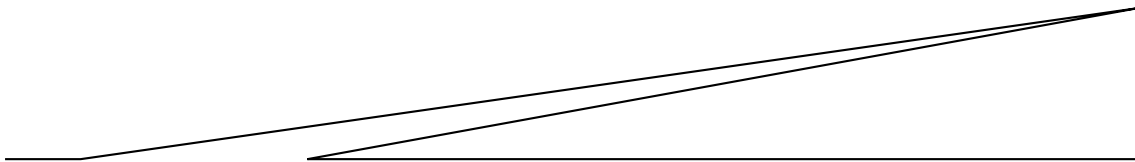
But for clarity of the pictures it will be easier to draw the tendrils using horizontal and vertical lines.

Each of the tendrils originates from somewhere on the left half, and goes all the way to the rightmost side. The rightmost point has no small connected neighbourhoods, as each neighbourhood contains some tendril coming from the left side.

At each stage s, we will have a "target" topological space which we are building towards. This target space will be a non-locally-connected space looking something like the space above with many tendrils. However, at each stage of the construction we will have only actually put finitely many points from the target space into the metric space being constructed. For example, suppose that the target space has a tendril
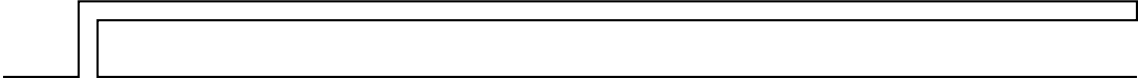
At a finite stage, we might have only placed points configured as

We might then decide to *collapse* the tendril, changing the target space to
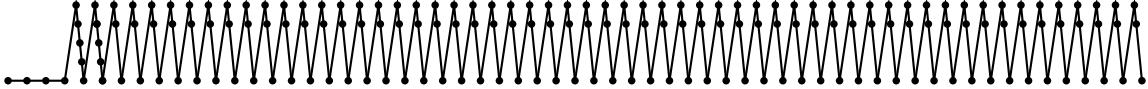
Since we are already committed to the finitely many points we have placed, and what we have said about the distances between them, they must all stay in the same positions in the new target space. But by changing the target space, we can incorporate the tendril into a (new) homeomorphic copy of the unit interval.

We will then introduce new tendrils, which may themselves get collapsed, after which we introduce more new tendrils, and so on. If we collapse all but finitely many tendrils infinitely often, then in the limit our space will be locally connected and homeomorphic to the unit interval. Otherwise, there will be infinitely many tendrils and our space will not be locally connected.

One remaining technical point is to explain why, in the case that we collapse all but finitely many tendrils, we end up with a space homeomorphic to the unit interval. As illustrated above, we end up with a sawtooth function even after collapsing a single tendril. We need to ensure that in the process of introducing and collapsing new tendrils, this sawtooth is not too wild, and instead comes to a nice limit. To do this, when we introduce new tendrils, they will closely follow the existing sawtooth space. In this way, even though the sawtooth gets more and more bumpy, it comes to a limit.

5.2   Construction for Σ_3^0-completeness

Consider the Σ_3^0 set
$$A = \{W \subseteq \omega \times \omega \mid \exists n \; |W_n| = \infty\}$$
where, as before, W_n is the nth column of W.

Given W, we must build a Polish space $X_W = \overline{M_W}$ such that $W \in A$ if and only if X_W is homeomorphic to $[0,1]$. The main idea is that whenever an element enters W_n we will collapse the nth tendril along with every greater tendril. Thus if for every n the set W_n is finite then for each n there is some point after which we stop collapsing the nth tendril. In that case, we have infinitely many tendrils, and X_m is not homeomorphic to $[0,1]$. Otherwise, if W_n is infinite for some n, then we collapse the nth tendril and every greater tendril infinitely often, and in the end have only finitely many tendrils. So X_W is homeomorphic to $[0,1]$.

For the following, fix the W for which we will construct X_W. For simplicity, we suppress the subscript W, writing, e.g., X for X_W and M for M_W. The construction will be a dynamic construction, and as before we use $W_n[s] = W_n \restriction s$ for the restriction of W_n to elements $< s$. We can view $W_n[s]$ as the portion of W_n that we have seen by stage s, just like if we were doing a many-one reduction in computability theory. (For the computability theorist, one can think of W_n as the set of stages at which a new element enters a c.e. set.)

We will work inside $\mathbb{R}^2$ and indeed inside $[-1,1] \times [0,1]$; we call these the x and y-coordinates, respectively. To simplify a later part of the construction, we will ensure that there is only one special point for each particular value of the x coordinate. Each special point of M will be a rational tuple.

5.2.1 The data

At each stage s, we will have the following data:

(1) The *main line*: A function $\mathfrak{m} : [-1,1] \to [0,1]$ with $\mathfrak{m}(-1) = 0$ and $\mathfrak{m}(1) = 0$.

(2) Infinitely many *tendrils* where the nth tendril consists of the data:

    (a) a *left attachment point* ℓ_n, given as an x-value;

    (b) a *right attachment point* r_n, given as an x-value;

    (c) a *top arc* $\mathfrak{t}_n : [\ell_n, 1] \to [0,1]$;

    (d) a *bottom arc* $\mathfrak{b}_n : [r_n, 1] \to [0,1]$.

The functions $\mathfrak{m}$, $\mathfrak{t}_n$, and $\mathfrak{b}_n$ should be thought of as representing the points on their graphs, e.g., the actual points in $\mathbb{R}^2$ making up the nth tendril will be the union of the graphs of $\mathfrak{t}_n$ and $\mathfrak{b}_n$. These functions will change stage-by-stage, and for the value at stage s we will write $\mathfrak{m}[s]$, $\mathfrak{t}_n[s]$, and $\mathfrak{b}_n[s]$. When the stage is understood, this may be omitted.

The data should satisfy the following conditions:

(1) Each tendril is attached to the main line on the left side, and is above the main line. For the nth tendril, we have:

    (a) $\ell_n < r_n < 0$: the left attachment point is to the left of the right attachment point, and both are to the left of the midpoint.

    (b) $\mathfrak{m}(\ell_n) = \mathfrak{t}_n(\ell_n)$: the top arc of the tendril attaches to the main line at the left attachment point.

