Computability up to homeomorphism

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Abstract

We study computable Polish spaces and Polish groups up to homeomorphism. We prove a natural effective analogy of Stone duality, and we also develop an effective definability technique which works up to homeomorphism. As an application, we show that there is a Δ^0_2 Polish space not homeomorphic to a computable one. We also prove that, for any computable ordinal α, there is an effectively closed set not homeomorphic to any $0^{(\alpha)}$-computable Polish space; this answers a question of Nies. We also prove analogous results for compact Polish groups.

1 Introduction

In this article we focus on the following general problem fundamental to computable mathematics:

Describe computably presentable mathematical structures.

Of course, to formally clarify the problem we need to restrict it to some natural class of mathematical structures and agree on what we mean by a computable presentation for such structures. For instance, Turing [Tur36, Tur37] suggested the following formal definition of a computable real: A real r is computable if there is an effective procedure (Turing machine) which, on input s, outputs a rational q such that $|q - r| < 2^{-s}$. Turing's definition has a natural generalisation to functions. Similarly, we say that a function $f : [0, 1] \to \mathbb{R}$ is computable if there is an effective procedure which, on input s, outputs a tuple of rationals $\langle q_0, \ldots, q_n \rangle$ such that $\sup_{x \in [0,1]} \{|f - \sum_{i=0}^n q_i x^i|\} < 2^{-s}$. Using the formal notion of computability for functions, we can use tools of computability theory to attack the general problem of computable presentability informally stated above. For instance, Myhill [Myh71] showed that there exists a computable function which is continuously differentiable, but its derivative is not computable. In contrast, Pour-El and Richards [PER83] showed that if the second derivative of a computable function f exists (but is not necessarily computable), then the derivative of f has a computable presentation. Results of this kind belong to a field of mathematics called *computable analysis*; see books [PER89, BHW08].

The ideas of Turing can be naturally extended beyond the space of reals to define the classical notion of a computable Polish space [Wei00]. Recall that a metrized Polish space (M, d) has a computable Polish presentation if there exists a countable metric space $((x_i)_{i \in \omega}, \tilde{d})$

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whose completion is *isometrically isomorphic* to (M, d) and, given i, j and n, we can compute $\tilde{d}(x_i, x_j)$ with precision 2^{-n}. In the case of a separable Banach space we also assume that the standard Banach space operations are computable; we omit the definition (see [PER89]). The study of computable presentations of metrized separable spaces has been central to computable analysis for decades. See books [PER89, Wei00], a tutorial survey [BHW08], and also, e.g., [BG09, NS15, GW07, CMS19, HRSS19] for recent results on computable Polish and Banach spaces.

As was first noted in [Mel13], the study of separable spaces up to isometric isomorphism can be viewed as a generalisation of (discrete) computable algebra [AK00, EG00]. With some effort, the techniques and ideas from computable algebra can be adjusted to separable spaces. Beginning with [Mel13] there have been several successful applications of effective algebraic techniques to separable spaces; see [CMS19, MS19, GMKT18, GMNT18, McN17], a PhD thesis [Bro19], and a recent survey [DM].

In the case of Polish *groups* the situation becomes more complex. We of course require the standard group operations to be computable with respect to the computable dense set (to be clarified), but this is not what makes the case of Polish groups different from Banach spaces. Since topological groups are typically studied up to topological isomorphism, we require that the completion of the computable presentation of G is merely algebraically *homeomorphic* to G. The relaxation of isometry to homeomorphism makes it essentially impossible to apply methods developed for Banach spaces and metric spaces up to isometry to Polish groups.

There were however some notable exceptions. For instance, working under the supervision of Nerode, La Roche [LR81] proved that the correspondence between computable algebraic number field extensions and profinite groups is uniformly effective. In particular, computable presentability of a profinite group is completely reduced to the similar problem for the corresponding field extension. Quite interestingly, the algorithmic techniques developed in [LR81] allowed La Roche to prove a theorem on free profinite groups that was new even in the purely algebraic (non-computable) setting, see [Jar74] for the earlier (and weaker) purely algebraic result. Based on the work of La Roche, Smith [Smi81, Smi79] studied "recursive presentations" of profinite groups; these are computable linear inverse systems of finite groups. These results and notions are of course naturally limited to the class of profinite groups, and there had been very little progress in computable topological groups theory for several decades (but see [GR93]).

