Describing finitely presented algebraic structures

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Abstract

Scott showed that for every countable structure $\mathcal{A}$, there is a sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$, called a Scott sentence for $\mathcal{A}$, whose models are exactly the isomorphic copies of $\mathcal{A}$. We say that the Scott complexity of a structure is the least complexity of a Scott sentence for that structure. This is a measure of the complexity of describing the structure up to isomorphism.

A finitely generated structure has Scott complexity at most Σ_3, and many finitely generated structures—e.g. all fields, vector spaces, and abelian groups—have Scott complexity at most d-Σ_2. We show that finitely generated commutative rings and finitely generated modules over Noetherian rings have d-Σ_2 Scott sentences. We also give a purely group-theoretic characterization of the finitely presented groups with a d-Σ_2 Scott sentence, and use this characterization to prove that almost all finitely presented groups have a d-Σ_2 Scott sentence. We use the characterization to show that the Baumslag-Solitar groups $BS(m,n)$ have d-Σ_2 Scott sentences.

Finally we answer a question of Alvir, Knight, and McCoy by showing that there is a computable finitely generated group with a d-Σ_2 Scott sentence, but no computable d-Σ_2 Scott sentence.

1 Introduction

Given a countable structure $\mathcal{A}$, Scott [Sco65] showed that we can describe $\mathcal{A}$ up to isomorphism among countable structures by a sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$. This logic is more expressive than elementary first-order logic and allows countable conjunctions and disjunctions. For example, the group $\mathbb{Z}$ is the only rank 1 torsion-free abelian group with an element that has no non-trivial divisibilities, and this description can be expressed in $\mathcal{L}_{\omega_1\omega}$. For example, to say that a group is rank 1, we write:

$$\forall g, h \bigvee_{(n,m)\neq(0,0)} ng = mh.$$

To measure the complexity of the structure $\mathcal{A}$, we want to write down the simplest possible description of $\mathcal{A}$. There is a hierarchy of sentences depending on the number of quantifier alternations, and counting infinite conjunctions the same as universal quantifiers and infinite disjunctions as existential quantifiers. The Σ_n sentences have n alternations of quantifiers, beginning with existential quantifiers; the Π_n sentences have n alternations of quantifiers, beginning with a universal quantifier; and the d-Σ_n sentences are the conjunction of a Σ_n and a Π_n sentence. The hierarchy is ordered as follows, from the simplest formulas on the left, to the most complicated formulas on the right:

$$\begin{array}{ccccccccccccc}
 & & \Sigma_1 & & & & \Sigma_2 & & & & \Sigma_3 & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\Sigma_0 & & & & \text{d-}\Sigma_1 & & & & \text{d-}\Sigma_2 & & & & \text{d-}\Sigma_3 \to \cdots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & \Pi_1 & & & & \Pi_2 & & & & \Pi_3 & &
\end{array}$$

The hierarchy continues through the transfinite. The description of $\mathbb{Z}$ given above is d-Σ_2, as the group axioms are Π_2, saying that a group is rank 1 torsion-free is Π_2, and saying that there is an element with no non-trivial divisibilities is Σ_2.

We say that the *Scott complexity* of a structure is the complexity of its simplest Scott sentence. Though there are other complexities of $\mathcal{L}_{\omega_1\omega}$ formulas (such as disjunctions of a Σ_α and Π_α formula), the Scott complexity of a structure will always be one of Σ_α, Π_α, and d-Σ_α [AGHTT].

In this paper, we consider the Scott complexity mostly of finitely presented algebraic structures (and sometimes related finitely generated but not finitely presented structures). This is part of a recent program of analysing the Scott complexity of finitely generated structures [CHKM06, CHK+12, KS18, Ho17, HTH18, HTH]. This began with the analysis of several examples in group theory, and in all cases the groups were shown to have d-Σ_2 Scott sentences. Calvert, Harizanov, Knight, and Miller [CHKM06] showed that finitely generated abelian groups—such as $\mathbb{Z}$, as described above—all have d-Σ_2 descriptions, and Carson et al. [CHK+12] showed that finitely generated free groups have d-Σ_2 descriptions. Knight and Saraph [KS18] remarked that every finitely generated structure has a Σ_3 description. If $\mathcal{A}$ is generated by a tuple $\bar{a}$, then a Σ_3 Scott sentence for $\mathcal{A}$ is:

there is a tuple $\bar{x}$, satisfying the same atomic formulas as $\bar{a}$ (i.e., for all atomic formulas true of $\bar{a}$, the formula is true of $\bar{x}$), such that every element is generated by $\bar{x}$ (i.e., for all y, there is a term t in the language such that $y = t(\bar{x})$).

They asked the natural question of whether every finitely generated group has a d-Σ_2 Scott sentence. Ho [Ho17] gave many more examples of groups with d-Σ_2 Scott sentences, including polycyclic groups (which include nilpotent groups and abelian groups) and the Baumslag-Solitar groups $B(1,n)$. The author finally resolved this question together with Ho [HTH18] by showing that there is a finitely generated group with no d-Σ_2 Scott sentence, and hence with Scott complexity Σ_3. These methods were generalized in [HTH] to show that finitely generated groups are universal among finitely generated structures, and to study pseudo Scott sentences (i.e., unique descriptions of a structure within the class of finitely generated structures).

Moving to general results, there are several nice characterizations of when a finitely generated structure has a d-Σ_2 Scott sentence.

Theorem 1.1 (A. Miller [Mil83], Harrison-Trainor and Ho [HTH18], Alvir, Knight, and McCoy [AKM])**.** *Let $\mathcal{A}$ be a finitely generated structure. The following are equivalent:*

1. *$\mathcal{A}$ has a d-Σ_2 Scott sentence.*

2. *$\mathcal{A}$ has a Π_3 Scott sentence.*

3. *$\mathcal{A}$ is the only model of its Σ_2 theory.*

4. *Some generating tuple of $\mathcal{A}$ is isolated by a Π_1 formula.*

5. *Every generating tuple of $\mathcal{A}$ is isolated by a Π_1 formula.*

6. *$\mathcal{A}$ does not contain a copy of itself as a proper Σ_1-elementary substructure.*

(2) is due to A. Miller [Mil83], (3) is due to the author and Ho [HTH], (4) and (5) are due to Alvir, Knight, and McCoy [AKM], and (6) is due to the author and Ho [HTH18]. Alvir, Knight, and McCoy also have a nice characterization of when a structure has a *computable* d-Σ_2 Scott sentence.

