Computable learning of natural hypothesis classes

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Abstract

This paper is about the recent notion of computably probably approximately correct learning, which lies between the statistical learning theory where there is no computational requirement on the learner and efficient PAC where the learner must be polynomially bounded. Examples have recently been given of hypothesis classes which are PAC learnable but not computably PAC learnable, but these hypothesis classes are unnatural or non-canonical in the sense that they depend on a numbering of proofs, formulas, or programs. We use the on-a-cone machinery from computability theory to prove that, under mild assumptions such as that the hypothesis class can be computably listable, any natural hypothesis class which is learnable must be computably learnable. Thus the counterexamples given previously are necessarily unnatural.

1 Introduction

In the setting of binary classification learning we consider Probably Approximately Correct (PAC)-learning. The learner must, on most training sets (*probably*), learn the binary classification with only small error (*approximately*). There have generally been two approaches. The first, due to Valiant [1984], is *efficient* PAC learning where there must be a polynomial-time learning algorithm whose error is also polynomially bounded. The second is the statistical Vapnik-Chervonenkis theory [Vapnik and Chervonenkis, 1971] which gives a classification, in terms of VC-dimension, of the hypothesis classes which may be PAC-learned. In this VC-theory, the learners are arbitrary functions and there is no requirement that the learners be implementable by an algorithm, let alone an efficient one.

A recent series of papers starting with Agarwal et al. [2020] and continuing with Sterkenburg [2022] and Delle Rose et al. [2023] has investigated an intermediate notion where the learner is required to be computable but without placing any resource bounds on the learner. Part of the motivation for this line of inquiry is the recent discovery that learnability in general [Ben-David et al., 2019] and PAC learning in particular [Caro, 2023] is sensitive to set-theoretic considerations such as the continuum hypothesis. To try to avoid this, Agarwal et al. [2020] introduced a new notion

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of *computably probably approximately correct (CPAC)* learning. Trivially CPAC learning implies PAC learning, but in Agarwal et al. [2020] it is shown that there are PAC-learnable hypothesis classes, i.e., with finite VC-dimension, which are not CPAC-learnable. Their counterexample is the hypothesis class

$$\mathcal{H} := \{h_{i,j} : \text{the } i\text{th proof is a proof of the } j\text{th formula of arithmetic}\}$$

where

$$h_{i,j}(k) = \begin{cases} 1 & k = 2i \\ 1 & k = 2j+1 \\ 0 & \text{otherwise} \end{cases}.$$

This counterexample has some unnatural features, not looking like many of the more geometric hypothesis classes one might want to learn in practice. One unnatural aspect is that there is no canonical meaning to "the *n*th formula of arithmetic" and which functions are in the hypothesis class depend on arbitrary choices like how we number the formulas of arithmetic. In contrast standard examples of learnable problems, like half planes, are more canonical; whether a hypothesis is the indicator function of a half plane does not depend on any arbitrary choices.

How often should we expect to run into PAC-learnable but not CPAC-learnable hypothesis classes in the standard course of learning theory, that is, without purposefully seeking them out? In this paper, we begin an investigation into this question: Must any separation of PAC and CPAC learning have some unnatural feature? The concept of naturality is inherently an informal one and so one of the chief difficulties is to formally capture some aspect of naturality. In this paper we appeal to a well-established "on-a-cone" formalism from computability theory which captures certain aspects, particularly *canonicity*, of certain classes of computable functions. Much of this paper consists of explaining the "on-a-cone" formalism and arguing that it appropriately captures certain aspects of naturality, with the proofs of our main results appearing in the appendices.

The full definitions are somewhat technical. We give a slightly informal treatment in this introduction, with the full details in Section 3 (including a review of the required ideas from in computability theory and descriptive set theory). Recall that an oracle is a "black box" provided to a computer which is capable of solving certain (often non-computable) problems in a single operation. Given a proof/statement about computers and programs, its relativization to an oracle x is the replacement of every instance of a computer or program in that proof/statement by a computer or program with oracle x. The "on-a-cone" formalism applies to concepts which relativize. Not all concepts relativize, and those that do not are outside of the domain of the "on-a-cone" formalism. Nevertheless, there are many examples of relativizing classes where the "on-a-cone" formalism is applicable. Formulas of arithmetic, as used in the Agarwal et al. [2020] example, relativize by adding a predicate; this is equivalent to relativizing the computational arithmetic hierarchy to an oracle. Thus the "on-a-cone" formalism includes such examples.

Our main result is that for hypothesis classes $\mathcal{H}$ consisting of computable listings of computable functions, which relativize in a well-behaved and degree-invariant way, PAC-learnability and CPAC-learnability coincide. What we mean by "relativize in a well-behaved and degree invariant way" will be defined shortly, but heuristically we expect that many or most natural hypothesis classes (and particularly geometric hypothesis classes) relativize in this well-behaved way, and so this gives evidence that though there are PAC-learnable but not CPAC-learnable hypothesis classes, one might hope not to encounter them without specifically searching them out.

1.1 Computability-theoretic notation

We fix an effective listing $(\varphi_e)_{e\in\mathbb{N}}$ of the partial computable functions. We will use W_e for the eth computably enumerable (c.e.) set, $W_e = \mathrm{dom}(\varphi_e)$. We use $2^{<\mathbb{N}}$ for the collection of finite binary strings, which is sometimes also denoted $\{0,1\}^*$, and $2^{\mathbb{N}}$ for Cantor space, the collection of infinite binary strings. We use lower case letters $b, x, y, z \in 2^{\mathbb{N}}$ for infinite binary strings. We identify an infinite binary string $x \in 2^{\mathbb{N}}$ with the corresponding subset $\{i : x(i) = 1\}$ of $\mathbb{N}$, and we can also think of these as languages or decision problems. These are our oracles. We say that a function, relation, or set is computable relative to x, or x-computable, if it can be computed using x as an oracle. Given an oracle x, by relativizing our listing of partial computable functions we have a listing $(\varphi_e^x)_{e\in\mathbb{N}}$ of the partial x-computable functions. Similarly W_e^x is the eth x-c.e. set.

Given $x, y \in 2^{\mathbb{N}}$ we write $x \leq_T y$ and say that x is Turing-reducible to y if we can compute x given an oracle y. This is a partial pre-ordering of the oracles. We write $x <_T y$ if $x \leq_T y$ and $y \not\leq_T x$. If $x \leq_T y$ and $y \leq_T x$ we say that x and y are Turing-equivalent and write $x \equiv_T y$. The Turing degrees are the equivalence classes of $2^{\mathbb{N}}$ modulo $\equiv_T$. Given $x, y \in 2^{\mathbb{N}}$ their join $x \oplus y$ is the set

$$x \oplus y = \{(0,n) : n \in x\} \cup \{(1,n) : n \in y\}.$$

This is a set of least Turing degree computing both x and y. Given an infinite sequence of sets $(x_i)_{i\in\mathbb{N}}$, their join is

$$\bigoplus_{i\in\mathbb{N}} \{(i,n) : n \in x_i\}.$$

Further details can be found in Soare [2016].

1.2 Relativizing learning theory

Let us begin by considering a hypothesis class, such as the PAC-learnable but not CPAC-learnable hypothesis class of Agarwal et al. [2020], which is *c.e.-represented*. This means that there is a computable listing of programs computing the functions in $\mathcal{H}$, i.e., $\mathcal{H} = \{\varphi_i : i \in W_e\}$. This is the setting for many of the results on CPAC learning. This definition of $\mathcal{H}$ relativizes to an oracle x as $\mathcal{H}^x = \{\varphi_i^x : i \in W_e^x\}$—essentially $\mathcal{H}^x$ is defined using the same programs as $\mathcal{H}$, but with a different oracle—and so we get a family of hypothesis classes. It is important here to remark that there are many possible ways to chose the listing $(\varphi_i)_{i\in\mathbb{N}}$, and we have non-canonically fixed one of them. We should differentiate between a function φ_i and its index i; $\mathcal{H}$ is the set of functions φ_i, and though it is represented using the indices i, these indices are not part of the hypothesis class.