16

(c) $\mathfrak{m}(r_n) = \mathfrak{b}_n(r_n)$: the bottom arc of the tendril attaches to the main line at the right attachment point.

(d) $\mathfrak{t}_n(1) = \mathfrak{b}_n(1)$: the top and bottom arcs of the tendril meet at $x = 1$.

(e) For $x \in (\ell_n, r_n]$ we have $\mathfrak{m}(x) < \mathfrak{t}_n(x)$, and for $x \in (r_n, 1)$ we have $\mathfrak{m}(x) < \mathfrak{b}_n(x) < \mathfrak{t}_n(x)$.

(2) The tendrils are attached to the main line in order from left to right, and are non-intersecting:

(a) $\ell_1 < r_1 < \ell_2 < r_2 < \cdots$.

(b) for $m < n$ and values where they are both defined, i.e., $x \in (\ell_n, 1)$, we have $\mathfrak{t}_n(x) < \mathfrak{b}_m(x)$.

Given the above data, the *target space* is the closed subspace which consists of the main line with the portion between ℓ_n and r_n replaced by the nth tendril. That is, the following points:

(1) the points $(x, \mathfrak{m}(x))$ for $x \notin (\ell_n, r_n)$ for any n.

(2) for each n,

(a) the points $(x, \mathfrak{t}_n(x))$ for $x \in [\ell_n, 1]$;

(b) the points $(x, \mathfrak{b}_n(x))$ for $x \in [r_n, 1]$.

Recall that the target space is *not* homeomorphic to $[0, 1]$, as it is the target in Π_3^0 outcome.

5.2.2 The construction at each stage

Recall that we are building a countable metric space $M = M_W$ as a subspace of $\mathbb{R}^2$ with the induced metric. We begin at stage 0 with M being almost empty, and at each stage we may put some points into M. Then the topological space $X = X_W$ will be the completion of M.

Let $\{2, p_{n,c}, q_{n,c} : n, c \in \mathbb{N}\}$ be a computable listing of all the primes with no repetitions.

Construction at stage 0: We begin the construction at stage 0 with the main line $\mathfrak{m}(x) = 0$ and the tendrils given by $\ell_n = -\frac{1}{2} - \frac{1}{2n}$, $r_n = -\frac{1}{2} - \frac{1}{2n+1}$, and

$$\mathfrak{t}_n : [\ell_n, 1] \to [0, 1]$$

$$x \mapsto \mathfrak{m}(x) + \frac{1}{2^n}\left(\frac{x - \ell_n}{1 - \ell_n}\right)$$

and

$$\mathfrak{b}_n : [r_n, 1] \to [0, 1]$$

$$x \mapsto \mathfrak{m}(x) + \frac{1}{2^n}\left(\frac{x - r_n}{1 - r_n}\right)$$

Put $(-1, 0)$ and $(1, 0)$ into the metric space $M = M_W$.

Construction at stage $s + 1$: Suppose that at stage $s + 1$, a new element enters $W_n[s + 1]$ that was not in $W_n[s]$. Then:

17

- First, we collapse all tendrils numbered n onward. We must incorporate the special points on these tendrils that we have already included in M into the main line. We define the new main line $\mathfrak{m}[s+1]$ as follows. For $x \in (-1, \ell_n]$, we keep the main line the same: $\mathfrak{m}[s+1](x) := \mathfrak{m}[s](x)$. Now consider all of the special points (x, y) which we have already put in M which are either on the main line (with $x \in [\ell_n, 1]$) or on the nth tendril or a greater one. Namely, these are all of the points (x, y) with $x \in [\ell_n, 1]$ and $y < \mathfrak{b}_{n-1}[s](x)$. Order these points from left to right as $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$. Recall that there is no repetition among the x_i, as no two special points share the same x-coordinate. Now on $[\ell_n, 1]$ let $\mathfrak{l}$ be the piecewise linear function which linearly interpolates between the points

$$(\ell_n, \mathfrak{m}[s](\ell_n)), (x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k), (1, 0).$$

  Let $\mathfrak{m}[s+1](x) = \max(\mathfrak{m}[s](x), \mathfrak{l}(x))$. (As we will see later, for each i, $y_i \geq \mathfrak{m}[s](x_i)$, so that each of the points (x_i, y_i) is on the graph of $\mathfrak{m}[s+1]$. Also, $(-1, 0)$ and $(1, 0)$ were put into M at stage 0, and so we have $\mathfrak{m}[s+1](-1) = \mathfrak{m}[s+1](1) = 0$.)

- Second, we must create new tendrils. Choose new values $\ell_n, r_n, \ell_{n+1}, r_{n+1}, \ldots$ such that $r_{n-1} < -\frac{1}{2s+2} < \ell_n < r_n < \ell_{n+1} < r_{n+1} < \cdots < -\frac{1}{2s+4} < 0$ and such that there are no special points between ℓ_n and r_n, between ℓ_{n+1} and r_{n+1}, and so on. Let

$$g := \sup_{x \in [\ell_n, 1]} \mathfrak{b}_{n-1}(x) - \mathfrak{m}[s+1](x).$$

  Think of $g > 0$ as the gap between the main line $\mathfrak{m}$ and the remaining tendrils above $\mathfrak{m}$ (which have not been collapsed). Then define

$$\mathfrak{t}_n : [\ell_n, 1] \to [0, 1]$$

$$x \mapsto \mathfrak{m}[s+1](x) + \frac{g}{2^{n+s}}\left(\frac{x - \ell_n}{1 + \ell_n}\right)$$

  and

$$\mathfrak{b}_n : [r_n, 1] \to [0, 1]$$

$$x \mapsto \mathfrak{m}[s+1](x) + \frac{g}{2^{n+s}}\left(\frac{x - r_n}{1 + r_n}\right)$$

  It is somewhat tedious but not hard to see that the new main line and tendrils satisfy all of the conditions required.

If no element enters any W_n at stage $s+1$, then we leave everything the same (e.g., $\mathfrak{m}[s+1] = \mathfrak{m}[s]$).