Theorem 1.2 (Alvir, Knight, and McCoy [AKM]). *Let $\mathcal{A}$ be a finitely generated structure. The following are equivalent:*

1. *$\mathcal{A}$ has a computable d-Σ_2 Scott sentence.*

2. *Some generating tuple of $\mathcal{A}$ is isolated by a computable Π_1 formula.*

3. *Every generating tuple of $\mathcal{A}$ is isolated by a computable Π_1 formula.*

There are indeed computable structures with a d-Σ_2 Scott sentence, but no computable d-Σ_2 Scott sentence; see Theorem 2.8 in this paper for one such example.

Though the focus so far has mostly been on groups or general algebraic structures, there are a few known results on other particular classes of structures [HTH18]: every finitely generated field and vector space has a d-Σ_2 Scott sentence, reflecting the strong structure theorems for these classes; and on the other hand there is a (non-commutative) ring with no d-Σ_2 structure.

Some of the main open questions in this area are:

Question 1 (Question 1.7 of [HTH18] and Question 1 of [AKM]). Does every finitely presented group have a d-Σ_2 Scott sentence?

Question 2 (Question 2 of [AKM]). Is there a precise sense in which most finitely generated groups have a d-Σ_2 Scott sentence?

Question 3 (Question 3 of [AKM]). Is there a computable finitely generated group with a d-Σ_2 Scott sentence but no computable d-Σ_2 Scott sentence?

Question 4 (Question 1.8 of [HTH18]). Does every finitely generated commutative ring have a d-Σ_2 Scott sentence?

Question 5 (Question 5.7 of [HTH18]). Give a characterization of the finitely generated structures which have a d-Σ_2 quasi Scott sentence.

Question 6 (See Section 5.3 of [HTH]). Is there a finitely generated structure with a d-Σ_2 quasi Scott sentence, but no d-Σ_2 Scott sentence?

In this paper, we look at various types of structures with finite presentations—such as finitely presented groups as well as finitely generated rings and modules over Noetherian rings—and prove a number of results, including answering Questions 2, 3, and 4 positively. Some of the results can be summarized by the following table, with the new results from this paper marked *:

Class of finitely generated structures	Every structure has a d-Σ_2 Scott sentence?
Abelian groups	yes
Free groups	yes
Torsion-free hyperbolic groups	yes*
Finitely presented groups	?
Groups	no
Vector spaces	yes
Fields	yes
Commutative rings	yes*
Rings	no
Modules over Noetherian rings	yes*
Modules	no*

On the topic of finitely presented groups, we do not resolve the question of whether every finitely presented group has a d-Σ_2 Scott sentence. We do however get a completely group-theoretic characterization, not involving any notions from logic, of when a finitely presented group has a d-Σ_2 Scott sentence.

Theorem 1.3. *Let G be a finitely presented group. Then the following are equivalent:*

1. *G does not have a d-Σ_2 Scott sentence,*
2. *G contains a proper subgroup $H \cong G$ with the property that for every finite set of non-identity elements $a_1, \ldots, a_n \in G$, there is a normal subgroup $H' \subseteq G$ such that $G = H' \rtimes H$ and $a_1, \ldots, a_n \notin H'$.*

It is already known that a finitely generated group which is coHopfian—which means that it is not isomorphic to any proper subgroup—has a d-Σ_2 Scott sentence. This was first proven by Ho [Ho17] and is a consequence of (6) in Theorem 1.1. The notion of a coHopfian group is the dual to the notion of a Hopfian group, which means that the group is not isomorphic to any proper quotient. From Theorem 1.3, we get the following corollary:

Corollary 1.4. *Every finitely presented Hopfian group has a d-Σ_2 Scott sentence, including:*

- *any finitely generated abelian group,*
- *any finitely-generated free group,*
- *any finitely presented residually finite group,*[1]
- *$B(1, n)$,*

[1] The fact that such groups are Hopfian is due to Mal'cev [Mal40].

- *any torsion-free hyperbolic group.*[2]

This unifies many of the known results about groups with d-Σ_2 Scott sentences, dividing such results into a group-theoretic component (arguing that they are Hopfian) and a logical component encapsulated by the corollary. Moreover, almost all finitely presented groups are torsion-free hyperbolic, and so almost all finitely presented groups have a d-Σ_2 Scott sentence, answering Question 2. There are various senses in which almost all finitely presented groups are torsion-free hyperbolic; one such sense is the few-relator model with various lengths:

Definition 1.5 (Gromov [Gro87])**.** Given positive integers m, k, and $\ell_1, \dots, \ell_k$, let $\mathcal{R}_{k,\ell_1,\dots,\ell_k}$ be the set of group presentations of the form

$$\langle a_1, \dots, a_m \mid r_1, \dots, r_k \rangle$$

where the r_i are reduced and of length ℓ_i. We say that almost all groups have property P if for any $\epsilon > 0$, there is an ℓ such that whenever $\min_i \ell_i \geq \ell$, the proportion of presentations in $\mathcal{R}_{k,\ell_1,\dots,\ell_k}$ with property P is greater than $1 - \epsilon$.

Gromov stated without proof that almost all groups are torsion-free hyperbolic in this sense, and a proof was later given by Ol'shanskiĭ [Ol'92].

There are known examples of finitely presented non-Hopfian groups. The simplest examples are the Baumslag-Solitar groups $BS(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$, some of which, such as $BS(2,3)$, are non-Hopfian [BS62]. We use Theorem 1.3 and known results about the structure of Baumslag-Solitar groups to prove that all such groups have d-Σ_2 Scott sentences. The purely group-theoretic nature of Theorem 1.3, as opposed to the more logical nature of Theorem 1.1, makes it possible to make use of the existing group-theoretic results. See Theorem 4.3.

Finally, we show in Theorem 2.8 that there is a computable module over a PID which has no computable d-Σ_2 Scott sentence; by Theorem 2.2, every such module has a d-Σ_2 Scott sentence. So there is a computable structure with a d-Σ_2 Scott sentence but no computable d-Σ_2 Scott sentence. By the universality among finitely generated structure of finitely generated groups [HTH18], there is also such a computable finitely generated group. This answers Question 3.

2 Modules

We begin the algebraic portion of this paper with modules over a ring R. The language for such structures is the same as the usual vector space language: a binary addition operator and for each $r \in R$ a unary operator for scalar multiplication.