If $\mathcal{H}$ is a *natural* hypothesis class, we might expect that there is some relationship between $\mathcal{H}$ and its relativizations $\mathcal{H}^x$. The reason for this is that standard computability-theoretic techniques and proofs relativize, and so if we prove something about $\mathcal{H}$, the same proof relativized to an oracle x proves the same thing about $\mathcal{H}^x$.[1] It is not true that *all* proofs or facts relativize to an oracle x, for

[1] One of the more well-known uses of oracles in computer science, which is spiritually similar to this paper, is the Baker-Gill-Solovay theorem [Baker et al., 1975] which says that there are oracles x with $\mathsf{P}^x = \mathsf{NP}^x$ and also oracles y with $\mathsf{P}^y \neq \mathsf{NP}^y$. Here, e.g., P^x is the class of decision problems solvable in polynomial time by a machine with oracle x. The Baker-Gill-Solovay theorem is a no-go theorem. It says that the P vs NP problem cannot be solved by proof techniques that relativize. This is because such a resolution, say a relativizable proof that $\mathsf{P} \neq \mathsf{NP}$, would relativize to the oracle x from the Baker-Gill-Solovay theorem to prove that $\mathsf{P}^x \neq \mathsf{NP}^x$, contradicting the fact that $\mathsf{P}^x = \mathsf{NP}^x$; similarly a relativizable proof that $\mathsf{P} = \mathsf{NP}$ would relativize to prove that $\mathsf{P}^y = \mathsf{NP}^y$ contradicting $\mathsf{P}^y \neq \mathsf{NP}^y$. However

example, it might be that $\varphi_1(5) = 2$ but, relativizing to the oracle x, we instead get $\varphi_1^x(5) = 3$, and furthermore relativizing to some other oracle y we might get that $\varphi_1^y(5)$ is undefined. However we would not expect a fact like $\varphi_1(5) = 2$ to appear as part a proof of a reasonable fact about $\mathcal{H}$. The fact that *some* function in $\mathcal{H}$ maps $5 \mapsto 2$ might be important, but the fact that this function has index 1 should not be relevant for any reasonable fact about $\mathcal{H}$. In general, proof techniques that are agnostic to the choice of the listing $(\varphi_i)_{i\in\mathbb{N}}$ of partial computable functions tend to relativize. This leads to our first assumption about the naturality of $\mathcal{H}$: If P is a reasonably natural property, then $\mathcal{H}$ has property P if and only if $\mathcal{H}^x$ has property P^x for all x. This property is not formally definable, as we cannot say what is means for a property P to be reasonably natural; nevertheless we will call such a hypothesis class $\mathcal{H}$ *naturally relativizing*. The "on-a-cone" technique is intended to be a formal way to capture this informal notion.

We will have a second assumption on the hypothesis class $\mathcal{H} = \{\varphi_i : i \in W_e\}$. While above we fixed an ordering $(\varphi_i)_{i\in\mathbb{N}}$ of the partial computable functions, it would be unnatural—especially, e.g., if $\mathcal{H}$ had some geometric interpretation—for the definition of $\mathcal{H}$ to depend on how this listing was chosen. We can capture this by considering relativizations of $\mathcal{H}$. Consider two oracles $x \equiv_T y$ which are of the same Turing degree. Then the partial x-computable functions and the partial y-computable functions coincide, so the listings $(\varphi_i^x)_{i\in\mathbb{N}}$ of partial x-computable functions and $(\varphi_i^y)_{i\in\mathbb{N}}$ of partial y-computable functions are listings of the same partial functions but in a different order. When we relativize $\mathcal{H}$ to x and y we obtain $\mathcal{H}^x = \{\varphi_i^x : i \in W_e^x\}$ and $\mathcal{H}^y = \{\varphi_i^y : i \in W_e^y\}$. For natural classes we might expect that these would consist of the same functions $\mathcal{H}^x = \mathcal{H}^y$. For example, if $\mathcal{H}$ was the half-planes with computable slopes, then $\mathcal{H}^x = \mathcal{H}^y$ would be the half-planes with x-computable and equivalently y-computable slopes. This property can be captured formally as *degree-invariance* of the hypothesis class: If $x \equiv_T y$ then $\mathcal{H}^x = \mathcal{H}^y$. In general if $\mathcal{H}^x$ is the set of x-computable functions satisfying some fixed condition independent of x then $\mathcal{H}^x$ will be degree-invariant. The example of Agarwal et al. [2020] is not degree-invariant, and this represents its dependence on the arbitrary choice of how to number the formulas of arithmetic. As we will describe in the following section, forms of degree-invariance have a long history of being used to capture the naturality of relativizing constructions in computability theory.

Now suppose that we have a hypothesis class $\mathcal{H} = \{\varphi_i : i \in W_e\}$ which is PAC-learnable and relativizes naturally and in a degree-invariant way. We prove that it is CPAC-learnable. As "naturally relativizing" cannot be captured in a formal way, our theorem is stated formally using "on-a-cone". We state here the informal version of our theorem with the formal version to follow once we have discussed the "on-a-cone" formalism and its history.

Theorem 1.1 (Informal). *Let $\mathcal{H} = \{\varphi_i : i \in W_e\}$ be a c.e.-represented hypothesis class which is naturally relativizing and degree-invariant. If $\mathcal{H}$ is PAC-learnable then it is (properly) CPAC-learnable.*

The formal statement of this theorem will appear in Section 1.4, together with the formal statements of two other theorems for hypothesis classes which are not c.e.-represented but rather represented in some other way.

almost all standard proof techniques relativize, and so P vs NP cannot be resolved by these standard techniques.

1.3 The history and motivation of “on-a-cone”

In this section we will discuss the history and motivation of the development of the “on-a-cone” formalism which will also elaborate on how the method can capture aspects of naturality. An expository article by Montalbán [2019] explains these ideas in more depth than we can here.

The origin of the “one-a-cone” method is in a famous still-open question of Sacks [1963]. After Turing had shown that the Halting problem

$$K = \{e : \text{the } e\text{th program halts}\}$$

is non-computable, many other problems were shown to be non-computable by encoding the Halting problem into them thereby showing that they are at least as hard as the Halting problem. Indeed whenever anyone has shown that a c.e. decision problem A of independent mathematical interest is undecidable they have done this by giving a reduction from the Halting problem so that $K \leq_T A$. Thus natural decision problems seem to either be computable or at least as hard as the Halting problem. On the other hand, there *are* problems of intermediate difficulty: Answering Post’s problem, Friedberg [1957] and Muchnik [1956] constructed a c.e. set A of intermediate Turing degree $0 <_T A <_T K$ where K is the Halting problem. The construction was a technical construction by hand using diagonalization and the resulting set of intermediate Turing degree (and any other such set later constructed) was neither canonical nor of independent mathematical interest. Sacks wanted to explain this. He noted that many natural c.e. sets W_e are *degree-invariant*: they give a procedure which, given an oracle x, relativizes to give a set W_e^x which is c.e. in x, such that if $x \equiv_T y$ are Turing-equivalent oracles, the relativized sets $W_e^x \equiv_T W_e^y$ are also Turing equivalent. For example, the Halting problem relativizes in this way, to

$$K^x = \{e : \text{the } e\text{th program with oracle } x \text{ halts}\}$$

and if $x \equiv_T y$ then $K^x \equiv_T K^y$. While not all natural definitions of c.e. sets relativize, those that do are (heuristically) degree-invariant. (One example of a non-relativizing natural definition of a c.e. set is Hilbert’s tenth problem of deciding whether a polynomial $p(\bar{x})$ with integer coefficients has an integer solution. One cannot relativize a polynomial to an oracle.) Restricting attention to those definitions that relativize, Sacks asked whether there was a degree-invariant c.e. operator W_e of intermediate degree $x <_T W_e^x <_T K^x$ for all x. This question is still open, but if one could show that no such operator exists then this would be evidence that there are no natural decision problems of intermediate Turing degree.