  We may now have a new target space (though if no elements entered any W_n, then our target space remains the same). We must add new points to our space. For each n, let c_n be the number of times that the nth tendril has been collapsed. Add to M the following points from the target space:

(1) for each $x \in (-1, 1)$, $x \neq 0$, of the form $t/2^s$ for any $t \in \mathbb{N}$, and such that $x \notin (\ell_n, r_n)$, add $(x, \mathfrak{m}(x))$ to M.

18

(2) for each $x \in [t_n, 1)$, $x \neq 0$, of the form $t/p_{n,c_n}^s$ for any $t \in \mathbb{N}$, add $(x, \mathfrak{t}_n(x))$ to M.

(3) for each $x \in [b_n, 1)$, $x \neq 0$, of the form $t/q_{n,c_n}^s$ for any $t \in \mathbb{N}$, add $(x, \mathfrak{b}_n(x))$ to M.

Note that p_{n,c_n} and q_{n,c_n} are unique primes not only for the nth tendril, but unique for this instance of the nth tendril (i.e., different than the primes used for previous instances of the nth tendril that have now been collapsed). Also, for each point (x, y) that we have added to M, $y \geq \mathfrak{m}[s+1](x)$. Thus the main line is a lower bound on all of the points in M.

This completes the construction.

5.3 Verification

Fix W for which we have constructed $X = W$. We begin with a list of properties of the construction.

Remark 5.1. We have the following:

(1) If at stage s we collapse the nth, $n+1$st, $\ldots$, tendril, then $\mathfrak{t}_n[s-1] \geq \mathfrak{l}$ where $\mathfrak{l}$ is as defined at stage s.

Note that $\mathfrak{t}_n[s-1]$ is a line, and $\mathfrak{l}$ is a linear interpolation of points all of which are on or below $\mathfrak{t}_n[s-1]$.

(2) At any stage s, the first tendril is above the second tendril, the second is above the third, and so on. All of the tendrils are above the main line.

Both points are easy to see by following the construction. The point of the gap g is to ensure that this is the case; also, note that the line $\mathfrak{l}$ is under any of the tendrils that are not collapsed.

(3) If a special point p is put in M at stage s, then p is above the main line $\mathfrak{m}[s]$.

This is because all of the tendrils are above the main line, and the points that are added to M are either on a collapsed tendril or on the main line.

(4) For each fixed x, the sequence $\mathfrak{m}[s](x)$ is increasing.

We define $\mathfrak{m}[s+1](x) = \max(\mathfrak{m}[s](x), \mathfrak{l}(x))$ or $\mathfrak{m}[s+1](x) = \mathfrak{m}[s](x)$ depending on what happens at stage $s+1$.

(5) If, at a stage s, a special point $p = (x, y)$ is in M and is on the main line, then it is on the main line at any greater stage. Moreover, from stage s on, x is never between any pair of left and right endpoints ℓ_i and r_i.

At stage s, suppose that we collapse the nth, $n+1$st, $\ldots$, tendrils. If $p = (x, y)$ with $x \in [-1, \ell_n]$ then $\mathfrak{m}[s+1](x) = \mathfrak{m}[s](x) = y$ and so p is still on the main line. Otherwise, if $x \in [\ell_n, 1]$, then p is on the line $\mathfrak{l}$ defined at stage s, and $\mathfrak{m}[s+1](x) = \max(\mathfrak{m}[s](x), \mathfrak{l}(x))$ but $\mathfrak{m}[s](x) = \mathfrak{l}(x) = y$.

For the moreover clause, note that if p is added directly onto the main line, then it is not between any pair of left and right endpoints. If on the other hand p is put into M

as part of a tendril and becomes part of the main line due to collapse, then any pair of left and right endpoints containing it are redefined. Finally, in either case, when any pair of left and right endpoints are redefined at any later stage, they do not contain between them any special points of M.

(6) If the nth tendril was first defined at stage t, and was never collapsed between stage t and stage $s > t$, then $\mathfrak{t}_n(x) - \mathfrak{m}[s](x) \le 2^{-t}$.

At stage t, we would have defined $\mathfrak{t}_n(x) = \mathfrak{t}_n[t](x)$ such that $\mathfrak{t}_n(x) - \mathfrak{m}[t](x) \le 2^{-t}$. But $\mathfrak{m}[t](x) \le \mathfrak{m}[s](x) \le \mathfrak{t}_n(x)$.

(7) If, at stage s, a new element enters W_n for the kth time, then $\|\mathfrak{m}[s] - \mathfrak{m}[s+1]\| \le 2^{-n-k}$.

(8) If ℓ_n or r_n was defined at stage s, then $\ell_n < r_n < -\frac{1}{2s+2}$.

This is because at stage 0 we initially define $\ell_n < r_n < -\frac{1}{2}$ and if, at stage $s + 1$ we redefine them, it is with $-\frac{1}{2s+2} < \ell_n < r_n < \ell_{n+1} < r_{n+1} < \cdots < -\frac{1}{2s+4} < 0$.

First, we will argue that the main line comes to a limit $\mathfrak{m}_\infty$.

Lemma 5.2. *The limit $\mathfrak{m}_\infty(x) = \lim_{s \to \infty} \mathfrak{m}[s](x)$ exists, and $\mathfrak{m}_\infty : [-1, 1] \to [0, 1]$ is continuous. $\mathfrak{m}_\infty(-1) = \mathfrak{m}_\infty(1) = 0$.*

Proof. For a fixed x, the sequence $(\mathfrak{m}[s](x))_{s \in \mathbb{N}}$ is increasing and bounded above (by 1), and hence converges. Thus $\mathfrak{m}[s]$ converges pointwise to a function $\mathfrak{m}_\infty : [-1, 1] \to [0, 1]$. Moreover, by Remark 5.1 (7), for each k there can only be finitely many stages s with $\|\mathfrak{m}[s] - \mathfrak{m}[s+1]\| > 2^{-k}$. Thus the sequence of functions $(\mathfrak{m}[s])$ is a Cauchy sequence under the uniform metric, and so converges uniformly to $\mathfrak{m}_\infty$ which is therefore a continuous function. The final claim of the lemma follows because for each s, $\mathfrak{m}[s](-1) = \mathfrak{m}[s](1) = 0$. $\qquad\square$

5.3.1 The case $W \in A$

Suppose that $W \in A$. We must argue that X_W is homeomorphic to $[0, 1]$. Let n be least such that W_n is infinite, and let s^* be a stage after which no element ever enters $W_1, \ldots, W_{n-1}$.