When R is a PID, there is a strong structure theorem for finitely generated R-modules: they can all be written in a unique way as a direct sum of a free module and of proper quotients of R. One can use this to show that every such module has a d-Σ_2 Scott sentence. It turns out that we do not even need to use the fact that R is a PID, but just that it is

[2] Any hyperbolic group is finitely presented.

Noetherian. This is surprising because we do not know of a good characterization theorem of modules over Noetherian rings, other than that they are finitely presented. The two main ingredients that go into the proof are the fact that every finitely generated module over a Noetherian ring is finitely presented (as a quotient of a free module by a finitely generated submodule) and the fact that every Noetherian R-module is Hopfian. These are both well-known facts, but we will remind the reader of the proof of the latter fact.

Theorem 2.1 (Well-known). *Noetherian R-modules are Hopfian: every epimorphism is an isomorphism.*

Proof. Suppose that $f\colon M \to M$ is a surjective homomorphism. Then $\ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \cdots$ is an ascending chain of ideals, and so $\ker f^n = \ker f^{2n}$ for some n. Let $x \in \ker f^n$; we claim that $x = 0$, so that $\ker f \subseteq \ker f^n = 0$. Since f^n is surjective, there is y such that $f^n(y) = x$. Then $f^{2n}(y) = f^n(x) = 0$, so $y \in \ker f^{2n} = \ker f^n$, and so $x = f^n(y) = 0$. □

We now argue that every finitely generated module over a Noetherian commutative ring has a d-Σ_2 Scott sentence.

Theorem 2.2. *Let R be a Noetherian commutative ring and M a finitely generated R-module. Then M has a d-Σ_2 Scott sentence.*

Proof. Suppose that $N \preceq_1 M$ where $N \cong M$. We will argue below that there is an R-linear map $g\colon M \to N \cong M$ which admits the inclusion $N \to M$ as a section. Since M is Hopfian, g must be injective and so $N = M$. By Theorem 1.1 this implies that M has a d-Σ_2 Scott sentence.

First, we fix a finite presentation of M over N. Let M be generated by elements $x_1, \ldots, x_n$. Consider the surjection $N \oplus R^n \to M$ which maps $N \to N \subseteq M$ and the ith basis element e_i of R^n to x_i. Let A be the kernel of this map. Then A is finitely generated, say by $a_1, \ldots, a_\ell \in N \oplus R^n$. Each a_i can be considered as a tuple $(\hat{a}_i, a_i^1, \ldots, a_i^n)$ with $\hat{a}_i \in N$ and $a_i^1, \ldots, a_i^n \in R$.

Now as witnessed by $x_1, \ldots, x_n$,

$$M \models (\exists u_1, \ldots, u_n) \bigwedge_{i=1}^{\ell} \hat{a}_i + \underline{a}_i^1 u_1 + \cdots + \underline{a}_i^n u_n = 0$$

where the $\hat{a}_i$ are parameters from N and the $\underline{a}_i^j$ are the unary scalar multiplication operators in the language. So, as $N \preceq_1 M$,

$$N \models (\exists u_1, \ldots, u_n) \bigwedge_{i=1}^{\ell} \hat{a}_i + \underline{a}_i^1 u_1 + \cdots + \underline{a}_i^n u_n = 0$$

Let $b_1, \ldots, b_n \in N$ be witnesses to this. Then the surjection $N \oplus R^n \to N$ which maps N to N and e_i to b_i has A in its kernel and so factors through $M \cong N \oplus R^n / A$. Note that this map admits the inclusion $N \to N \oplus R^n / A$ as a section. □

The fact that the ring R is Noetherian is important, as we can find a finitely generated R-module over a non-Noetherian ring R with no d-Σ_2 Scott sentence. We use perhaps the most natural choice of R, $R = \mathbb{Z}[X_1, X_2, \ldots]$.

Theorem 2.3. *There is a finitely generated $\mathbb{Z}[X_1, X_2, \ldots]$-module with no d-$\Sigma_2$ Scott sentence.*

Proof. Let M be the free abelian group on

$$\{a_i \mid i \in \omega\} \cup \{b_{i,\sigma} \mid i \in \omega, \sigma \in \mathbb{Z}^{<\omega}, |\sigma| \geq 1\}.$$

Let $R = \mathbb{Z}[X, Y_i \mid i \in \mathbb{Z}]$. Make M into an R-module by having X act as

$$X \cdot a_i = a_{i+1}, \qquad X \cdot b_{i,\langle \ell \rangle} = a_i, \qquad \text{and for } |\sigma| \geq 2 \qquad X \cdot b_{i,\sigma} = b_{i,\sigma^-}$$

where σ^- is σ with the last entry removed, and having Y act as

$$Y_j \cdot a_i = b_{i,\langle j \rangle} \qquad \text{and} \qquad Y_j \cdot b_{i,\sigma} = b_{i,\sigma^\frown\langle j \rangle}.$$

This module is finitely generated by a_0.

Let N be the group generated by a_1; this is exactly the free abelian group on

$$\{a_i \mid i \geq 1\} \cup \{b_{i,\sigma} \mid i \geq 1, \sigma \in \mathbb{Z}^{<\omega}\}.$$

Note that $N \cong M$ via the map induced by $a_i \mapsto a_{i+1}$ and $b_{i,\sigma} \mapsto b_{i+1,\sigma}$. Moreover, $N \neq M$ as $a_0 \notin N$. We claim that $N \prec_1 M$.

Suppose that $\bar{u} \in M$ and $\bar{v} \in N$, and $\varphi(\bar{x}, \bar{y})$ is a quantifier-free formula with

$$M \models \varphi(\bar{u}, \bar{v}).$$

There is k such that $\bar{u}$ and $\bar{v}$ are in the subgroup of M generated by

$$\{a_i \mid i \in \omega\} \cup \{b_{i,\sigma} \mid i \in \omega, \sigma \in \{-k, \ldots, k\}^{<\omega}\}$$

and φ is a formula in the language of $R_k = \mathbb{Z}[X, Y_{-k}, \ldots, Y_k]$-modules. Note that M is also an R_k-module, and let M_k denote the subgroup containing $\bar{u}$ and $\bar{v}$ mentioned previously as an R_k-submodule of M.