In an extension of Sacks’s Question, known as Martin’s conjecture, Martin introduced the “on-a-cone” formalism which we will make use of in this paper (though we will not need to introduce the full statement of Martin’s conjecture[2]). The idea is as follows. Given a statement P about computation, we can relativize P to an oracle x to give the statement P^x by, for example, replacing any instance of “computable” by “x-computable”. The informal claim is that if P is a reasonably natural statement which relativizes then any proof or disproof of P should relativize, and so P is true if and only if P^x is true. In contrast, unnatural statements might change their truth value as we

[2]Martin’s conjecture says, informally, that the natural functions from Turing degrees to Turing degrees are the iterates of the Turing jump. While the conjecture remains open and is one of the most important problems in computability theory, various special cases have been proved by Steel [1982] and Slaman and Steel [1988] and most recently by Lutz and Siskind [2025].

relativize to different oracles. An example of an unnatural statement might be "the 104th program with oracle x halts" which may be true for certain oracles x and false for others. This statement can be viewed as unnatural because there is no canonical meaning to "the 104th program". (Recall that statements which do not relativize are not subject to this paradigm. For example, the statement "Hilbert's tenth problem is not decidable" does not relativize to give "Hilbert's tenth problem is not x-decidable" for all oracles x because Hilbert's tenth problem does not relativize.) We will use the adjective *naturally relativizing* for concepts which are both natural and relativize appropriately.

While one can formally define what it means for a statement to relativize, what it means to be natural is not easily definable. While for unnatural statements P the truth value of P^x might change as we relativize to different oracles x, Martin [1968, 1975] showed that if P is "reasonably definable" (Borel) then as we relativize to more and more powerful oracles x, eventually the truth value of P^x stabilizes to a limit (in the sense that there is some oracle b such that for all stronger oracles $x \geq_T b$ the truth value of P^x is the same). This is the truth value of P *on a cone*. The idea is that if P is a natural relativizing statement, then the truth or falsity of P relativizes, and P is true if and only if P^x is true for all x if and only if P is true on a cone. Thus:

> *If we prove a theorem on a cone, then we should think that the theorem holds computably (with no oracle) when applied to naturally relativizing objects.*

We note that this is a heuristic statement rather than a formal one.

Somewhat recently, following work of Montalbán [2013], the "on-a-cone" approach has been applied in computable structure theory. See Montalbán [2015], Csima and Harrison-Trainor [2017], Harrison-Trainor [2018], Damaj and Harrison-Trainor [n.d.] for other uses in computable structure theory and particularly the books Montalbán [2021, n.d.]. In general the technique has captured a useful distinction. For example Bazhenov et al. [2025] give an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ which has "intermediate degree spectrum", but does not have "intermediate degree spectrum on a cone". The function constructed is not canonical, and, e.g., the value of $f(2)$ depends on the listing of computer programs one chooses. Following this, Damaj and Harrison-Trainor [n.d.] constructed a function $f\colon \mathbb{N} \to \mathbb{N}$ which has "intermediate degree spectrum on a cone". This function is explicitly given, e.g., $f(2) = 2$ and $f(3) = 4$.

1.4 Learning theory on a cone

Returning to learning theory, consider a hypothesis class $\mathcal{H}$ which has a definition that relativizes, giving a family of hypothesis classes $\mathcal{H}^x$ relative to each oracle x. This includes the c.e.-represented hypothesis classes discussed earlier as well as further ways of defining hypothesis classes. We suppose that $\mathcal{H}$ is degree-invariant in the sense that $x \equiv_T y$ then $\mathcal{H}^x = \mathcal{H}^y$. Finally, suppose that $\mathcal{H}^x$ is PAC-learnable for all oracles x, or for all oracles x on a cone. As we relativize to different oracles x, $\mathcal{H}^x$ may be CPAC-learnable for certain oracles x, and not CPAC-learnable for other oracles x. What is the limiting behaviour as x becomes more and more powerful? Is $\mathcal{H}^x$ is CPAC-learnable on a cone?

We prove that $\mathcal{H}^x$ is CPAC-learnable on a cone under three different assumptions. We have explained many of the terms but the reader should refer to Sections 2 and 3 for the formal definitions. We begin with the assumption that $\mathcal{H}^x$ is c.e.-represented for each x.

Theorem 1.1 (Formal)**.** *Let $\mathcal{H}^x$ be a degree-invariant family of hypothesis classes such that $\mathcal{H}^x$ is c.e.-represented and PAC-learnable for all x. Then $\mathcal{H}^x$ is properly SCPAC-learnable on a cone: There is an oracle b such that for all $x \geq_T b$, $\mathcal{H}^x$ is properly SCPAC-learnable.*

This is a formal theorem with all of the terms formally defined. The informal version of Theorem 1.1 stated earlier follows from this formal statement by an informal argument as follows. Suppose that the c.e.-represented hypothesis class $\mathcal{H} = \{\varphi_i : i \in W_e\}$ is naturally relativizing, degree-invariant, and PAC-learnable. Then as described in Section 1.2 we can relativize $\mathcal{H}$ to an oracle x to obtain a hypothesis class $\mathcal{H}^x = \{\varphi_i^x : i \in W_e^x\}$, and (a) this is degree-invariant and (b) any proof of a *reasonable* fact about $\mathcal{H}$ relativizes to prove the corresponding fact about $\mathcal{H}^x$. Since $\mathcal{H}$ is PAC-learnable, we expect that any proof of this relativizes and that $\mathcal{H}^x$ is PAC-learnable for all oracles x. If the original hypothesis $\mathcal{H}$ was not CPAC-learnable, we expect the proof of this to relativize as well so that for all oracles x the hypothesis class $\mathcal{H}^x$ would not be CPAC-learnable. But by Theorem 1.1 for all sufficiently powerful oracles x (on a cone) $\mathcal{H}^x$ is CPAC-learnable. This is a contradiction; we conclude that $\mathcal{H}$ is CPAC-learnable. Thus we obtain the informal version of this theorem stated previously: If $\mathcal{H}$ is a c.e.-represented hypothesis class which is naturally relativizing and degree-invariant, then if $\mathcal{H}$ is PAC-learnable then it is properly CPAC-learnable.

What if our hypothesis class $\mathcal{H}$ is not c.e.-represented? For our second theorem, we assume that $\mathcal{H}$ is topologically closed. This means that the limit of any sequence of functions in the hypothesis class is itself in the hypothesis class, a natural condition which would hold of many geometric examples. We require that the definition of the hypothesis class $\mathcal{H}$ relativizes in a reasonably definably (Borel) way. (The formal definitions will appear in Section 3.) In this case we show that $\mathcal{H}^x$ is CPAC-learnable on a cone.

Theorem 1.2. *Let $\mathcal{H}^x$ be a degree-invariant Borel family of hypothesis classes such that $\mathcal{H}^x$ is closed and PAC-learnable for all x. Then $\mathcal{H}^x$ is properly SCPAC-learnable on a cone: There is an oracle b such that for all $x \geq_T b$, $\mathcal{H}^x$ is properly SCPAC-learnable.*

Finally, we assume that the hypothesis class is not set-theoretically pathological. Formally, this theorem is stated under the assumption of the axiom of determinacy. This is a set-theoretic axiom which is incompatible with the axiom of choice. The axiom of determinacy is often though of as being an axiom of "definable mathematics". Under this assumption, we can prove that all degree-invariant relativizing families of hypothesis classes which are PAC-learnable are improperly CPAC-learnable on a cone. This differs from the previous two theorems in that we only get improper learnability.