Beginning with [MM18], there have been a few successful applications of effective algebraic techniques to topological group theory beyond profinite groups. If a group is not profinite then we follow [MM18] and define its computable presentation to be a computable Polish presentation of the underlying space which makes the group operations $\cdot$ and $^{-1}$ computable; see Definition 2.3 for formal details[1]. The main difficulty in such more general investigations is that there is still no general machinery, so every new result seems to require a new method. For example, Melnikov [Mel18] used Pontryagin duality, computable pregeometries, and a result of Dobrica [Dob83] to partially reduce the study of computable compact topological abelian groups to the theory of computable discrete abelian groups.

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[1]We note here that every "recursive" profinite group (in the sense of computable inverse limits) can naturally be viewed as a computably metrized Polish one, and passing from a computable Polish presentation to a "recursive" one requires $0'$, and this is sharp [Mel18].

Greenberg, Melnikov, Nies and Turetsky [GMNT18] used ideas from descriptive set theory, the above-mentioned result of La Roche [LR81], methods of higher recursion theory [Sac90], and the jump inversion technique from effective algebra to study computable totally disconnected groups.

One of the key obstacles here is that essentially nothing is known about computable presentability of Polish *spaces* up to homeomorphism, and our goal is to initiate such a theory. All that is known to date can be found in the very recent short survey [Sel20]. For instance, Selivanov [Sel20] introduced the notion of the degree spectrum of a Polish space up to homeomorphism, and he also answered the first main question for algebraic domains up to homeomorphism; however, the result for algebraic domains does not have implications for Polish spaces since the notion of computability there is rather different. There are of course some results in the literature on computable *topological* spaces (e.g. [WG09]) which naturally hold up to homeomorphism, such as the effective metrization theorem [GW07], but computable presentability of metrized Polish spaces up to homeomorphism seems completely unexplored[2].

We will prove several general results and will develop elements of definability which work up to homeomorphism. Although these are only the first steps, our new techniques will allow us to answer several fundamental questions, including:

(1) Is there a Δ_2^0-presented Polish space not homeomorphic to a computable one?[3]

(2) Is every effectively closed set homeomorphic to a computable Polish space?[4]

We will then apply our techniques to answer similar questions for Polish groups. As we mentioned above, essentially nothing is known about computable presentability up to homeomorphism. Nonetheless, the reader will perhaps be surprised that the two main questions above were open since they are so fundamental and basic. Of course, the analogy of the first question up to isometry is trivial: Just take two points at distance a real number α coding the halting set. But the question is no longer straightforward up to homeomorphism. For instance, Greenberg and Montalbán [GM08] showed that every hyperarithmetical compact countable Polish space has a computable copy. The proof relies on the computability-theoretic analysis of the Cantor-Bendixson process due to Friedman (unpublished notes); in particular, for a countable space the Cantor-Bendixson rank must be hyperarithmetical, and since a countable compact space is homeomorphic to an ordinal the result of Greenberg and Montalbán follows from the similar classical result for ordinals (see, e.g., [AK00]). Thus, if we want to get an example of a Δ_2^0 Polish space not homeomorphic to a computable one, the space must contain a non-trivial perfect kernel. Regarding the second question, we will see that computable presentability of an effectively closed set is related with the ability to decide whether a given open set intersects it.

Now to the results.

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[2]Working independently, Kihara, Hoyrup and Selivanov [HKS] have recently established a number of interesting results on degree spectra of Polish spaces up to homeomorphism.

[3]Takayuki Kihara, Mathieu Hoyrup and Victor Selivanov have recently announced an independent solution.

[4]Thanks to Andre Nies for asking this question. We also thank Alex Galicki for several discussions related to this topic. Alex is involved in a related project which is still in progress.

Stone spaces. Recall that compact and totally disconnected Polish spaces are also called profinite spaces or Stone spaces. The well-known Stone duality states that a countable discrete Boolean algebra B is dual to the profinite space $\widehat{B}$ of its ultrafilters, in the sense that $B \cong_{iso} C$ iff $\widehat{B} \cong_{hom} \widehat{C}$. Greenberg and Montalbán [GM08] (essentially) observed that Stone duality holds arithmetically. Passing from a computably metrized Stone space to the respective Boolean algebra is of course the harder direction, and it seems to require at least one Turing jump. Interestingly, we discovered that the duality holds computably:

Theorem 1.1. *Let B be a (countable, discrete) Boolean algebra. Then the following are equivalent:*