Consider also the subgroup of M generated by

$$\{a_i \mid i \geq 1\} \cup \{b_{i,\sigma} \mid i \geq 1, \sigma \in \{-k, \ldots, k\}^{<\omega}\} \cup \{b_{1,\langle k+1 \rangle^\frown\sigma} \mid \sigma \in \mathbb{Z}^{<\omega}\}.$$

This subgroup is also an R_k-submodule of M; denote it by N_k. Note that $N_k \subseteq N$.

Now consider the isomorphism $f\colon M_k \to N_k$ of R_k-modules which maps

$$\begin{aligned} a_0 &\mapsto b_{1,\langle k+1 \rangle} & \qquad a_i &\mapsto a_i, \quad \text{for } i \geq 1 \\ b_{0,\sigma} &\mapsto b_{1,\langle k+1 \rangle^\frown\sigma} & \qquad b_{i,\sigma} &\mapsto b_{i,\sigma}, \quad \text{for } i \geq 1 \end{aligned}$$

It is easy to check that this is in fact an isomorphism and that it fixes $\bar{v} \in N$. Note that $\bar{u}, \bar{v} \in M_k$ and $f(\bar{u}), \bar{v} \in N_k \subseteq N$. So

$$N \models \varphi(f(\bar{u}), \bar{v}).$$

This shows that $N \prec_1 M$. □

On the other hand, it is surprising that we seem to get nothing out of the assumption that R is a PID. Indeed, $\mathbb{Z}$ (as a $\mathbb{Z}$-module, i.e., an abelian group) has no Scott sentence simpler than d-Σ_2. We do not know of any computability-theoretic way to use the classification of finitely generated modules over PIDs to obtain a result that one could not obtain for modules over Noetherian rings.

Question 7. Is there a computability-theoretic way in which modules over PIDs are simpler than modules over Noetherian rings?

One possible approach is to consider computable structures and computable Scott sentences, but there seems to be nothing here to differentiate PIDs from Noetherian rings. When R is a nicely computable Noetherian ring, we can get a computable Scott sentence. On the other hand, there are PIDs which are not "nicely computable", and there are finitely generated modules over those PIDs with no computable d-Σ_2 Scott sentence.

The desired notion of "nicely computable" has been called submodule computable.

Definition 2.4. Let R be a Noetherian ring. A finitely generated R-module M is *submodule computable* if there exist computable procedures which when applied to a finite set $\{v_1, \ldots, v_p\}$ of words in M yield

1. a finite presentation of the submodule $L \subseteq M$ generated by $v_1, \ldots, v_n$, and
2. an algorithm to decide membership in L.

A ring R is *submodule computable* if every finitely presented R-module is submodule computable.

Baumslag, Cannonito, and Miller showed that there are many examples of such rings: any finitely generated commutative algebra over $\mathbb{Z}$ or over any computable field is submodule computable.

Theorem 2.5 (Baumslag, Cannonito, and Miller, Theorem 2.7 of [BCM81])**.** *$\mathbb{Z}$, $\mathbb{Q}$, and all other computable fields are submodule computable. If R is a commutative submodule computable ring and A is a finitely generated commutative R-algebra, then A is submodule computable.*

Furthermore, over such a ring, we can computably analyse homomorphisms, finding their kernels and determining whether they are injective or surjective.

Theorem 2.6 (Baumslag, Cannonito, and Miller, Lemma 2.3 of [BCM81])**.** *Let R be a submodule computable ring, and let L and M be finitely presented R-modules given by finite presentations. Then there is a recursive procedure to determine of an arbitrary function f from the given generators of M to words of L whether or not f defines an R-module homomorphism; if so the procedure yields a set of generators for the kernel of f and determines whether f defines a monomorphism or an epimorphism (hence also an isomorphism). The procedure is uniform in the given data.*

Over such a ring, we can come up with a computable d-Σ_2 Scott sentence for any finitely generated R-module. Note that all such R-modules are computable structures. We use the characterization in Theorem 1.2 due to Alvir, Knight, and McCoy; it says that it suffices to show that a generating tuple is defined by a computable Π_1 formula. The argument essentially explicitly identifies the formulas we used before in Theorem 2.2.

Theorem 2.7. *Let R be a finitely generated submodule computable commutative ring and M finitely generated R-module. Then M has a computable d-Σ_2 Scott sentence.*

Proof. Let $\bar{x} = (x_1, \dots, x_n)$ be a set of generators for M. We claim that the orbit of $\bar{x}$ is defined by a computable Π_1 formula, which implies that M has a computable d-Σ_2 Scott sentence. Since R is submodule computable, M has a computable copy. So the atomic type of $\bar{x}$ is computable. Fix a generating set $x_1, \dots, x_n$ for M.

Suppose that $y_1, \dots, y_n \in M$ generate a proper submodule N of M with $(N, \bar{y}) \cong (M, \bar{x})$. Let R^{2n} be the free module on generators $a_1, \dots, a_n, b_1, \dots, b_n$. Consider the map which takes $a_i \mapsto y_i$ and $b_i \mapsto x_i$. Let A be the kernel of this map, so that it induces an isomorphism $R^{2n}/A \to M$. Note that the map $N \to R^{2n}/A$ which takes $y_i \mapsto a_i$ is an embedding. Let $w_1(a_1, \dots, a_n, b_1, \dots, b_n), \dots, w_\ell(a_1, \dots, a_n, b_1, \dots, b_n)$ generate A. Then

$$M \vDash \exists v_1, \dots, v_n \bigwedge_k w_k(y_1, \dots, y_n, v_1, \dots, v_n) = 0.$$

On the other hand, as argued previously in Theorem 2.2, there is no map $R^{2n}/A \to N$ induced by $a_i \mapsto y_i$ and $b_i \mapsto z_i$ for any z_i. (If there was such a map, then the argument in Theorem 2.2 shows that $N = M$.) So for each $\bar{z}$, there is some i such that $w_i(y_1, \dots, y_n, z_1, \dots, z_n) \neq 0$. Thus

$$N \vDash \neg\exists v_1, \dots, v_n \bigwedge_k w_k(y_1, \dots, y_n, v_1, \dots, v_n) = 0$$

or equivalently

$$N \vDash \forall v_1, \dots, v_n \bigvee_k w_k(y_1, \dots, y_n, v_1, \dots, v_n) \neq 0.$$