Theorem 1.3 (ZF + Axiom of Determinacy)**.** *Let $\mathcal{H}^x$ be a degree-invariant family of hypothesis classes such that $\mathcal{H}^x$ is PAC-learnable for all x. Then $\mathcal{H}^x$ is improperly SCPAC-learnable on a cone: There is an oracle b such that for all $x \geq_T b$, $\mathcal{H}^x$ is improperly SCPAC-learnable.*

The use of determinacy in the proof is local, that is, we only use the fact that certain sets associated to the hypothesis class are determined. Undetermined sets are pathological, like unmeasurable sets, and so even under ZFC we would expect that for natural hypothesis classes all sets involved satisfy determinacy, and so the theorem would still go through.

Moreover, even though the axiom of determinacy is incompatible with the standard axioms chosen for mathematics, restricted amounts of determinacy are compatible with choice and vice versa. Borel determinacy was proved by Martin [1969/70], and analytic determinacy follows from large cardinal hypotheses[3] and so is generally thought to be consistent with ZFC. If the family of hypothesis classes are reasonably definable (Borel) then we require only the axiom of analytic determinacy, and thus the following theorem is (very likely) consistent with ZFC.

Corollary 1.4 (ZFC + Analytic Determinacy)**.** *Let $\mathcal{H}^x$ be a degree-invariant Borel family of hypothesis classes such that for all x on a cone, $\mathcal{H}^x$ is PAC-learnable. Then $\mathcal{H}^x$ is improperly SCPAC-learnable on a cone: There is an oracle b such that for all $x \geq_T b$, $\mathcal{H}^x$ is improperly SCPAC-learnable.*

The proof of Theorem 1.1 will appear in Section 4. The proofs of Theorems 1.2 and 1.3 will appear in Appendices A and B respectively. In Section 5 we also prove a negative result, giving a degree-invariant Borel family of hypothesis classes that are PAC-learnable but not properly CPAC-learnable.

2 Learning theory preliminaries and notation

2.1 Classical statistical learning theory

We begin with a brief overview of the PAC-learning framework, primarily to fix our notation. See, e.g., Shalev-Shwartz and Ben-David [2022] for a recent text covering this material.

Let $\mathcal{X} = \mathbb{N}$ stand for the *domain* or *feature space*. While we fixed $\mathcal{X} = \mathbb{N}$, one should think of $\mathbb{N}$ as representing any desired countable feature space using a numbering or Gödel encoding. Let $\mathcal{Y} = \{0, 1\}$ be the *label space*. A *hypothesis* is a function $h : \mathcal{X} \mapsto \mathcal{Y}$, and a *hypothesis class* is a collection of hypotheses $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$.

The training data is assumed to be generated from some distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, and the performance of a hypothesis h on this distribution is denoted by the *true error* $L_{\mathcal{D}}(h) = \Pr_{(x,y)\sim\mathcal{D}}[h(x) \neq y]$. The training data, called a *sample*, S, is a finite sequence of feature-label pairs $S \in \mathcal{S} = \bigcup_{n\in\mathbb{N}}(\mathcal{X} \times \mathcal{Y})^n$. The error of a hypothesis on the sample, also called the *empirical risk*, is $L_S(h) = \frac{1}{m} \sum_{i=1}^m \mathbb{1}[h(x_i) \neq y_i]$ where $\mathbb{1}[h(x_i) \neq y_i]$ is the indicator function for $h(x_i) \neq y_i$.

Definition 2.1. We say a hypothesis class $\mathcal{H}$ is *(properly) PAC-learnable* if there is a learner function $A : \mathcal{S} \mapsto \mathcal{H}$ and a sample size function $m : (0, 1)^2 \mapsto \mathbb{N}$ such that for every distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, for every error ϵ, confidence $\delta \in (0, 1)^2$, and for every $m \geq m(\epsilon, \delta)$, we have that

$$\Pr_{S\sim\mathcal{D}^m}\left[\, L_{\mathcal{D}}(A(S)) \leq \min_{h\in\mathcal{H}} L_{\mathcal{D}}(h) + \epsilon \,\right] \geq 1 - \delta.$$

By default, we require the learner to output hypotheses inside the hypothesis class, but we can relax this assumption to allow it to output hypotheses even outside of the hypothesis class, and we call this case *improper* PAC learning.

The fundamental theorem of statistical learning gives a powerful combinatorial characterization of the PAC-learnable classes in terms of VC dimension, the definition of which is as follows.

Definition 2.2. Given some collection of points $X = \{x_1, x_2, ..., x_k\} \subseteq \mathcal{X}$, we say $\mathcal{H}$ *shatters* X if for all $y_1, y_2, ..., y_k \in \mathcal{Y}$ there is a hypothesis $h \in \mathcal{H}$ such that $h(x_i) = y_i$ for all i.

[3] Analytic determinacy equivalent to the existence of $x^\sharp$ for every x [Harrington, 1978].

Definition 2.3. The *VC dimension* of $\mathcal{H}$, $VC(\mathcal{H})$, is the maximum k such that $\mathcal{H}$ can shatter a set $X \subseteq \mathcal{X}$ of size k. If $\mathcal{H}$ can shatter arbitrarily large sets, we say its VC dimension is infinite.

Theorem 2.4 (Fundamental theorem of statistical learning)**.** *A hypothesis class $\mathcal{H}$ is PAC-learnable if and only if the VC-dimension of $\mathcal{H}$ is finite. Moreover, if $\mathcal{H}$ is PAC-learnable then any empirical risk minimizer for $\mathcal{H}$ witnesses this.*

Here, an empirical risk minimizer is a learner of the following type.

Definition 2.5. A learner $A : \mathcal{S} \mapsto \mathcal{H}$ is called an *empirical risk minimizer (ERM)* for $\mathcal{H}$ if given any sample $S \in \mathcal{S}$ the learner outputs a hypothesis with the minimum possible empirical risk, i.e., $A(S) = \arg\min_{h \in \mathcal{H}} L_S(h)$.

2.2 Computable learning theory

We are now ready to give the basic definitions and theorems of computable learning theory from Agarwal et al. [2020]. All of these definitions and theorems relativize to any oracle, and given an oracle x we use a superscript ^{x} for the relativization, e.g., CPAC^x is the relativization of CPAC to oracle x. We will give the relativizations of the first few definitions in footnotes, but then after this we leave it to the reader.

Definition 2.6. A hypothesis class $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ is called *(properly) computably PAC-learnable (or CPAC-learnable)* if it is PAC-learnable and there is a computable learner $A : \mathcal{S} \mapsto \mathcal{H}$ as witness.[4]

Definition 2.7. A hypothesis class $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ is called *(properly) strongly computably PAC-learnable (or SCPAC-learnable)* if it is PAC-learnable and there is a computable learner $A : \mathcal{S} \mapsto \mathcal{H}$ and a computable sample size function $m \colon (0,1)^2 \to \mathbb{N}$ witnessing this.[5]

We also extend *improper* PAC learning to CPAC and SCPAC learning in the natural way.

To characterize computable learning in a style similar to the fundamental theorem of statistical learning, we need a few definitions.

Definition 2.8. Let $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ be a hypothesis class with $VC(\mathcal{H}) \leq d \in \mathbb{N}$. A *$d$-witness* for $\mathcal{H}$ is a function $w : \mathcal{X}^{d+1} \mapsto \mathcal{Y}^{d+1}$ such that for each $\bar{u}$, $w(\bar{u}) = \bar{\ell}$ for some $\bar{\ell}$ such that there is no $h \in \mathcal{H}$ with $h(x_i) = \ell_i$ for all i.