For $i = 1, \ldots, n - 1$, let $\ell_i, r_i, \mathfrak{b}_i, \mathfrak{t}_i$ be the data of the ith tendril at stage s^*. The tendril is never collapsed at any later stage, so this is the final data associated with the tendril. Let Y consist of:

(1) the points $(x, \mathfrak{m}_\infty(x))$ for $x \in [-1, \ell_1]$;

(2) for each $1 \le i < n - 1$,

    (a) the points $(x, \mathfrak{m}_\infty(x))$ for $x \in [r_i, \ell_{i+1}]$;

(3) the points $(x, \mathfrak{m}_\infty(x))$ for $x \in [r_{n-1}, 1]$;

(4) for each $i < n$,

    (a) the points $(x, \mathfrak{t}_i(x))$ for $x \in [\ell_i, 1]$;

    (b) the points $(x, \mathfrak{b}_i(x))$ for $x \in [r_i, 1]$.

Each of these is the graph of a continuous function over a closed interval and hence home-omorphic to $[0, 1]$. Thus Y is the union of finitely many closed arcs, which are connected end-to-end and otherwise disjoint, and so Y is homeomorphic to $[0, 1]$. Now we must argue that $X = X_W$ is equal to Y, that is, that $M = M_W$ is a subset of Y and that $\overline{M} = Y$.

First we argue that $M \subseteq Y$. Suppose that a point (x, y) is put into M at stage s. There are three possible ways that it might be put into M. If it is put into the main line, then by Remark 5.1 (5) (x, y) will be part of the main line at every later stage (and will never be between any pair of left and right endpoints). If (x, y) is added to M as part of the ith tendril, then either that tendril is never later collapsed, in which case it remains in that tendril, or the tendril is at some later point collapsed to the main line, and (x, y) remains in the main line at all stages after that.

Now we will argue that the closure $X = \overline{M}$ covers all of Y. Given $(x, y) \in Y$ and $\epsilon > 0$, we break into cases. Suppose that (x, y) is a point of the form $(x, \mathfrak{m}_\infty(x))$ on the main line, with $x \notin (\ell_i, r_i)$ for any $i \le n$. Choose $\delta > 0$ and a stage s such that

(1) if $|x - x'| < \delta$ then $|\mathfrak{m}_\infty(x) - \mathfrak{m}_\infty(x')| < \epsilon/3$;

(2) at stage s, either $(x - \delta, x]$ or $[x, x + \delta)$ is disjoint from any interval (ℓ_i, r_i);

(3) $2^{-s} < \delta$;

(4) $2^{-s} < \epsilon/3$;

(5) $\|\mathfrak{m}_\infty - \mathfrak{m}[s]\| < \epsilon/3$.

It takes some argument to see that we can do this. We can choose δ satisfying (1) since $\mathfrak{m}_\infty$ is uniformly continuous (as it is continuous on a compact set). Moreover, (1) is maintained if we shrink δ. We obtain (2) as follows. If $x \ge 0$, then (2) is clear for $[x, x + \delta)$. Otherwise, if $x < 0$, we argue as follows. By shrinking δ, we may assume that for any $s \ge s^*$ (2) holds restricted solely to $i \le n$. By taking s sufficiently large that $x + \delta < -\frac{1}{2s}$, and moreover taking s to be a sufficiently large stage such that an element enters W_n at stage s, by Remark 5.1 (8) at stage s we have that $x + \delta < -\frac{1}{2s} < \ell_n < r_n < \cdots$ and moreover this remains true at any stage $s' \ge s$. Increasing s further, we can obtain (3), (4), and (5).

Then at each stage s, there is a point of the form $x' = t/2^s$ within 2^{-s} of x and not in any interval (ℓ_i, r_i). Then $(x', \mathfrak{m}[s](x'))$ is added to M at stage s. Since $|x - x'| < 2^{-s} < \delta$, $|\mathfrak{m}(x) - \mathfrak{m}(x')| < \epsilon/3$. Also, $|\mathfrak{m}(x') - \mathfrak{m}[s](x')| < \epsilon/3$. Thus the distance between $(x, \mathfrak{m}(x))$ and $(x', \mathfrak{m}[s](x'))$ is at most

$$|x - x'| + |\mathfrak{m}(x) - \mathfrak{m}[s](x')| \le |x - x'| + |\mathfrak{m}(x) - \mathfrak{m}(x')| + |\mathfrak{m}(x') - \mathfrak{m}[s](x')| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

If (x, y) is a point of the form $(x, \mathfrak{t}_i(x))$ or $(x, \mathfrak{b}_i(x))$, $i < n$, a simpler version of the above argument works, where the stage s is taken to be large enough that the ith tendril has stabilized. (Also, 2^{-s} is replaced by p_{i,c_i}^{-s} or p_{i,c_i}^{-s}.)

Remark 5.3. The fact that $\mathfrak{m}(-1) = \mathfrak{m}(1) = 0$ means that the homeomorphism $[0, 1] \to X$ maps 0 to $(-1, 0)$ and 1 to $(1, 0)$.

5.3.2 The case $W \notin A$

Suppose that $W \notin A$, so that for each n, W_n is finite. We will argue that X_W is not locally connected, whence it is not homeomorphic to $[0, 1]$.

For each n, there is a stage s_n after which no element enters $W_1, \ldots, W_n$. Thus, after stage s_n, the nth tendril remains the same. It is easy to see that the nth tendril—that is, the points $(x, \mathfrak{t}_n(x))$ for $x \in [\ell_n, 1]$ and $(x, \mathfrak{b}_n(x))$ for $x \in [r_n, 1]$—are in $X = X_W$. There are infinitely many tendrils and we argue that X is not locally connected.

Indeed, let B be any open set containing $(1, 0)$ and of diameter at most, say, $1/2$. Then for sufficiently large n, say $n \geq N$, the tip of the nth tendril, $(1, \mathfrak{t}_n(1)) = (1, \mathfrak{b}_n(1))$ is contained within B. However the left and right endpoints of the tendril lie outside of B since the diameter of B is at most $1/2$. The tip of each tendril $(n \geq N)$ is in a different connected component of B. Thus B is not connected.