Call this formula $\varphi_{\bar{y}}$. Note that as $(N, \bar{y}) \cong (M, \bar{x})$,

$$M \vDash \varphi_{\bar{y}}(\bar{x})$$

but as seen above,

$$M \nvDash \varphi_{\bar{y}}(\bar{y}).$$

Let χ be the Π_1 formula which is the conjunction of the atomic type of $\bar{x}$. Consider the Π_1 formula

$$\chi(\bar{u}) \wedge \bigwedge\!\!\!\bigwedge_{\langle \bar{y} \rangle \cong M, \langle \bar{y} \rangle \neq M} \varphi_{\bar{y}}(\bar{u}).$$

This defines the orbit of $\bar{x}$; indeed, if it holds of a tuple $\bar{y}$, then since $M \vDash \chi(\bar{y})$, $\bar{y}$ generates a submodule N of M with $(N, \bar{y}) \cong (M, \bar{x})$; and since $M \vDash \varphi_{\bar{y}}(\bar{y})$, $N = M$. The formula is computable by Theorems 2.5 and 2.6. $\square$

On the other hand, when R is not submodule computable, it may be possible to have a computable R-module with no computable d-Σ_2 Scott sentence. We give an example when R is even a PID, and the R-module is free (in fact, the R-module is R itself).

Theorem 2.8. *There is a computable PID R which does not have a computable d-Σ_2 Scott sentence as a module over itself.*

Proof. We will define below a c.e. set S. Let $(p_i)_{i\in\omega}$ be a computable listing of the primes, and let U be the multiplicative set generated by $\{p_n \mid n \in S\}$. Let R be the localization $R = \mathbb{Z}_U$. The R-module M will be R itself.

Let $(\varphi_i(x))_{i\in\omega}$ be a listing of the computable Σ_1 formulas. We will construct S stage-by-stage while simultaneously constructing R stage-by-stage. We can produce a computable copy of R uniformly from an enumeration of S, so we do not have to explicitly construct R. Begin with $S_0 = \varnothing$. At stage $s+1$, we will already have constructed S_s and produced $R_s \subseteq R$ from it. Find the least $i \notin S$, $i \leq s$, with $R_s \models \varphi_i(p_i)$ (by this, we mean that we can already tell in R_s that that p_i satisfies one of the first s disjunctions of φ). If we find such an i, put $i \in S_{s+1}$.

This construction ensures that for every i, $R \models \varphi_i(p_i)$ if and only if p_i is in the same orbit as 1 (the automorphism taking p_i to 1 being the multiply-by-$\frac{1}{p_i}$ map). So no single computable Π_1 formula can define the orbit of 1 in R. □

3 Rings and Algebras

The situation for rings and algebras is quite similar to that of modules, and the proofs use similar ideas.

Fix a commutative ring R, and consider the finitely generated commutative R-algebras in the language with a constant for each element of R. The case of finitely generated commutative rings is exactly the case of $\mathbb{Z}$-algebras. The language is the standard language of rings together with, for each $r \in R$, a unary operator for scalar multiplication by r. In the case $R = \mathbb{Z}$, this adds no new expressive power to the language of rings.

First suppose that R is Noetherian, so that every finitely generated R-module is also Noetherian. We are in a similar situation to that of modules over R: every finitely generated R-algebra is finitely presented and Hopfian. We can show that every such R-algebra has a d-Σ_2 Scott sentence.

Theorem 3.1. *Let R be a Noetherian commutative ring, and A a finitely generated R-algebra. Then A has a d-Σ_2 Scott sentence.*

Proof. Suppose that $B \leq_1 A$ where $B \cong A$. We will argue below that there is a surjection of rings $g: A \to B \cong A$ which admits the inclusion $B \to A$ as a section. Since A is Hopfian, g must be injective and so $B = A$. By Theorem 1.1 this implies that A has a d-Σ_2 Scott sentence.

Let A be generated by elements $x_1, \ldots, x_n$ over R. Write $A \cong B[X_1, \ldots, X_n]/I$, identifying x_i with the image of X_i in the quotient. Here I is a finitely generated ideal with generators $I = (f_1, \ldots, f_\ell)$ with each f_i a polynomial over B. Then

$$A \models (\exists u_1, \ldots, u_n)\big[f_1(u_1, \ldots, u_n) = \cdots = f_\ell(u_1, \ldots, u_n) = 0\big].$$

So, as $B \preceq_1 A$ and each f_i is a polynomial over B,

$$B \models (\exists u_1, \dots, u_n)\big[f_1(u_1, \dots, u_n) \wedge \dots \wedge f_\ell(u_1, \dots, u_n)\big].$$

Let $a_1, \dots, a_n \in B$ be such that $f_i(a_1, \dots, a_n) = 0$ for $i = 1, \dots, \ell$. Then the surjection $B[X_1, \dots, X_n] \to B$ which maps $X_i \mapsto a_i$ has $I = (f_1, \dots, f_\ell)$ in its kernel and so factors through $A \cong B[X_1, \dots, X_n]/I$. Note that this map admits the inclusion $B \to B[X_1, \dots, X_n]/I$ as a section. □

Once again, this fails if the ring R is not Noetherian.

Theorem 3.2. *There is a finitely generated $\mathbb{Z}[X_1, X_2, \dots]$-algebra with no d-$\Sigma_2$ Scott sentence.*

Proof. Let A be the polynomial ring over $\mathbb{Z}$ in indeterminates

$$\{X\} \cup \{Y_i \mid i \in \mathbb{Z}\} \cup \{a_i \mid i \in \omega\} \cup \{b_{i,\sigma} \mid i \in \omega, \sigma \in \mathbb{Z}^{<\omega}, |\sigma| \geq 1\}.$$

Let $R = \mathbb{Z}[X, Y_i \mid i \in \mathbb{Z}]$. Let B be A modulo the ideal generated by

$$X \cdot a_i - a_{i+1}, \qquad X \cdot b_{i,\langle \ell \rangle} - a_i, \qquad X \cdot b_{i,\sigma} - b_{i,\sigma^-}, \qquad Y_j \cdot a_i - b_{i,\langle j \rangle}, \qquad Y_j \cdot b_{i,\sigma} - b_{i,\sigma^\frown \langle j \rangle}$$

where for the third element, $|\sigma| \geq 2$ and σ^- is σ with the last entry removed. So, in B, we have

$$X \cdot a_i = a_{i+1}, \qquad X \cdot b_{i,\langle \ell \rangle} = a_i, \qquad \text{and} \qquad X \cdot b_{i,\sigma} = b_{i,\sigma^-}$$

as well as

$$Y_j \cdot a_i = b_{i,\langle j \rangle} \qquad \text{and} \qquad Y_j \cdot b_{i,\sigma} = b_{i,\sigma^\frown \langle j \rangle}.$$

Note that B is an R-algebra, and indeed, the inclusion $R \to B$ is an injection. B is finitely generated by a_0.