If $VC(\mathcal{H}) = d \in \mathbb{N}$, recall that means that no $d+1$ dimensional point can get shattered by $\mathcal{H}$, so for all $d+1$ dimensional points there exists some labelling not expressible by $\mathcal{H}$. A d-witness w is just a function that takes every $d+1$ dimensional point to this label, hence witnessing that $VC(\mathcal{H}) \leq d$. If $VC(\mathcal{H})$ is infinite then there is no d-witness function for any $d \in \mathbb{N}$.

Definition 2.9. A learner $A : \mathcal{S} \mapsto \mathcal{H}$ is called an *asymptotic empirical risk minimizer (asymptotic ERM)* for $\mathcal{H}$ if it outputs only hypotheses in $\mathcal{H}$, and if there is an infinite sequence $\{\epsilon_m \in [0,1]\}_{m=1}^{\mathbb{N}}$ converging to 0 as $m \mapsto \mathbb{N}$ such that for every sample $S \in \mathcal{S}$ we have that:

$$L_S(A(S)) \leq \inf_{h \in \mathcal{H}} L_S(h) + \epsilon_{|S|}.$$

[4]The relativization is that $\mathcal{H}$ is CPAC^x if it is PAC-learnable and there is an x-computable learner $A : \mathcal{S} \mapsto \mathcal{H}$ as witness.

[5]The relativization is that $\mathcal{H}$ is SCPAC^x if it is PAC-learnable and there is an x-computable learner $A : \mathcal{S} \mapsto \mathcal{H}$ and an x-computable sample size function $m \colon (0,1)^2 \to \mathbb{N}$ witnessing this.

We can now present the effective versions of the fundamental theorem of statistical learning given by Sterkenburg [2022] and Delle Rose et al. [2023]. The different notions of computable learning are no longer equivalent.

Theorem 2.10 (Effective fundamental theorem of statistical learning)**.** *Let $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ be a hypothesis class of finite VC dimension. Then $\mathcal{H}$ is:*

(1) proper SCPAC-learnable $\Longleftrightarrow$ there is a computable ERM for $\mathcal{H}$

(2) proper CPAC-learnable $\Longleftrightarrow$ there is a computable asymptotic ERM for $\mathcal{H}$

(3) improper SCPAC-learnable $\Longleftrightarrow$ improper CPAC-learnable $\Longleftrightarrow$ there is a computable d-witness function for $\mathcal{H}$ (for some $d \in \mathbb{N}$)

Part (1) is Proposition 1 of Sterkenburg [2022], (2) is Proposition 2 of Delle Rose et al. [2023], and (3) is Theorem 3 of Delle Rose et al. [2023]. Since (3) also proves that improper SCPAC and improper CPAC are equivalent conditions, we will simply write improper S/CPAC when discussing these conditions. Also, from Theorem 3 of Sterkenburg [2022] and Theorem 4 of Delle Rose et al. [2023], PAC $\not\Longrightarrow$ improper S/CPAC $\not\Longrightarrow$ proper CPAC $\not\Longrightarrow$ proper SCPAC (though clearly each of the reverse implications hold), so these are all distinct classes.

In the effective setting, one would generally require that the hypothesis class be effectively represented in some form. In Agarwal et al. [2020], the following notions of an effective hypothesis class was used.

Definition 2.11. A hypothesis class $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ is *computably enumerably representable* (c.e.r) if there is a c.e. set of (indices for) Turing machines such that the set of functions they compute is equal to $\mathcal{H}$, i.e., $\mathcal{H} = \{\varphi_i : i \in W_e\}$ for some c.e. set W_e. Similarly, it is *computably representable* (c.r.) if there exists a computable set of Turing machines such that the set of functions they compute is equal to $\mathcal{H}$.

Another possible effective representation of a hypothesis class is as an effectively closed set. A *tree* is a set $T \subseteq \{0,1\}^* = 2^{<\mathbb{N}}$ of finite binary strings closed under initial segments. A *path* h through T is an infinite binary string such that all initial segments of h belong to T. The set of all paths through T is denoted $[T]$. Since $\mathcal{X} = \mathbb{N}$ and $\mathcal{Y} = \{0,1\}$, paths through T can be identified with hypotheses $h\colon \mathcal{X} \to \mathcal{Y}$.

Definition 2.12. A hypothesis class $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ is *effectively closed* if there is a computable tree $T \subseteq \{0,1\}^* = 2^{<\mathbb{N}}$ such that $\mathcal{H} = [T]$ is the set of paths through T.

One should think of the tree T as defining $\mathcal{H}$ by specifying the finite partial functions that hypotheses can extend. The closed sets of Cantor space $2^{\mathbb{N}}$ are exactly the sets of the form $[T]$ for trees T, hence the term effectively closed.

3 The "on-a-cone" approach

Given a set $A \subseteq 2^{\mathbb{N}}$, we say that A is *degree-invariant* if whenever $x \in A$ and $y \equiv_T x$, $y \in A$. If A is degree invariant, we can identify it with the corresponding set of Turing degrees $\{\deg_T(x) : x \in A\}$.

Definition 3.1. Given $x \subseteq \mathbb{N}$, the *cone above* x is

$$C_x = \{y : y \geq_T x\}.$$

One should think of a cone as a large set. For one, they are cofinal in the Turing degrees. Also, any two cones intersect, and so a set cannot both contain a cone and be disjoint from a cone. One can define the $\{0, 1\}$-valued *Martin's measure* by setting $\mu(A) = 1$ if A contains a cone, and $\mu(A) = 0$ if A is disjoint from a cone. It is not clear that all sets A are Martin-measurable, and this is an important topic in descriptive set theory: For which classes of sets must every degree-invariant set either contain a cone or be disjoint from a cone?

In addition to our computability-theoretic notation from Section 1.1 we will also need some notions from descriptive set theory. First, let us recall the Borel sets. We work in Cantor space $2^{\mathbb{N}}$ with the standard topology. The basic clopen sets are the sets $[\sigma]$ of all extensions of a finite string σ, and the open sets are all unions of the basic clopen sets. The closed sets are their compliments, equivalently, a set is closed if and only if it is the set of infinite paths through a tree $T \subseteq 2^{<\mathbb{N}}$. The Borel sets are the smallest class of sets containing the open sets and closed under compliments and countable intersections and unions. One can think of the Borel sets as those with a reasonable constructive definition.

Given a set $A \subseteq 2^{\mathbb{N}}$, one can consider the two-player Gale-Stewart game $\mathcal{G}_A$ with A as the payoff set. For this paper, one can take the precise definitions as a black box. This game is an infinitely long game of perfect information. Gale and Stewart showed that if A is open or closed, then one of the two players has a winning strategy. However one can prove using the axiom of choice that there is a set A such that neither player has a winning strategy in $\mathcal{G}_A$. If one of the two players has a winning strategy in $\mathcal{G}_A$, then we say that A is *determined*. While one can use the axiom of choice to construct an undetermined set, Martin [1975] proved that all Borel sets are determined.

Theorem 3.2 (Borel determinacy, Martin [1975]). *Every Borel set is determined.*

The following theorem, together with Borel determinacy, says that every Borel set either contains a cone or is disjoint from a cone, and in particular has either measure 1 or measure 0.

Theorem 3.3 (Martin [1968]). *If a degree-invariant set $A \subseteq 2^{\mathbb{N}}$ is determined, then it either contains a cone or is disjoint from a cone. In particular, every degree-invariant Borel subset of $2^{\mathbb{N}}$ either contains a cone or is disjoint from a cone.*

In particular, for such a set A, it cannot be the case that both A and its complement are cofinal in the Turing degrees. Recall that the $\{0, 1\}$-valued *Martin's measure* puts $\mu(A) = 1$ if A contains a cone, and $\mu(A) = 0$ if A is disjoint from a cone. Martin's measure is a countably additive measure on invariant Borel sets.

Remark 3.4. Martin's measure is countably additive, so that if $\mu(\bigcup_i A_i) = 1$, and each A_i is determined (e.g., Borel), then for some i, $\mu(A_i) = 1$. That is, if $\bigcup_i A_i$ contains a cone then, for some i, A_i contains a cone. This will be an important property later in the paper.