5.4 The construction for Π^0_4-completeness

Let B be the $\mathbf{\Pi}^0_4$-Wadge-complete set

$$B = \{U \in \omega \times \omega \times \omega \mid \forall m \; U_m \in A\}.$$

Given U, for each m, let X_{U_m} be the metric space produced by the construction just given when applied to U_m; that is, X_{U_m} is not locally connected if $U_m \notin A$, and X_{U_m} is homeomorphic to $[0, 1]$ if $U_m \in A$. Moreover, in the latter case, the homeomorphism maps $0 \mapsto (-1, 0)$ and $1 \mapsto (1, 0)$. We may assume, by an easy rescaling, that $X_{U_m} \subseteq [0, 1] \times [0, 1]$ with $0 \mapsto (0, 0)$ and $1 \mapsto (0, 1)$.

Let Y_U be the space which contains, for each m, a copy of X_{U_m} which is scaled down by a factor of $1/2^m$ and shifted so that the point corresponding to $(1, 0)$ in X_{U_m} matches up with the point corresponding to $(0, 0)$ in $X_{U_{m+1}}$. Moreover, we "twist" each subsequent copy in a new direction so that there is no other intersection. Without the twists, this looks like:
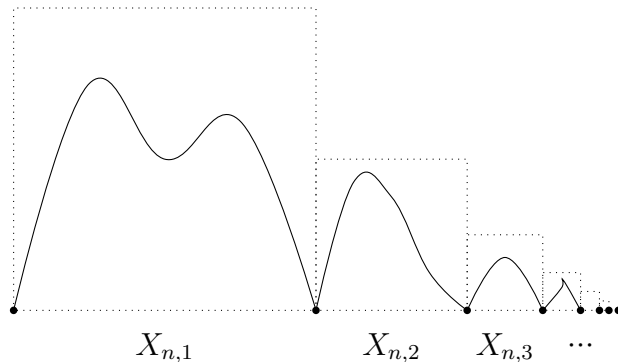
We work in c_0, the space of sequences in $\mathbb{R}^\mathbb{N}$ which converge to 0. Let $\mathbf{u}$ be $(1, 0, 0, 0, \ldots)$ and let $\mathbf{w}_1, \mathbf{w}_2, \ldots$ be a list of the other standard basis elements of c_0. These $\mathbf{w}_i$ will be the direction for the twists. For each m, Y_U will contain the following points:

$$\left\{ \left(\frac{1}{2} + \cdots + \frac{1}{2^{m-1}} + \frac{1}{2^m} x \right) \mathbf{u} + \frac{1}{2^m} y \mathbf{w}_m : (x, y) \in X_{U_m} \right\}$$

Y_U will also contain $(1,0)$ as the limit point of the above points. It is not hard to see that Y_U is homeomorphic to $[0,1]$ if and only if each X_{U_m} is homeomorphic to $[0,1]$. Thus Y_U is homeomorphic to $[0,1]$ if and only if $U \in B$, as desired.

Remark 5.4. In the next section, we will use the fact that Y_U always includes $(0,0,0,\dots)$ and $(1,0,0,\dots)$ and, in the case that it is homeomorphic to $[0,1]$, then these are the two endpoints.

6 The circle S^1

We now show that $\mathrm{HCopies}(S^1)$ is Π_4^0. The strategy is similar to that for $[0,1]$, and we reuse conditions such as CPCT and CONN from Section 3. Our new condition describes the circle by saying that any four points can be arranged in order around the circle, where we use the same ideas of betweenness as before to identify the order in which they are arranged.

Definition 6.1. $X = \overline{(M,d)}$ satisfies CIRC if for any special points x_0, x_1, x_2, x_3 (where we consider $0,1,2,3 \in \mathbb{Z}/4\mathbb{Z}$) there is some reordering of them such that

(1) for all $i \in \mathbb{Z}/4\mathbb{Z}$ and $\delta > 0$ there is $\rho < \delta$ such that for all ϵ there is an ϵ-path $u_1 = x_i, \dots, u_n = x_{i+1}$ with no u_j in $\overline{B}_\rho(x_{i+2}) \cup \overline{B}_\rho(x_{i+3})$.[5]

(2) for all $i \in \mathbb{Z}/4\mathbb{Z}$ and $\delta > 0$ there is ϵ such that for any ϵ-path $u_1 = x_i, \dots, u_n = x_{i+2}$, for some j either $u_j \in B_\delta(x_{i+1})$ or $u_j \in B_\delta(x_{i-1})$.

Note that this is Π_4^0.

Lemma 6.2. *If $X = \overline{(M,d)}$ is homeomorphic to S^1, then it satisfies CIRC (as well as NDEGEN, CPCT, CONN, and LC).*

Proof. All of these properties other than CIRC follow from Section 3. So we now verify CIRC. Given any four points on S^1, list them in order around the circle as x_0, x_1, x_2, x_3.

For (1), given i and $\delta > 0$, choose $\rho < \delta$ as follows. Let C be the segment of S^1 between x_i and x_{i+1} not containing x_{i+2} and x_{i+3}. Choose $\rho < \delta$ such that 2ρ is smaller than the distance between this segment and either of x_{i+2} or x_{i+3}. Then for any ϵ, we can choose an ϵ-path approximating the arc C between x_i and x_{i+1} (in the sense that each element of the ϵ-path is within ρ of C).

For (2), given i and $\delta > 0$, if either x_i or x_{i+1} are in $B_\delta(x_{i-1})$ or $B_\delta(x_{i+1})$, then (2) holds trivially. Otherwise, $X - B_\delta(x_{i+1}) - B_\delta(x_{i-1})$ is closed and disconnected, and x_i and x_{i+2} are in different connected components. Choosing ϵ to be smaller than the distance between clopen (in $X - B_\delta(x_{i+1}) - B_\delta(x_{i-1})$) separating sets containing each of them, there is no ϵ-path from x_i to x_{i+2} in $X - B_\delta(x_{i+1}) - B_\delta(x_{i-1})$. $\square$

Lemma 6.3. *If $X = \overline{(M,d)}$ satisfies CPCT, CONN, LC, and CIRC then for any four special points, there is some ordering x_0, x_1, x_2, x_3 and arcs $x_0 \to x_1 \to x_2 \to x_3 \to x_0$ homeomorphic to S^1. Moreover, any path from x_i to x_{i+2} must pass through x_{i+1} or x_{i-1}.*

---

[5]Recall that we use $\overline{B}_\rho(x)$ for the closed ball rather than the closure of the open ball.