Let C be the sub-algebra generated by

$$\{X\} \cup \{Y_i \mid i \in \mathbb{Z}\} \cup \{a_i \mid i \geq 1\} \cup \{b_{i,\sigma} \mid i \geq 1, \sigma \in \mathbb{Z}^{<\omega}, |\sigma| \geq 1\}.$$

Note that $a_0 \notin C$ and that $B \cong C$ by the map

$$X \mapsto X, \qquad Y_i \mapsto Y_i, \qquad a_i \mapsto a_{i+1}, \qquad b_{i,\sigma} \mapsto b_{i+1,\sigma}.$$

We claim that $C \prec_1 B$.

Suppose that $\bar{u} \in B$ and $\bar{v} \in C$, and $\varphi(\bar{x}, \bar{y})$ is a quantifier-free formula with

$$B \models \varphi(\bar{u}, \bar{v}).$$

Since $R \subseteq C$, we may include the elements of R that appear in the formula φ as elements of $\bar{v}$ to assume that φ is just a formula in the language of rings. There is k such that $\bar{u}$ and $\bar{v}$ are in the subring of B generated by

$$\{X\} \cup \{Y_i \mid -k \leq i \leq k\} \cup \{a_i \mid i \in \omega\} \cup \{b_{i,\sigma} \mid i \in \omega, \sigma \in \{-k, \dots, k\}^{<\omega}\}.$$

Let B_k denote this subring.

Consider also the subring of B generated by

$$\{X\} \cup \{Y_i \mid -k \le i \le k\} \cup \{a_i \mid i \ge 1\} \cup \{b_{i,\sigma} \mid i \ge 1, \sigma \in \{-k,\ldots,k\}^{<\omega}\} \cup \{b_{1,\langle k+1\rangle^\frown\sigma} \mid i \in \omega, \sigma \in \mathbb{Z}^{<\omega}\}.$$

This is a subring of C; denote it by C_k.

Now consider the isomorphism $f\colon B_k \to C_k$ which maps

$$\begin{aligned} X &\mapsto X & Y_i &\mapsto Y_i, \quad \text{for } -k \le i \le k \\ a_0 &\mapsto b_{1,\langle k+1\rangle} & a_i &\mapsto a_i, \quad \text{for } i \ge 1 \\ b_{0,\sigma} &\mapsto b_{1,\langle k+1\rangle^\frown\sigma} & b_{i,\sigma} &\mapsto b_{i,\sigma}, \quad \text{for } i \ge 1 \end{aligned}$$

It is easy to check that this is in fact an isomorphism and that it fixes $\bar{v}$. Note that $\bar{u}, \bar{v} \in B_k$ and $f(\bar{u}), \bar{v} \in C_k \subseteq C$. So

$$C \models \varphi(f(\bar{u}), \bar{v}).$$

This shows that $C \prec_1 B$. So B has no d-Σ_2 Scott sentence. □

As for computable Scott sentences, we can use an argument similar to Theorem 2.7. We will need the following facts:

Theorem 3.3 (Corollary 2.9 of [BCM81])**.** *Suppose the commutative ring R is submodule computable. There is a computable procedure which, when applied to a finite presentation of a commutative R-algebra A and a finite set $u_1, \ldots, u_\ell$ of words of A yields a finite presentation on the given generators for the R-subalgebra of A generated by $u_1, \ldots, u_\ell$.*

Theorem 3.4 (Corollary 2.11 of [BCM81])**.** *Suppose the commutative ring R is submodule computable. Let A and B be finitely generated commutative R-algebras given by finite presentations. Then there is an effective procedure to determine of an arbitrary function ψ from the given generators of A to words of B whether ψ defines a homomorphism and, if so, find a set of generators for the kernel of ψ and determine whether ψ defines a monomorphism. The procedure is uniform in the given data.*

Theorem 3.5 (Corollary 6.45 of [BW93])**.** *Given R either a computable field or $\mathbb{Z}$, there is an algorithm to decide, given $f, g_1, \ldots, g_\ell \in R[x_1, \ldots, x_n]$, whether f is in $R[g_1, \ldots, g_\ell]$.*

Proof. Corollary 6.45 of [BW93] contains the proof when R is a computable field. The proof goes through for $\mathbb{Z}$ as well, see [KRK84]. □

Then we prove:

Theorem 3.6. *Let A be a finitely generated algebra over a computable field k or over $\mathbb{Z}$. Then R has a computable d-Σ_2 Scott sentence.*

Proof. Let $\bar{a} = (a_1, \ldots, a_n)$ be a set of generators for A over k. We claim that the orbit of $\bar{a}$ is defined by a computable Π_1 formula, which implies that A has a computable d-Σ_2 Scott sentence. Since k is submodule computable, A has a computable copy. So the atomic type of $\bar{a}$ is computable.

Suppose that $b_1, \ldots, b_n \in A$ generate a proper subalgebra B of A which is isomorphic to A. Let $k[\bar{x}, \bar{y}] = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring on $2n$ variables. Consider the map which takes $x_i \mapsto a_i$ and $y_i \mapsto b_i$. Let I be the kernel of this map, so that it induces an isomorphism $k[\bar{x}, \bar{y}]/I \to A$. Note that the map $B \to k[\bar{x}, \bar{y}]/I$ which takes $b_i \mapsto x_i$ is an embedding. Let $p_1(\bar{x}, \bar{y}), \ldots, p_\ell(\bar{x}, \bar{y})$ generate I. Then

$$M \vDash (\exists \bar{v})[p_1(\bar{b}, \bar{v}) = \cdots = p_\ell(\bar{b}, \bar{v}) = 0].$$

On the other hand, as argued previously

$$A \vDash (\forall \bar{v})[p_1(\bar{b}, \bar{v}) \neq 0 \vee \cdots \vee \cdots = p_\ell(\bar{a}, \bar{v}) \neq 0].$$

Call this formula $\varphi_{\bar{b}}$.