(1) *B has a computable copy;*

(2) *the Stone space $\widehat{B}$ of B has a computable Polish presentation.*

The proof of the theorem is not difficult but is subtle, and it makes an essential use of the well-known result of Downey and Jockusch [DJ94]. We note that Kihara, Hoyrup and Selivanov [HKS] have recently and independently investigated the effective content of Stone duality. Combine the theorem above with the classical theorem of Feiner [Fei70a] to obtain:

Corollary 1.2. *There exists a Δ_2^0-presented profinite Polish space not homeomorphic to any computable Polish space.*

This answers the first main question. Theorem 1.1 essentially completely reduces the theory of computable Stone spaces to computable Boolean algebras; see book [Gon97] for an excellent but somewhat dated exposition of the latter. We state only one of the many corollaries: Every low_4 profinite Polish space is homeomorphic to a computable one (follows from [KS00]).

Effectively closed subspaces. Recall that a closed subspace of a computable Polish space is Π_1^0 or *effectively closed* if its complement is computably enumerable; i.e., there is a computably enumerable set of open balls which makes up the complement. Effectively closed subspaces of 2^ω are called Π_1^0-classes and have been studied extensively. However, not much is known about effectively closed spaces of arbitrary spaces; see, e.g., [CR99, YMT99] for a few results. Note that an effectively closed space does not have to contain a dense set computable in the ambient space. If an effectively closed C does contain a dense uniformly computable sequence then C is called *effectively overt*. Equivalently, a closed set is effectively overt if the set of basic open balls intersecting the set is computably enumerable (folklore; see, e.g., [HRSS19, Mel18] for details). If a set is effectively overt then of course it has a computable presentation under the induced metric. It turns out that effective overtness is essential in producing a computable presentation of a Π_1^0 closed set.

Theorem 1.3. *For each computable ordinal α, there is a computable Polish space M and a Π_1^0 subspace C of M such that C is not homeomorphic to any $0^{(\alpha)}$-computable metric space.*

This answers the second main question raised above in a strong way. It also follows from Theorem 1.3 that there exists a Π_1^0 subspace C which is not $0^{(\alpha)}$-overt. We leave open:

Question 1. Is there an effectively closed X not homeomorphic to any hyperarithmetically represented space?

Locally connected spaces. Although we settled both main questions in general, we would like to test these ideas on spaces from other natural classes. After all, Stone spaces are very specific topological spaces, they are just representations of the well-studied class of countable Boolean algebras. One would naturally expect that techniques required to build examples of Stone spaces will likely be too specific to be of any use outside of this narrow class of spaces.

In particular, the classes of connected and locally connected spaces seem to be on the other end of the technical "spectrum". Recall that there is no logical implication between connectedness and local connectedness. We focus on the locally connected case. As we mentioned above, methods developed in the proof of Theorem 1.1 seem to be of little help. However, the definability techniques developed in the proof of Theorem 1.3 for Π^0_1 sets are more versatile and will allow us to prove:

Theorem 1.4. *There is a Δ^0_2 locally compact and locally path-connected space which is not homeomorphic to any computable Polish space.*

In contrast with Corollary 1.2, the proof of the theorem above is much more direct, in the sense that shows how to build such spaces "by hand" without outsourcing to effective algebraic techniques. Its proof is a priority construction combined with definability, and the complexity of guessing in the proof is at the level of Π^0_3. We conjecture that any construction of a locally connected space witnessing the theorem above must involve a difficult guessing. This is because definability up to homeomorphism in such spaces seems to require at least three quantifiers. The proof of the theorem above is the first example of a $0'''$ argument in computable analysis that we are aware of, but the good news is that there will not be much injury and therefore no special training in $0'''$ arguments is necessary to understand the proof. We leave open:

Question 2. Is there a Δ^0_2 connected Polish space not homeomorphic to a computable one?

Topological groups. We test our methods on the class of Polish groups. We apply techniques similar to those used in Theorem 1.1 for Stone spaces to prove:

Theorem 1.5. *There exists a Δ^0_2 compact Polish abelian group not topologically isomorphic to any computable Polish group.*

The group witnessing the theorem is not profinite but all the information is coded into the profinite factor. In particular, the group is not connected and is merely locally connected. We strongly suspect that with some extra work one can design a profinite group witnessing the theorem above, but we leave it for future work. The connected case seems more challenging.

Question 3. Is there a Δ^0_2 connected Polish group not topologically isomorphic to any computable Polish group?

Finally, we finish the introduction with the theorem which is a version of the second main result Theorem 1.3 for Polish groups.