It is not hard to see why this remark is true. Suppose that for all i $\mu(A_i) = 0$, i.e., that no A_i contains a cone. For each i there is a cone with base b_i disjoint from A_i. Let $b = \bigoplus b_i$. Then the cone with base b is disjoint from every A_i, and so disjoint from $\bigcup_i A_i$. Thus, being disjoint from a cone, $\mu(\bigcup_i A_i) = 0$.

Essentially, one should think that for any Borel degree-invariant set A, one can classify A as either *large* (if it contains a cone) or *small* (if it is disjoint from a cone).

We can now consider learning on a cone, beginning with two motivating examples. First, we have the example from Agarwal et al. [2020] of a PAC-learnable but not CPAC-learnable hypothesis class. Though that hypothesis class used provability in arithmetic, the example can be more simply formulated as follows. (This formulation was also used in Sterkenburg [2022] where it was shown that it was not even improperly CPAC-learnable.) Let $K \subseteq \mathbb{N}$ be the halting problem $K = \{e : \text{the } e\text{th program halts}\}$ (though any non-computable c.e. set could also be used). Then K has a computable approximation K_s where K_s is the set of (indices for) programs which have halted after s steps of computation. Let $\mathcal{H} := \{h_{s,e} : e \in K_s\}$ where

$$h_{s,e}(k) = \begin{cases} 1 & k = 2e \\ 1 & k = 2s+1 \\ 0 & \text{otherwise} \end{cases}.$$

Since K_s is computable, we can computably list out $\mathcal{H}$, and $\mathcal{H}$ has VC dimension at most 2. One can argue that $\mathcal{H}$ is not CPAC-learnable. Towards a contradiction, assume that there is a proper CPAC learner for $\mathcal{H}$ in the agnostic setting. Given e, let $\mathcal{D}_i$ be a distribution with all weight on $(2e, 1)$. Then almost surely any training sample contains a sequence of $(2e, 1)$ with label 1. If $e \in K$, then for sufficiently long samples (of a fixed length known ahead of time) the learner must output a function $h_{s,e}$ where $e \in K_s$. Otherwise, it may output any function from $\mathcal{H}$. So we can use the learner to compute K: for each e, feed in a sufficiently long sample sequence consisting of $(2e, 1)$ with label 1 to the learner, for which it outputs a function $h = h_{s,e'}$. If $h(2e) = 1$ (so that $e' = e$) then $e \in K$. Otherwise, $e \notin K$.

We can relativize $\mathcal{H}$ using relativization of K to an oracle x,

$$K^x = \{e : \text{the } e\text{th program with oracle } x \text{ halts}\},$$

and then defining $\mathcal{H}^x = \{h_{s,e} : e \in K^x_s\}$. The argument above relativizes to show that $\mathcal{H}^x$ is PAC-learnable, but not properly CPAC-learnable relative to x in the agnostic setting. If $x \equiv_T y$ are Turing-equivalent, then everything computable with oracle x is computable with oracle y and vice versa. However, even though $K^x \equiv_T K^y$ are Turing-equivalent, we might have $K^x \neq K^y$ and so $\mathcal{H}^x \neq \mathcal{H}^y$. This is because the ith program with oracle x might be completely different from the ith program with oracle y, and so one might halt while the other does not. This is essentially capturing the fact that there is no canonical choice for listing out all programs, and the set K, and thus what functions $h_{s,i}$ are in $\mathcal{H}$, depends on this listing. If we choose a different listing, then the hypothesis class $\mathcal{H}$ obtained will be different. The hypothesis class $\mathcal{H}$ does not relativize in a degree-invariant way because $x \equiv_T y \not\Longrightarrow \mathcal{H}^x = \mathcal{H}^y$.

On the other hand, consider a natural hypothesis class such as the class $\mathcal{G}$ of computable half planes (where by computable we mean that the parameters of the dividing line are given by computable real numbers). This class relativizes to, given an oracle x, the hypothesis class $\mathcal{G}^x$ of x-computable half planes. This is degree-invariant, because the x-computable half planes are the same as the y-computable half planes. For all x, $\mathcal{G}^x$ is PAC-learnable and CPAC-learnable relative to x. Thus $\mathcal{G}$ is CPAC-learnable on a cone. A general heuristic is that hypothesis classes given by properties of functions are degree-invariant while hypothesis classes given as properties of programs or indices are not. Whether a function represents a half-plane or not does not depend on the program computing the function, but solely on the function itself. On the other hand, whether a hypothesis $h_{s,e}$ is in $\mathcal{H} = \{h_{s,e} : e \in K_s\}$ depends on the program e, and how long it takes to run.

Keeping these motivating examples in mind, we are now ready for the formal definitions.

Definition 3.5. Let $x \mapsto \mathcal{H}^x$ be a function $2^{\mathbb{N}} \to \mathcal{G}(2^X)$ which associates to each oracle $x \subseteq \mathbb{N}$ a hypothesis class. We denote this function by $\mathcal{H}$ and say that $\mathcal{H}$ is a *family of hypothesis classes*. We say that $\mathcal{H}$ is:

(1) *degree-invariant* if whenever $x \equiv_T y$, $\mathcal{H}^x = \mathcal{H}^y$.

(2) *Borel* if its graph $\Gamma(\mathcal{H}) = \{(x, f) : f \in \mathcal{H}^x\}$ is Borel.

(3) *closed* if each $\mathcal{H}^x$ is closed (and similarly for open, $\mathbf{\Pi}^0_2$, and so on).

Definition 3.6. Given $\mathcal{H}^x$ a degree-invariant family of hypothesis classes, and P a property of hypothesis class, we say that $\mathcal{H}^x$ is P on a cone if there is b such that for all $x \geq_T b$, $\mathcal{H}^x$ has property P. For example:

(1) $\mathcal{H}^x$ is *PAC-learnable on a cone* if there is a cone of x's on which $\mathcal{H}^x$ is PAC-learnable, that is, there is a cone with base b such that for all $x \geq_T b$, $\mathcal{H}^x$ is PAC-learnable.

(2) $\mathcal{H}^x$ is *not-PAC-learnable on a cone* if there is a cone of x's on which $\mathcal{H}^x$ is not PAC-learnable, that is, there is a cone with base b such that for all $x \geq_T b$, $\mathcal{H}^x$ is not PAC-learnable.

(3) $\mathcal{H}^x$ is *CPAC-learnable on a cone* if there is a cone of x's on which $\mathcal{H}^x$ is CPAC^x-learnable, that is, there is a cone with base b such that for all $x \geq_T b$, there is an x-computable PAC-learner for $\mathcal{H}^x$.

(4) $\mathcal{H}^x$ is *c.e.-represented on a cone* if there is a cone with base b such that for all $x \geq_T b$, $\mathcal{H}^x$ is c.e.-represented relative to x, that is, represented by an x-c.e. set of x-computable functions.

Note that for arbitrary degree-invariant $\mathcal{H}$, $\mathcal{H}^x$ being not-PAC-learnable on a cone (there is a cone on which $\mathcal{H}^x$ is not PAC-learnable) is not equivalent to $\mathcal{H}^x$ not being PAC-learnable on a cone (there is not a cone on which $\mathcal{H}^x$ is PAC-learnable). This is because if $\mathcal{H}^x$ is not Borel, then $\{x : \mathcal{H}^x \text{ is PAC-learnable}\}$ might not be Borel, and hence might both not contain a cone and not be disjoint from a cone.

If $\mathcal{H}^x$ is Borel, then because being PAC-learnable is a Borel property, $\{x : \mathcal{H}^x \text{ is PAC-learnable}\}$ is a Borel set and so either $\mathcal{H}^x$ is PAC-learnable on a cone or $\mathcal{H}^x$ is not-PAC-learnable on a cone. One of the main technical issues in the results in this paper is whether the sets involved are determined and hence whether we can conclude that they contain or are disjoint from a cone.