Let χ be the Π_1 formula which is the conjunction of the atomic type of $\bar{a}$. Consider the Π_1 formula

$$\chi(\bar{u}) \wedge \bigwedge_{\langle \bar{b} \rangle \cong A, \langle \bar{b} \rangle \neq A} \varphi_{\bar{b}}(\bar{u}).$$

This defines the orbit of $\bar{a}$. It is computable by the theorems cited above. □

4 Groups

For finitely generated algebras and modules over Noetherian rings, we used the fact that all such structures are finitely presented and Hopfian to show that they have d-Σ_2 Scott sentences. It is not true that every non-abelian group is finitely presented, Hopfian, or has a d-Σ_2 Scott sentence. It is not even true that every finitely presented group is Hopfian, e.g., the Baumslag-Solitar group $B(2,3)$. But we can still get something by applying the methods used above for finitely presented groups. We will use the following consequence of (the proof of) the splitting lemma for non-abelian groups.

Lemma 4.1 (Splitting Lemma)**.** *Let A and B be groups. Given a surjection $f\colon A \to B$, the following are equivalent:*

1. *There exists a morphism $g : B \to A$ such that fg is the identity on B,*
2. *B splits as a semidirect product $B = \ker(f) \rtimes g(B)$.*

Now we prove the following characterization of when a finitely presented group has a d-Σ_2 Scott sentence. The equivalence of (1) and (2) was already known for finitely generated structures in general (see Theorem 1.1); the new content is (3), which is a purely group-theoretic characterization and relies specifically on the assumption that G is finitely generated.

Theorem 1.3. *Let G be a finitely presented group. Then the following are equivalent:*

1. *G does not have a d-Σ_2 Scott sentence,*
2. *G contains a proper subgroup $H \cong G$ as a 1-elementary substructure,*

3. *G contains a proper subgroup $H \cong G$ with the property that for every finite set of non-identity elements $a_1, \dots, a_n \in G$, there is a normal subgroup $H' \subseteq G$ such that $G = H' \rtimes H$ and $a_1, \dots, a_n \notin H'$.*

Proof. (1) is equivalent to (2) by Theorem 1.1. We will show that (2) implies (3) and that (3) implies (2). Only the fact that (2) implies (3) uses the fact that G is finitely presented as opposed to just finitely generated.

To see that (3) implies (2), we will argue that the subgroup H in (3) is a 1-elementary substructure of G. Let φ be a quantifier-free formula and let $\bar{g} \in G$ and $\bar{h} \in H$ be such that

$$G \vDash \varphi(\bar{g}, \bar{h}).$$

By writing φ in disjunctive normal form, we may assume that φ is a conjunction of atomic formulas. By expanding the tuple $\bar{g}$ and $\bar{h}$, we may assume that φ is a conjunction of formulas of the following forms, where t_i, t_j, and t_ℓ are elements of $\bar{g}, \bar{h}$:

$$t_i = t_j, \qquad t_i \neq t_j, \qquad t_i + t_j = t_\ell.$$

We may also remove any conjunct that only involves $\bar{h}$ and not $\bar{g}$, as $\bar{h}$ will be fixed. After doing this, we can write

$$\varphi(\bar{x}, \bar{h}) \equiv \psi(\bar{x}, \bar{h}) \wedge \left[\bigwedge_{(i,j) \in I} g_i \neq g_j \right] \wedge \left[\bigwedge_{(i,j) \in J} g_i \neq h_j \right]$$

for some sets of indices I and J, where ψ contains all of the formulas of the first and third type above (i.e., all of the positive formulas which are maintained under homomorphism). By (3), there is a normal subgroup $H' \subseteq G$ such that:

- $G = H' \rtimes H$,
- for $(i, j) \in I$, $g_i g_j^{-1} \notin H'$, and
- for $(i, j) \in J$, $g_i h_j^{-1} \notin H'$.

Let $\pi \colon G \to H$ be the projection of G onto H. Let $g_i' = \pi(g_i)$ and $\bar{g}'$ be the tuple of g_i'. We claim that $H \vDash \varphi(\bar{g}', \bar{h})$. As π is a homomorphism, $H \vDash \psi(\bar{g}', \bar{h})$. For $(i, j) \in I$, we also have that $g_i' \neq g_j'$ as $g_i' g_j'^{-1} \notin H'$. Similarly, for $(i, j) \in J$, we have that $g_i' \neq h_j'$. Thus $H \vDash \varphi(\bar{g}', \bar{h})$ and so $H \preceq_1 G$.

We will now argue that (2) implies (3). Suppose that $H \cong G$, $H \preceq_1 G$, and fix $a_1, \dots, a_k \in G - \{e\}$. We will argue below that there is a surjective homomorphism $g \colon G \to H \cong G$ which admits the inclusion $f \colon H \to G$ as a section, and moreover so that $a_1, \dots, a_k \notin \ker f$. The splitting lemma for non-abelian groups says that in this case $G = \ker f \rtimes H$.

Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_\ell \rangle$ be a finite presentation. Let F_n be the free group on $x_1, \dots, x_n$ and let $b_1, \dots, b_n$ be generators of H. (Recall that $H \cong G$, so we can just take $b_1, \dots, b_n$ to be the isomorphic images of the n generators of G.) Let $v_1, \dots, v_k \in F_n$ and

$w_1, \dots, w_n \in F_n$ be words whose images under the quotient map $F_n \to G$ are $a_1, \dots, a_k$ and $b_1, \dots, b_n$, i.e., such that $a_i \equiv v_i \pmod{r_1, \dots, r_\ell}$ and $b_i \equiv w_i \pmod{r_1, \dots, r_\ell}$. Then

$$G \cong \langle H * F_n \mid b_1 = w_1, \dots, b_n = w_n; r_1, \dots, r_\ell \rangle.$$

Let A be the kernel of the quotient map $H * F_n \to G$. Now as witnessed by $x_1, \dots, x_n$,

$$G \vDash (\exists u_1, \dots, u_n) \left[\left(\bigwedge_{i=1}^{n} b_i = w_i(u_1, \dots, u_n) \right) \wedge \left(\bigwedge_{i=1}^{\ell} r_i(u_1, \dots, u_n) = 0 \right) \wedge \left(\bigwedge_{i=1}^{n} v_i(u_1, \dots, u_n) \neq 0 \right) \right].$$

So, as $H \preceq_1 G$, the same is true of H. Let $c_1, \dots, c_n \in H$ be witnesses to this. Then the surjection $H * F_n \to H$ which maps H to H and x_i to c_i has A in its kernel and so factors through $G \cong H * F_n / A$. Note that this map admits the inclusion $H \to H * F_n / A$ as a section. Finally, $a_i \in G$ maps to $v_i(c_1, \dots, c_n) \neq 0$, so a_i is not in the kernel of this map. □

It was remarked in the introduction—see Corollary 1.4—that every finitely presented Hopfian group has a d-Σ_2 Scott sentence. There are examples of finitely presented non-Hopfian groups, such as the Baumslag-Solitar group $B(2,3)$. But (3) in our characterization is much stronger than simply being non-Hopfian, and so we can apply our characterization to show that $B(2,3)$ has a d-Σ_2 Scott sentence.