Theorem 1.6. *For every computable α there exists an effectively closed compact (thus, profinite) subgroup of S_∞ not homeomorphic to a Δ^0_α Polish group.*

The proof of this theorem extends an argument from [GMNT18] and should not be hard to understand to anyone familiar with the standard techniques of computable structure theory [AK00].

2 Formal Definitions

Recall that a real α is *computable* (Turing [Tur36, Tur37]) if there exists a Turing machine that, given $n \in \mathbb{N}$, outputs a rational r within 2^{-n} of α. A Polish space (M, d) is *computable* if there exists a sequence $(\alpha_i)_{i \in \mathbb{N}}$ of M-points which is dense in M and such that, for every $i, j \in \mathbb{N}$, the distance $d(\alpha_i, \alpha_j)$ is a computable real, uniformly in i and j [Wei00]. Given a computable presentation of a Polish space, we call the points $(\alpha_i)_{i \in \mathbb{N}}$ *special points*.

Definition 2.1. Let f be a continuous function between Polish metric spaces M and N. A *name* of f is any collection of pairs of basic open balls (B, C) such that $f(B) \subseteq C$, and for every $x \in M$ and every $\epsilon > 0$ there exists $(B, C) \in \Psi$ such that $B \ni x$ and $r(C) < \epsilon$.

Definition 2.2. A function $f : M \to N$ between computably presented Polish spaces M, N is *computable* if it possesses a c.e. name.

A function is continuous iff it has an X-c.e. name, where X is some oracle.

In a metric space, we say that a Cauchy sequence (x_i) is *fast* if $d(x_i, x_{i+1}) < 2^{-i-1}$. The above definition of a computable map is equivalent to saying that f is represented by a Turing functional that maps fast Cauchy sequences to fast Cauchy sequences (folklore). The definition below was first suggested in [MM18].

Definition 2.3. A computable Polish group is a triple (G, Φ, Ψ), where G is a computable Polish presentation of the underlying metric space and Φ and Ψ are (indices for) c.e. names of group-operations $\cdot$ and $^{-1}$ upon G.

3 Effectively closed subspaces

Theorem 3.1. *For each computable ordinal α, there is a computable Polish space M and a Π^0_1 subspace C of M such that C is not homeomorphic to any $0^{(\alpha)}$-computable Polish space.*

Proof. The proof relies on a definability technique. To develop the technique we first need to prove several lemmas. Some of these lemmas (such as the lemma below) are really folklore, but for completeness sake we include their proofs. The problem with dealing with a computable presentation of a Polish space is that we only really have access to a dense subset of special points, so we cannot for example talk about a path from one point to another. We can however approximate such a path by taking a discrete series of short steps.

Definition 3.2. Let (M, d) be a Polish space. Given special points x, y, an ϵ-path from x to y is a sequence of special points $x = u_0, u_1, \ldots, u_n = y$ such that $d(u_i, u_{i+1}) < \epsilon$.

The existence of a path yields the existence of an ϵ-path for every $\epsilon > 0$.

Lemma 3.3. *Let (M, d) be a Polish space with special points $\langle q_i \rangle_{i \in \omega}$. Suppose that there is a path between special points r and s. Then for every $\epsilon > 0$, there is an ϵ-path from r to s.*

Proof. Let $[0, 1]$ be the unit line, and let L be the image of $[0, 1]$ in X under a continuous map f which sends $0 \mapsto r$ and $1 \mapsto s$. Then L is compact in X as the continuous image of a compact set. Consider the collection of all open balls with radius $\epsilon/2$ around all special points of M; this must cover all of M, and hence in particular covers L. Since L is compact, we can choose special points $p_1, \ldots, p_n$ such that

$$L \subseteq B_{\epsilon/2}(p_1) \cup \cdots \cup B_{\epsilon/2}(p_n).$$

For each i, let $U_i = f^{-1}(B_{\epsilon/2}(p_i))$. This is an open set in $[0, 1]$, and $[0, 1] = \bigcup_i U_i$. Then we can pick $i_0, \ldots, i_n$ and

$$0 < a_{i_1} < b_{i_0} < a_{i_2} < b_{i_1} < a_{i_3} < b_{i_2} < \cdots < a_{i_n} < b_{i_{n-1}} < 1$$

such that $[0, b_{i_0}) \in U_{i_0}$, $(a_{i_j}, b_{i_j}) \in U_{i_j}$, and $(a_{i_n}, 1] \in U_{i_n}$. (Note that the subscripts are somewhat out of order in the above, so that the intervals overlap.) We claim that $r, p_{i_1}, p_{i_2}, \ldots, p_{i_n}, s$ is an ϵ-path. Indeed, for each j, $f(b_{i_j}) \in B_{\epsilon/2}(p_{i_j}) \cap B_{\epsilon/2}(p_{i_{j+1}})$; so

$$d(p_{i_j}, p_{i_{j+1}}) \leq d(p_{i_j}, f(b_{i_j})) + d(f(b_{i_j}), p_{i_{j+1}}) < \epsilon. \qquad \square$$