Proof. Since $X = \overline{(M,d)}$ satisfies CPCT, CONN, and LC it is compact, connected, locally connected, path connected, and locally path connected. Given any four special points, order them as x_0, x_1, x_2, x_3 to satisfy CIRC. By (1), for each i, x_i and x_{i+1} are in the same connected component, and so there is a path C_i from x_i to x_{i+1}.

Next we must argue that any two of these arcs do not intersect except possibly at their starting and ending points. If not, and, say, C_i and C_{i+1} intersect, let y be the point of intersection. (There are other combinatorial possibilities for the intersection, but a similar argument works in each case.) If y was a special point, then $y, x_{i+1}, x_{i-1}, x_{i+2}$ will not satisfy (2) of CIRC, because there are paths between any two of y, x_{i+1}, x_{i-1} avoiding the other and x_{i+2}. If y is not a special point, then because X is locally path connected we can choose some special point y' sufficient close to y such that there is a path from y to y' disjoint from any of the x's and disjoint from C_{i-1} and C_{i+2}. Then a similar argument applies to show that $y', x_{i+1}, x_{i-1}, x_{i+2}$ will not satisfy (2) of CIRC.

The same argument works for the moreover clause. $\qquad\square$

Lemma 6.4. *If $X = \overline{(M,d)}$ satisfies NDEGEN, CPCT, CONN, LC, and CIRC, then X is homeomorphic to S^1.*

Proof. Since $X = \overline{(M,d)}$ satisfies CPCT, CONN, and LC it is non-degenerate, compact, connected, locally connected, path connected, and locally path connected.

Let x_0, x_1, x_2, x_3 be four special points, as in Lemma 6.3, and for each i, let C_i be the arc from x_i to x_{i+1} with the four arcs forming a homeomorphic copy C of S^1. We argue that any other point a must lie on C. Suppose not, that a does not lie on C.

Now there is a special point y near a, and an arc from y to a disjoint from C. There must also be an arc from y to C. There are several possibilities, in all of which we get a contradiction:

(1) The arc from y to C first meets C at a point other than the x_i. Say it meets C_i. Then y, x_i, x_{i+1}, x_{i+2} contradict Lemma 6.3. (Indeed, there is an arc between any two of y, x_i, x_{i+1} avoiding the third and x_{i+2}).

(2) Otherwise, suppose that the arc from y to C first meets C at x_i, and that any arc from y to C must pass through x_i. Then y, x_i, x_{i-1}, x_{i+1} contradict Lemma 6.3.

(3) Otherwise, suppose that in addition to the arc from y to x_i, there is also an arc from y that first meets C at x_{i+2}. Then $y, x_{i-1}, x_{i+1}, x_{i+2}$ contradicts Lemma 6.3 because there is a path between any two of y, x_{i-1}, x_{i+1} avoiding the third and x_{i+2}.

(4) Finally, suppose that in addition to the arc from y to x_i, there is also an arc from y that first meets C at x_{i+1} (or the symmetric case of x_{i-1}). Choose a special point z such that there is an arc from z to the middle of C_i that is disjoint from any of the other arcs. Then y, z, x_i, x_{i+2} contradicts Lemma 6.3 because there is a path between any two of y, z, x_i avoiding the third and x_{i+2}.

This completes the proof that any other point a must lie on C; that is, C is in fact all of X and so X is homeomorphic to S^1. $\qquad\square$

Theorem 6.5. *The set*
$$\mathrm{HCopies}(S^1) = \{Y : Y \cong_{\mathrm{homeo}} S^1\}$$
is $\mathbf{\Pi}^0_4$-*Wadge-complete.*

Proof. The other lemmas of this section show that this set is Π^0_4. To show $\mathbf{\Pi}^0_4$-completeness, we use the same Π^0_4 set B from Section 5.4. Let Y_U be the topological space constructed there, with $U \in B$ if and only if Y_U is homeomorphic to $[0,1]$. By Remark 5.4, $Y_U \subseteq \mathbb{R}^\mathbb{N}$ always contains $(0,0,0,\dots)$ and $(1,0,0,\dots)$ and, in the case that it is homeomorphic to $[0,1]$, these are the endpoints. We can easily construct from Y_U a space Z_U such that Z_U is homeomorphic to S^1 if and only if Y_U is homeomorphic to $[0,1]$: In $\mathbb{R} \times \mathbb{R}^\mathbb{N}$, let

$$Z_U = \{(t,0,0,\dots) : t \in [0,1]\} \cup \{(1,t,0,\dots) : t \in [0,1]\} \cup \{(t,1,0,\dots) : t \in [0,1]\} \cup \{(0,\mathbf{u}) : \mathbf{u} \in Y_U\}.$$

Essentially, to construct Z_U, we have taken a segment of S^1 and replaced it by Y_U. $\qquad \square$

7   The real line $\mathbb{R}$

In this section we will prove that $\mathbb{R}$ has topological Scott complexity $\mathbf{\Pi}^1_1$.

Theorem 7.1. $\mathrm{HCopies}(\mathbb{R})$ *is* $\mathbf{\Pi}^1_1$-*Wadge-complete.*

The proof is in two parts; first we give a Π^1_1 characterisation of $\mathbb{R}$, and second we prove a $\mathbf{\Pi}^1_1$-hardness result.

7.1   A Π^1_1-characterization of $\mathbb{R}$

Theorem 7.2. $\mathrm{HCopies}(\mathbb{R})$ *is* Π^0_4 *within the locally compact spaces. In particular,* $\mathrm{HCopies}(\mathbb{R})$ *is* Π^1_1.

Proof. We begin by working within the locally compact spaces. Given a locally compact but non-compact metric space (X,d), we can form the one-point compactification by adding a new point ∞. We can form a metric $\hat{d}$ on the compactification as follows [Man89]. Fix a point $p \in X$, and let $h(x) = \frac{1}{1+d(p,x)}$. Let

$$\hat{d}(x,y) = \min(d(x,y), h(x) + h(y)), \text{ and } \hat{d}(\infty, x) = h(x).$$

This construction lets us form, from a presentation of a locally non-compact space, a presentation of the one-point compactification. Moreover, this is computable.