Originally introduced by Baumslag and Solitar [BS62] to provide examples of non-Hopfian groups, the Baumslag-Solitar groups $B(m,n)$ are now important counter-examples and test cases in combinatorial group theory. The group $B(m,n)$ is the one-relator group with presentation

$$\langle a, t \mid t a^m t^{-1} = a^n \rangle.$$

Baumslag and Solitar proved:

Theorem 4.2 (Baumslag and Solitar [BS62])**.** *$B(m,n)$ is Hopfian if and only if $m = 1$, $n = 1$, or m and n have the same prime divisors.*

In particular, $B(2,3)$ is non-Hopfian; the map $a \mapsto a^2$, $t \mapsto t$ an epimorphism which is not an isomorphism.

Theorem 4.3. *Every Baumslag-Solitar group $B(m,n)$ has a d-Σ_2 Scott sentence.*

Proof. If $BS(m,n)$ is Hopfian, then we are done. So we may assume that $|m|, |n| \neq 1$. Let $G = BS(m,n)$ and suppose to the contrary that $G = H' \rtimes H$, where $H \cong BS(m,n)$. We use the following lemma:

Lemma 4.4 (See Lemma 2.1 of [KRK12] or Propositions 1 and 2 of [Mol91])**.**

1. *If $x \in BS(m,n)$ and two powers of x are conjugate, then x is conjugate to a power of a.*
2. *The elements a^p and a^q are conjugate in $BS(m,n)$ if and only if $m \mid p$ and $n \mid q$, or $m \mid q$ and $n \mid q$.*

Let $f: G \to H$ be an isomorphism. Then $f(a)$ is conjugate to a power of itself in G, so by the lemma, $f(a)$ is conjugate to a power of a, say $f(a) = ua^k u^{-1}$. Without loss of generality, replacing H by $\{u^{-1}hu \mid h \in H\}$, by we may assume that $f(a) = a^k$. Writing $b = a^k = f(a)$ and $t = f(s)$ we have $H = \langle b, s \mid sb^m s^{-1} = b^n \rangle$.

By the splitting lemma, let $g: G \to H$ be a homomorphism which is the identity on H. Now $g(a)$ is conjugate to a power of itself in H, so by the lemma applied in H, it is conjugate in H to a power of b; we can write $g(a) = vb^\ell v^{-1}$ with $v \in H$. We also have $g(a^k) = g(b) = b$ since $a^k = b \in H$. So

$$b = g(a^k) = g(a)^k = vb^{k\ell}v^{-1}.$$

Now b and $b^{k\ell}$ are conjugate in H, and so either m or n divides 1. This contradicts the fact that $|m|, |n| \neq 1$. $\square$

References

[AGHTT] Rachel Alvir, Noam Greenberg, Matthew Harrison-Trainor, and Dan Turetsky. Scott complexity of countable structures. Preprint.

[AKM] R. Alvir, J. F. Knight, and C. F. D. McCoy. Complexity of scott sentences. Preprint.

[BCM81] Gilbert Baumslag, Frank B. Cannonito, and Charles F. Miller, III. Computable algebra and group embeddings. *J. Algebra*, 69(1):186–212, 1981.

[BS62] Gilbert Baumslag and Donald Solitar. Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.*, 68:199–201, 1962.

[BW93] Thomas Becker and Volker Weispfenning. *Gröbner bases*, volume 141 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993. A computational approach to commutative algebra, In cooperation with Heinz Kredel.

[CHK+12] J. Carson, V. S. Harizanov, J. F. Knight, K. Lange, C. F. D. McCoy, A. S. Morozov, S. Quinn, C. Safranski, and J. Wallbaum. Describing free groups. *Trans. Amer. Math. Soc.*, 364(11):5715–5728, 2012.

[CHKM06] W. Calvert, V. S. Harizanov, J. F. Knight, and S. Miller. Index sets of computable models. *Algebra Logika*, 45(5):538–574, 631–632, 2006.

[Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Ho17] Meng-Che Ho. Describing groups. *Proc. Amer. Math. Soc.*, 145(5):2223–2239, 2017.

[HTH] Matthew Harrison-Trainor and Meng-Che Ho. Finitely generated groups are universal. Preprint.

[HTH18] Matthew Harrison-Trainor and Meng-Che Ho. On optimal Scott sentences of finitely generated algebraic structures. *Proc. Amer. Math. Soc.*, 146(10):4473–4485, 2018.

[KRK84] Abdelilah Kandri-Rody and Deepak Kapur. Algorithms for computing Gröbner bases of polynomial ideals over various Euclidean rings. In *EUROSAM 84 (Cambridge, 1984)*, volume 174 of *Lecture Notes in Comput. Sci.*, pages 195–206. Springer, Berlin, 1984.

[KRK12] M. Kasal's-Ruis and I. V. Kazachkov. Two remarks on the first-order theories of Baumslag-Solitar groups. *Sibirsk. Mat. Zh.*, 53(5):1007–1012, 2012.

[KS18] Julia F. Knight and Vikram Saraph. Scott sentences for certain groups. *Arch. Math. Logic*, 57(3-4):453–472, 2018.

[Mal40] A. Malcev. On isomorphic matrix representations of infinite groups. *Rec. Math. [Mat. Sbornik] N.S.*, 8 (50):405–422, 1940.

[Mil83] Arnold W. Miller. On the Borel classification of the isomorphism class of a countable model. *Notre Dame J. Formal Logic*, 24(1):22–34, 1983.

[Mol91] D. I. Moldavanskiĭ. On the isomorphisms of Baumslag-Solitar groups. *Ukrain. Mat. Zh.*, 43(12):1684–1686, 1991.

[Ol'92] A. Yu. Ol'shanskiĭ. Almost every group is hyperbolic. *Internat. J. Algebra Comput.*, 2(1):1–17, 1992.

[Sco65] Dana Scott. Logic with denumerably long formulas and finite strings of quantifiers. In *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, pages 329–341. North-Holland, Amsterdam, 1965.