The converse is not true in general. For example there might be two points r and s at distance 1 from each other, such that for each $n \in \mathbb{N}$ there is a discrete set of n points at distance $1/n$ from each other forming an $1/n$-path from r to s; but no continuous path from r to s. However, this is only possible due to a failure of compactness for the path-components of r and s.

Lemma 3.4. *Let (M, d) be a Polish space with special points $\langle q_i \rangle_{i \in \omega}$. Suppose that each path-component of M is compact and open. If special points r, s are not in the same path-component, then there is an ϵ such that there is no ϵ-path between r and s.*

Proof. Let C be the path-component of r. Since C is open, its compliment is closed, and since C is compact there is a distance ϵ between C and C^c. Then there is no $\epsilon/2$-path from r to s, as given any path $r = u_0, u_1, \ldots, u_n = s$ there must be a first i such that $u_i \in C$ and $u_i \notin C$, and so $d(u_i, u_{i+1}) \geq \epsilon$. $\qquad \square$

The main coding components in our proof will be designed using k-*stars* which are defined below.

Definition 3.5. A k-star is a topological space homeomorphic to k copies of the interval $[0, 1]$ all joined at one end in a single point. Note that a k-star is not homeomorphic to a k'-star for $k \neq k'$.

We are ready to state and prove the main definability lemma.

Lemma 3.6. *Let (M, d) be a Polish space with special points $\langle q_i \rangle_{i \in \omega}$. Suppose that M is homeomorphic to the disjoint union of stars, each of which is compact and open. A special point r is contained within an n-star with $n \geq \ell$ if and only if*

($*$) *there are distinct points $p_1, p_2, \ldots, p_\ell$ in the same path-component as r and a $\delta > 0$ such that for each i, j, k and for every $\epsilon < \delta$ there is an ϵ-path $p_i = u_0, u_1, \ldots, u_n = p_j$ from p_i to p_j such that $u_0, \ldots, u_n \notin B_\delta(p_k)$.*

Proof. First we show that this is true of an ℓ-star. Let $p_1, \ldots, p_\ell$ be the points on each arm of the stars. Given p_i, p_j, and p_k, let δ be sufficiently small that $B_\delta(p_k)$ does not intersect the arms containing p_i and p_j, and also does not intersect S^c. Then there is a path between p_i and p_j in $M - B_\delta(p_k)$, so by Lemma 3.4, for every $\epsilon < \delta$ there is an ϵ-path from p_i to p_j which avoids $B_\delta(x_k)$. (Note that as $B_\delta(p_k)$ is open, $M - B_\delta(p_k)$ satisfies the hypotheses of Lemma 3.4.)

Let S be an n-star in M. If $n < \ell$, then given any distinct special points $p_1, \ldots, p_\ell$, two of them are on the same arm of the star (or one of these points is the center of the star). So we can choose p_i, p_j, p_k such that by removing p_k, the star divides into two connected components, one containing p_i, and the other containing p_j. We must show that for every δ, there is an $\epsilon < \delta$ such that for every ϵ-path $p_i = u_0, u_1, \ldots, u_n = p_j$ from p_i to p_j, there is some $u_i \in B_\delta(p_k)$. We may assume that δ is sufficiently small that $p_i, p_j \notin B_\delta(p_k)$ (otherwise it is trivial).

Now we can write S as a disjoint union $C_i \cup C_j \cup B_\delta(x_k)$ where C_i and C_j are closed sets containing x_i and x_j respectively. Then C_i and C_j are compact, and so we can choose ϵ smaller than the distance between C_i and C_j, and also smaller than the distance between S and the compliment of S. Then any ϵ-path $p_i = u_0, \ldots, u_n = p_j$ in M must have $u_0, \ldots, u_n \in S$ (since the distance between S and the compliment of S is greater than ϵ). Also, since $u_0 \in C_i$, $u_n \in C_j$, and the distance between C_i and C_j is greater than ϵ, for some s, $u_s \in B_{\delta(p_k)}$. So there is no ϵ-path from p_i to p_j avoiding $B_\delta(p_k)$. $\square$

For the computability-theoretic part of our proof we rely on the following simple lemma.