Then a locally compact space is homeomorphic to $\mathbb{R}$ if and only if it is non-compact and its one-point compactification is homeomorphic to S^1. (This is because S^1 is homogeneous, and S^1 minus any point is homeomorphic to $\mathbb{R}$.) The former is a Σ^0_3 condition and the latter is Π^0_4 as proved earlier in this paper.

The set of locally compact spaces is Π^1_1 [MN13], and thus $\mathrm{HCopies}(\mathbb{R})$ is Π^1_1. $\qquad \square$

Using the fact that $\mathrm{HCopies}([0,1])$ is $\mathbf{\Pi}^0_4$-Wadge-complete, it is easier to prove that $\mathrm{HCopies}(\mathbb{R})$ is $\mathbf{\Pi}^0_4$-Wadge-complete within the set of locally compact spaces. The proof is essentially the same idea as Theorem 6.5 where we take a standard copy of $\mathbb{R}$ and replace a subinterval by the construction of Section 5.4

Proposition 7.3. HCopies($\mathbb{R}$) *is $\mathbf{\Pi}^0_4$-Wadge-complete within the set of locally compact spaces.*

7.2 $\mathbf{\Pi}^1_1$-completeness

Now we must show that it is $\mathbf{\Pi}^1_1$-hard. Given a tree $T \subseteq \omega^{<\omega}$, we will produce a space $X = X_T$ such that T is well-founded if and only if X is homeomorphic to $\mathbb{R}$.

Let (σ_n) be a computable listing of $\omega^{<\omega}$ with the property that if $\sigma \prec \tau$ then σ appears before τ in this list.

We work within the ambient space c_0 of sequences in $\mathbb{R}^\mathbb{N}$ that converge to 0, with the norm $\|\cdot\|_\infty$. Let $\mathbf{u}_\sigma$ (for $\sigma \in \omega^{<\omega}$) and $\mathbf{v}_n$ (for $n \in \omega$) and $\mathbf{w}$ be distinct standard basis elements. For convenience, let

$$\chi_\sigma = \begin{cases} \frac{1}{|\sigma|} & \sigma \in T \\ 1 & \text{otherwise} \end{cases}$$

Given τ, let

$$\mathbf{x}_\tau = \sum_{\sigma \leq \tau} \chi_\sigma \mathbf{u}_\sigma.$$

Given $\pi \in \omega^\omega$, note that $\lim_{n \to \infty} \mathbf{x}_{\pi \restriction n}$ converges if and only if π is a path through T. If it does converge, then it converges to

$$\mathbf{y}_\pi = \sum_n \frac{1}{n} \mathbf{u}_{\pi \restriction n}.$$

Let

$$\psi_n(t) = \begin{cases} 0 & t \leq n \\ 2(t - n) & n \leq t \leq n + \frac{1}{2} \\ 2(n + 1 - t) & n + \frac{1}{2} \leq t \leq n + 1 \\ 0 & n \geq n + 1 \end{cases}$$

That is, $\psi_n(t)$ is 0 at $t = n$, increases linearly to 1 at $t = n + \frac{1}{2}$, and then decreases linearly back to 0 at $t = n + 1$.

Define a continuous function $p : \mathbb{R} \to c_0$ as follows:

- For $t \leq 0$, let $p(t) = e^{-t}\mathbf{w}$.

- For $0 \leq t \leq 1$, let
$$p(t) = t\mathbf{x}_{\sigma_1} + e^{-t}\mathbf{w}$$

- For $1 \leq n \leq t \leq n + 1$, let
$$p(t) = (n + 1 - t)\mathbf{x}_{\sigma_n} + (t - n)\mathbf{x}_{\sigma_{n+1}} + \psi_n(t)\mathbf{v}_n + e^{-t}\mathbf{w}.$$

Note that $p(n) = \mathbf{x}_{\sigma_n}$. Let $X = X(T)$ be the closure of the image of p. That is, let $M = p[\mathbb{Q}]$ be the countable metric space that is the image of the rationals, and let $X = \overline{M} = \overline{p[\mathbb{R}]}$.

It is easy to see that p is injective, continuous (even uniformly continuous), and computable from T. In fact, p is a homeomorphism onto its image (but its image is not necessarily closed in c_0).

Lemma 7.4. *p is a homeomorphism onto its image.*

Proof. We must show given a sequence $t_i \in \mathbb{R}$, and $s \in \mathbb{R}$, that $t_i \to s$ if and only if $p(t_i) \to p(s)$. One direction follows from the fact that p is continuous, and the other—that if $p(t_i) \to p(s)$ then $t_i \to s$—can be seen using the term $e^{-t}\mathbf{w}$ that always appears in $p(t)$. $\quad\square$

We now argue that p is a homeomorphism onto the *closure* of its image if and only if its image is closed if and only if T is well-founded.

Lemma 7.5. *Suppose that T is well-founded. Then the image of p is closed.*

Proof. Suppose that $p(t_i) \to \mathbf{z}$, and t_i does not converge. Write

$$\mathbf{z} = \sum_\sigma a_\sigma \mathbf{u}_\sigma + \sum_n b_n \mathbf{v}_n + c\mathbf{w}.$$

We must have $c = \lim_{t \to \infty} e^{-t_i}$ and so, since the t_i do not converge, $c = 0$ and $t_i \to \infty$. Since $t_i \to \infty$, we must have that $b_n = 0$ for all n.

Thus

$$\mathbf{z} = \sum_\sigma a_\sigma \mathbf{u}_\sigma.$$

By passing to a subsequence, we may assume that for each i there is $n_i \in \mathbb{N}$ such that $t_i \in [n_i - \frac{1}{2}, n_i + \frac{1}{2}]$, and moreover we may assume that the n_i form a strictly increasing sequence. Also, since $\mathbf{z}$ does not involve any of the $\mathbf{v}_n$, it must be that

$$\lim_{i \to \infty} |n_i - t_i| = 0.$$

So in fact we may assume that $t_i = n_i$, and $p(n_i) = \mathbf{x}_{\sigma_{n_i}}$.