4 C.e.-represented hypothesis classes

We show that if a natural hypothesis class is c.e.-represented and PAC-learnable, then it is SCPAC-learnable.

Theorem 1.1. *Let $\mathcal{H}^x$ be a degree-invariant family of hypothesis classes that is c.e.-represented on a cone and PAC-learnable on a cone. Then $\mathcal{H}^x$ is properly SCPAC-learnable on a cone.*

Proof. Since $\mathcal{H}^x$ is c.e.-represented on a cone (say the cone above y_1) and PAC-learnable on a cone (say the cone above y_2), then on the cone above $y_1 \oplus y_2$ $\mathcal{H}^x$ is both c.e.-represented and PAC-learnable. In the rest of the proof, for simplicity of notation, we will omit the oracle $y_1 \oplus y_2$ and assume that $\mathcal{H}^x$ is c.e.-represented and PAC-learnable for all x, and prove that for all x the hypothesis class $\mathcal{H}^x$ is properly SCPAC-learnable relative to x. Moreover, because Martin's measure is countably additive, we may assume that there is a single e such that for all x the class $\mathcal{H}^x$ is c.e.-represented by W_e^x (see Remark 3.4). Indeed,

$$2^{\mathbb{N}} = \bigcup_e \{x : \mathcal{H}^x \text{ is represented by the c.e. set } W_e^x\}$$

and each set in the union is Borel, and so one of these sets (say the one corresponding to e) must contain a cone. For all x on this cone, $\mathcal{H}^x = \{\varphi_n^x : n \in W_e^x\}$.

Let $S \in \mathcal{S}$ be a sample of size m. We can partition the Turing degrees as follows:

$$2^{\mathbb{N}} = \bigcup_{i=1}^{m} \left\{x : \min_{h \in \mathcal{H}^x} L_S(h) = \frac{i}{m}\right\}.$$

These sets are degree-invariant (because $\mathcal{H}^x$ is) and Borel because $\mathcal{H}^x$ is c.e.-represented by W_e^x. Then one of them contains a cone C_S with base b_S, say the set corresponding to i_S.

Define the function $f : \mathcal{S} \mapsto \mathbb{Q}$ such that $f(S) = \frac{i_S}{m}$. Then $f(S) = \min_{h \in \mathcal{H}^x} L_S(h)$ for all $x \geq_T b_S$. So, given any sample $S \in \mathcal{S}$, f takes S to the minimal empirical risk value that is reached on S on the cone above b_S.

Let $b = (\bigoplus_{S \in \mathcal{S}} b_S) \bigoplus f$ and let $x \geq_T b$. By the relativized version of Theorem 2.10 (1), $\mathcal{H}^x$ is properly SCPAC^x if and only if $\mathcal{H}^x$ is PAC-learnable (which we know that it is) and there is an x−computable ERM for $\mathcal{H}^x$. Such an x-computable ERM exists: Given a sample $S \in \mathcal{S}$, compute $f(S)$, and look for an index $n \in W_e^x$ such that $L_S(\varphi_n^x) = f(S)$. This $\varphi_n^x \in \mathcal{H}^x$ is the hypothesis with the minimum empirical risk. Output φ_n^x. ◻

5 Negative Results

In this section we give a negative result: An example of a degree-invariant Borel family of hypothesis classes that is PAC-learnable but not properly CPAC-learnable on a cone (but is improperly CPAC-learnable on a cone). The reader should keep in mind that while these are in some sense natural from a computational standpoint, in terms of relativizing on a cone, they are still, e.g., not c.e.-represented or closed and are not exactly natural from the standpoint of learning theory. The proof appears in Appendix C.

Theorem 5.1. *There is a degree-invariant Borel family of hypothesis classes $\mathcal{H}^x$ such that for all x, $\mathcal{H}^x$ is PAC-learnable and consists only of x-computable functions, but $\mathcal{H}^x$ is not properly CPAC-learnable on any cone.*

These hypothesis classes $\mathcal{H}^x$ are $\mathbf{\Sigma}_2^0/F_\sigma$ because they are countable, but this is making strong use of the fact that this is boldface $\mathbf{\Sigma}_2^0$ rather than lightface Σ_2^0.

Definition 5.2. A hypothesis class $\mathcal{H}$ is Σ_2^0-represented if there is a Σ_2^0 listing of the hypotheses in $\mathcal{H}$, i.e., a Σ_2^0 set U such that $\mathcal{H} = \{\varphi_e : e \in U\}$. Π_2^0-represented hypothesis classes are defined similarly.

Question 5.3. Is there a natural hypothesis class which is Σ^0_2-represented (or Π^0_2-represented) and PAC-learnable but not CPAC-learnable?

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A Proof of Theorem 1.2

Theorem 1.2. *Let $\mathcal{H}^x$ be a degree-invariant Borel family of hypothesis classes that is closed and PAC-learnable on a cone. Then $\mathcal{H}^x$ is properly SCPAC-learnable on a cone.*

We make use of the following proposition whose proof we delay to Appendix A.

Proposition A.1. *Let $\mathcal{H}^x$ be a degree-invariant Borel family of hypothesis classes such that for all x, $\mathcal{H}^x$ is closed. Then there is a tree T with no dead ends such that $\mathcal{H}^x = [T]$ for all x on a cone.*

Proof. Given $\mathcal{H}^x$ be a degree-invariant Borel family of hypothesis classes such that for all x, $\mathcal{H}^x$ is closed, we must show that there is a tree T with no dead ends such that $\mathcal{H}^x = [T]$ for all x on a cone. When we say that $\mathcal{H}^x$ is a Borel family, recall that we mean that the set $\mathcal{H} = \{(x, h) : h \in \mathcal{H}^x\}$ is Borel. The sets $\mathcal{H}^x$ are the sections of this set, and are all closed.

We use a theorem of Kunugi and Novikov from descriptive set theory (see Theorem 28.7 of Kechris [1995], particularly in the form of Exercise 28.9) which says that given a Borel set $\mathcal{H} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that every section $\mathcal{H}^x$ is closed, there is a Borel map $x \mapsto T_x \subseteq 2^{<\mathbb{N}}$ where T_x is a tree with $[T_x] = \mathcal{H}^x$. In the general form of this theorem, the sections $\mathcal{H}^x$ are subsets of Baire space $\mathbb{N}^{\mathbb{N}}$ and the trees T_x might not be pruned. In our case, $\mathcal{H}^x$ is a subset of Cantor space $2^{\mathbb{N}}$ and so the trees T_x are finitely branching and we may assume that they are pruned. This corresponds to replacing T_x by

$$T_x^* = \{\sigma \in T \mid \text{ for all } n \geq |\sigma|, \text{there is } \tau \in T_x \text{ with } \tau \succ \sigma \text{ of length } n\}.$$

Let

$$B_\sigma = \{x \in 2^{\mathbb{N}} : \sigma \in T_x\}.$$

Both B_σ and its compliment are Borel and so one of the two contains a cone. Let b_σ be the base of this cone.

Let

$$T = \{\sigma : B_\sigma \text{ contains a cone}\}.$$

Let $b = \bigoplus b_\sigma$. Then, for all $x \geq_T b$,

$$\{\sigma : \text{there is an } f \in \mathcal{H}^x \text{ with } f \succ \sigma\} = T.$$

Since $\mathcal{H}^x$ is closed, $\mathcal{H}^x = [T]$. ◻

Theorem 1.2. By the proposition, there is a tree T with no dead ends such that $\mathcal{H}^x = [T]$ for all x on a cone, say with base b. Increasing the cone from b to $b^* = b \oplus T$, for all x on the cone above b^*, we argue that $\mathcal{H}^x$ is properly SCPAC^x.