Now it is not hard to see that the only way to have $\lim_{i \to \infty} p(n_i)$ exist is to have the σ_{n_i} lie on a path π through T, and $\lim_{i \to \infty} p(n_i) = \mathbf{y}_\pi$. Thus T has a path.

So we conclude that if T is well-founded, then $\lim_{i \to \infty} p(t_i)$ converges if and only if t_i converges; in particular, p is a homeomorphism of $\mathbb{R}$ onto its image and its image is closed in c_0. $\quad\square$

Thus if T is well-founded, $X = \overline{M} = \overline{p[\mathbb{R}]} = p[\mathbb{R}]$, and this is homeomorphic to $\mathbb{R}$ by Lemma 7.4.

Lemma 7.6. *Suppose that T has an infinite path π. Then there is an increasing integer sequence n_i such that $\lim_{i \to \infty} p(n_i)$ converges.*

Proof. Let π be a path through T, and let n_i be an increasing integer sequence such that $\sigma_{n_i} = \pi \!\restriction\! i$. Then $p(n_i) = \mathbf{x}_{\pi \restriction i} + e^{-n_i}\mathbf{w}$, and

$$\lim_{i \to \infty} p(n_i) = \mathbf{y}_\pi = \sum_n \frac{1}{n} \mathbf{u}_{\pi \restriction n}.$$

Thus p maps a non-convergent sequence to a convergent sequence, and so cannot be a homeomorphism onto its image. $\quad\square$

If T is not well-founded, then $X = \overline{M} = \overline{p[\mathbb{R}]}$ contains a homeomorphic image of $\mathbb{R}$ as a proper dense subset. But $\mathbb{R}$ does not contain a homeomorphic image of $\mathbb{R}$ as a proper dense subspace (indeed, $\mathbb{R}$ has no proper connected dense subsets). Thus X is not homeomorphic to $\mathbb{R}$.

References

[AGHTT21] Rachael Alvir, Noam Greenberg, Matthew Harrison-Trainor, and Dan Turetsky. Scott complexity of countable structures. *J. Symb. Log.*, 86(4):1706–1720, 2021.

[BEF⁺] Peter Burton, Chris Eagle, Alec Fox, Isaac Goldbring, Matthew Harrison-Trainor, Tim McNicholl, Alexander Melnikov, Russell Miller, and Teerawat Thewmorakot. Effective Gelfand duality. In preparation.

[BM20] Tyler Brown and Timothy H. McNicholl. Analytic computable structure theory and L^p-spaces part 2. *Arch. Math. Logic*, 59(3-4):427–443, 2020.

[BYDNT17] Itaï Ben Yaacov, Michal Doucha, André Nies, and Todor Tsankov. Metric Scott analysis. *Adv. Math.*, 318:46–87, 2017.

[CHK⁺12] J. Carson, V. Harizanov, J. Knight, K. Lange, C. McCoy, A. Morozov, S. Quinn, C. Safranski, and J. Wallbaum. Describing free groups. *Trans. Amer. Math. Soc.*, 364(11):5715–5728, 2012.

[CMS19] Joe Clanin, Timothy H. McNicholl, and Don M. Stull. Analytic computable structure theory and L^p spaces. *Fund. Math.*, 244(3):255–285, 2019.

[DSR20] Gabriel Debs and Jean Saint Raymond. The descriptive complexity of connectedness in Polish spaces. *Fund. Math.*, 249(3):261–286, 2020.

[EFP⁺13] George A. Elliott, Ilijas Farah, Vern I. Paulsen, Christian Rosendal, Andrew S. Toms, and Asger Törnquist. The isomorphism relation for separable C*-algebras. *Math. Res. Lett.*, 20(6):1071–1080, 2013.

[Eng89] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[HTH18] Matthew Harrison-Trainor and Meng-Che Ho. On optimal Scott sentences of finitely generated algebraic structures. *Proc. Amer. Math. Soc.*, 146(10):4473–4485, 2018.

[HTM24] Matthew Harrison-Trainor and Alexander Melnikov. An arithmetical characterization of closed surfaces. *Transactions of the AMS*, 2024. Early view.

[LE65] E. G. K. Lopez-Escobar. An interpolation theorem for denumerably long formulas. *Fund. Math.*, 57:253–272, 1965.

[Man89] Mark Mandelkern. Metrization of the one-point compactification. *Proceedings of the American Mathematical Society*, 107(4):1111–1115, 1989.

[MN13] Alexander G. Melnikov and André Nies. The classification problem for compact computable metric spaces. In *The nature of computation*, volume 7921 of *Lecture Notes in Comput. Sci.*, pages 320–328. Springer, Heidelberg, 2013.

[Mon] Antonio Montalbán. Computable structure theory: Beyond the arithmetic. draft, available at https://math.berkeley.edu/~antonio/CSTpart2_DRAFT.pdf.

[MW12] Charles McCoy and John Wallbaum. Describing free groups, Part II: Π_4^0 hardness and no Σ_2^0 basis. *Trans. Amer. Math. Soc.*, 364(11):5729–5734, 2012.

[Nad92] Sam B. Nadler, Jr. *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1992. An introduction.

[NS15] André Nies and Slawomir Solecki. Local compactness for computable Polish metric spaces is Π_1^1-complete. In *Evolving computability*, volume 9136 of *Lecture Notes in Comput. Sci.*, pages 286–290. Springer, Cham, 2015.

[Sco65] Dana Scott. Logic with denumerably long formulas and finite strings of quantifiers. In *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, pages 329–341. North-Holland, Amsterdam, 1965.

[Tur36] Alan M. Turing. On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.

[Tur37] Alan M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. A Correction. *Proceedings of the London Mathematical Society*, 43:544–546, 1937.

[Wad83] William Wilfred Wadge. *Reducibility and determinateness on the Baire space.* ProQuest LLC, Ann Arbor, MI, 1983. Thesis (Ph.D.)–University of California, Berkeley.

[Wil70] Stephen Willard. *General topology.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.,, 1970.

[Zie16] Joseph Zielinski. The complexity of the homeomorphism relation between compact metric spaces. *Advances in Mathematics*, 291:635–645, 2016.