By Theorem 2.10 (1) relativized to x, $\mathcal{H}^x$ is properly SCPAC^x learnable if and only if it is PAC-learnable and there is an x-computable *ERM* for $\mathcal{H}^x$. $\mathcal{H}^x$ is PAC-learnable by assumption, and it is easy to x-compute an ERM for $\mathcal{H}^x = [T]$ using T as follows. Given a sample S, the tree T is enough to tell us all of the possible labellings of S by functions in $\mathcal{H}^x = [T]$. We can pick one with the least error, and since T has no dead ends, we can compute a path in T agreeing with this labelling (say, the leftmost such path in T). ◻

B Proof of Theorem 1.3

Theorem 1.3 (ZF + AD)**.** *Let $\mathcal{H}^x$ be a degree-invariant family of hypothesis classes that is PAC-learnable on a cone. Then $\mathcal{H}^x$ is improperly SCPAC-learnable on a cone.*

Proof. Given $\mathcal{H}^x$ a degree-invariant family of hypothesis classes that is PAC-learnable on a cone, we must show that for all x on a cone, $\mathcal{H}^x$ is improperly SCPAC. As we did previously, we may assume for simplicity that $\mathcal{H}^x$ is PAC-learnable for all x. Thus each $\mathcal{H}^x$ has finite VC-dimension; moreover, by invariance, if $x \equiv_T y$, then $\mathcal{H}^x = \mathcal{H}^y$ have the same VC-dimension. Thus, for each d, the set $VC_d = \{x : VC(\mathcal{H}^x) = d\}$ is degree-invariant and Borel. It follows that there is a number d and a cone on which all $\mathcal{H}^x$ have VC-dimension d. So as a simplification we may assume that all $\mathcal{H}^x$ have VC-dimension d.

Given x, because $VC(\mathcal{H}^x) = d$, for each $\bar{u} \in \mathbb{N}^{d+1}$ there is a labelling $\bar{\ell} \in 2^{d+1}$ witnessing that $\bar{u}$ cannot be shattered by $\mathcal{H}^x$, i.e., the functions in $\mathcal{H}^x$ cannot give the labelling i to $\bar{u}$. Moreover, there are finitely many such labels, so there is a lexicographically least such label. Call this labelling $w_x(\bar{u})$.

Fix $\bar{u} \in \mathbb{N}^{d+1}$ and observe that we can partition the Turing degrees as follows:

$$2^{\mathbb{N}} = \bigcup_{\bar{\ell} \in 2^{d+1}} \{x : w_x(\bar{u}) = \bar{\ell}\}$$

These sets are degree-invariant and so (using the axiom of determinacy) there is a cone of x's on which $w_x(\bar{u}) = \bar{\ell}$. Doing this for each $\bar{u}$, and taking the intersection of all of these cones, we get a function w_x and a cone such that for all x on that cone, $w_x = w$. Increasing the cone, we may also assume that w is computable.

By the relativized version of Theorem 2.10 (3) $\mathcal{H}^x$ is improper SCPAC^x learnable if and only if there is an x-computable witness function for $\mathcal{H}^x$. We have just shown that, for all x on a cone, w is an x-computable witness function and thus $\mathcal{H}^x$ is improper SCPAC^x learnable. □

Corollary B.1 (ZF + Analytic Determinacy)**.** *Let $\mathcal{H}^x$ be a degree-invariant Borel family of hypothesis classes that is PAC-learnable on a cone. Then $\mathcal{H}^x$ is improperly SCPAC-learnable on a cone.*

Proof. Since each $\mathcal{H}^x$ is Borel, the sets

$$\bigcup_{\bar{\ell} \in 2^{d+1}} \{x : x \geq_T y \wedge w_x(\bar{u}) = \bar{\ell}\}$$

are $\mathbf{\Pi}^1_1$ and hence determined by Analytic Determinacy. This suffices to complete the proof of the previous theorem without assuming any more determinacy. □

C Proof of Theorem 5.1

Theorem 5.1. *There is a degree-invariant Borel family of hypothesis classes $\mathcal{H}^x$ such that for all x, $\mathcal{H}^x$ is PAC-learnable and consists only of x-computable functions, but $\mathcal{H}^x$ is not properly CPAC-learnable on any cone.*

Proof. We must construct a degree-invariant Borel family of hypothesis classes $\mathcal{H}^x$ such that for all x, $\mathcal{H}^x$ is PAC-learnable and consists only of x-computable functions, but $\mathcal{H}^x$ is not CPAC-learnable on any cone.

Let $\mathcal{F}$ be the set of all functions $f\colon \mathbb{Q} \to \{0,1\}$ such that:

(1) there are p, q such that $f(p) = 0$ and $f(q) = 1$;

(2) if $p < q$, then if $f(p) = 1$ then $f(q) = 0$;

(3) there is no maximal element p with $f(p) = 0$ and no minimal element q with $f(q) = 1$.

It is easy to see that $\mathcal{F}$ has VC dimension 1, so any subset of $\mathcal{F}$ will also have VC dimension 1. $\mathcal{F}$ is also Borel.

To each $f \in \mathcal{F}$ we can associate a unique irrational r such that $f(p) = 0$ for $p < r$ and $f(p) = 1$ for $p > r$. It is easy to see that $\mathcal{F}$ has VC dimension 1. Define, for each x, $\mathcal{H}^x = \{f \in \mathcal{F} : f \equiv_T x\}$. Note that this is degree-invariant and that each $\mathcal{H}^x$ has VC dimension 1 (hence is PAC). Moreover, each $\mathcal{H}^x$ is countable and hence F_σ.

Now we argue that $\mathcal{H}^x$ is not properly CPAC-learnable on a cone. Suppose towards a contradiction that $\mathcal{H}^x$ was properly CPAC-learnable on a cone. Then, for each x on a cone, there would be an x-computable asymptotic ERM for $\mathcal{H}^x$. Given an x-computable function which is a purported asymptotic ERM for $\mathcal{H}^x$, we can check in a Borel way whether it is in fact an asymptotic ERM (i.e., whether the computable function satisfies Definition 2.9). Given x, choose one of these asymptotic ERMs for $\mathcal{H}^x$ as follows. List the samples $S \in \mathcal{S}$ as $S_1, S_2, \ldots$, and list the rationals as $q_1, q_2, q_3, \ldots$. Then we can think of an asymptotic ERM as a binary sequence $\mathbb{N} \times \mathbb{N} \to \{0,1\}$ via $(m,n) \mapsto (S_m, q_n) \mapsto A(S_m)(q_n)$. For each x, choose the lexicographically least asymptotic ERM A_x for $\mathcal{H}^x$. If $x \equiv_T y$, then $A_x = A_y$, and because we chose A_x and A_y using only their properties as functions (and not the programs that compute them) we will have chosen the same asymptotic ERM $A_x = A_y$.

Now for each sample S and $q \in \mathbb{Q}$, consider

$$\{x : A_x(S)(q) = 0\} \text{ and } \{x : A_x(S)(q) = 1\}.$$

This is degree-invariant (since the choice of A_x is degree-invariant). One of these two must contain a cone, say with base $b_{S,q}$ and let $g_S(q)$ record the corresponding value 0 or 1. Let $b = \bigoplus b_{S,q}$. For all $x \geq_T b$, and all $S \in \mathcal{S}$, $A_x(S) = g_S$ is the same function. But for $x \not\equiv_T y$, $\mathcal{H}^x$ and $\mathcal{H}^y$ are disjoint, and so for most $x \geq_T b$, $A_x(S) \notin \mathcal{H}^x$. Thus, for those x, A_x cannot be an asymptotic ERM for $\mathcal{H}^x$. But of course A_x was chosen to be an asymptotic ERM for all x, yielding a contradiction. So it is not true that $\mathcal{H}^x$ is properly CPAC-learnable on a cone. □