Lemma 3.7. *Let R be a hyperarithmetic relation. Then there is a computable sequence of trees $T_n \subseteq \omega^{<\omega}$ such that if $n \in R$, then T_n has a single path, and if $n \notin R$, then T_n has no path.*

Proof. Let α be such that R is $0^{(\alpha)}$-computable. It is well-known that for each computable ordinal α, there is a computable tree T with a single path $f \equiv_T 0^{(\alpha)}$. (By Proposition II.4.1 of [Sac90], $0^{(\alpha)}$ is a Π^0_2 singleton, and following Theorem 3.1 of [JM69] we can replace $0^{(\alpha)}$ by a lexicographically least Skolem function $f \equiv_T 0^{(\alpha)}$ such that f is a Π^0_1 singleton.) Let Φ be a Turing functional such that $R = \Phi^f$. Then for each n, let T_n be the tree with $\sigma \in T_n$ if and only if $\sigma \in T$ and $\Phi^\sigma(n) \uparrow$ or $\Phi^\sigma(n) = 1$. Then T_n has at most one path since $T_n \subseteq T$ and T has at most one path, and if T_n has a path, that path is f. If $n \in R$, then f is still a path of T_n; and if $n \notin R$, then for some $\sigma \preceq f$, $\Phi^\sigma(n) = 0$, and so $\sigma \notin T_n$ and f is not a path of T_n. $\square$

We return to the proof of the theorem.

We will build M as a computable Polish space with special points $(q_i)_{i\in\omega}$ and metric d. The Π^0_1 subspace C of M will be the disjoint union of stars, each of which is compact and open.

Let $(X_n, d_n, (r^n_i)_{i\in\omega})$ be a list of the (possibly partial) $0^{(\alpha)}$-computable metric spaces. Let R be the hyperarithmetic relation with $n \in R$ if and only if X_n is a metric space which

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contains a special point r satisfying $(*)$ of Lemma 3.6 for $\ell = n+3$, but not satisfying $(*)$ of Lemma 3.6 for $\ell = n+4$. Then $n \in R$ if and only if X_n has an $n+3$-star. (In $(*)$, we ask that $p_1, \ldots, p_\ell$ are in the same path component as r; we express this by saying that for each i and for every ϵ there is an ϵ-path from r to p_i, as in Lemma 3.4. If X_n is homeomorphic to C, then it will satisfy the hypotheses of Lemmas 3.4 and 3.6. If it is not homeomorphic to C then the guessing procedure will still make sense although its outcome may be topologically meaningless.)

Using Lemma 3.7, let $T_n \subseteq \omega^{<\omega}$ be a computable sequence of trees such that if $n \in R$ then T_n has no path, but if $n \notin R$ then T_n has a unique path. Let M be the disjoint union, over $n \in \omega$, of $\omega^\omega \times S_{n+3}$ where S_{n+3} is a particular chosen computable presentation of an $n+3$-star. Set each component $\omega^\omega \times S_{n+3}$ to be at distance 1 from each other such component. As a metric space, we use the sum metric on each component $\omega^\omega \times S_{n+3}$, i.e., if (f,x) and (g,y) are in the same component, we set $d((f,x),(g,y)) = d_{\omega^\omega}(f,g) + d_{S_{n+3}}(x,y)$.

Now we will define the Π^0_1 set C. Whenever we see $\sigma \notin T_n$, put $[\sigma] \times S_{n+3} \notin C$. To see that we can do this effectively, we must note that we can write $[\sigma] \times S_{n+3}$ as an effective union of basic open balls

$$\bigcup_{n \in \omega} \bigcup_{p \in S_{n+3}} B_{2^{-|\sigma|}}(\sigma\hat{\ }n, p)$$

where p ranges over special points in the chosen computable presentation of S_{n+3}. If $n \in R$, then C is disjoint from $\omega^\omega \times S_{n+3}$, and so does not have an $n+3$-star; and if $n \notin R$, then $C \cap \omega^\omega \times S_{n+3} = \{f\} \times S_{n+3}$ where f is the path through T_n, and so C has an $n+3$-star. Thus C is homeomorphic to the the disjoint union of an $n+3$-star for $n \notin R$, and each of these stars is compact and open.

We claim that C is not homeomorphic to any $0^{(\alpha)}$-computable metric X_n. Indeed, suppose to the contrary that X_n was homeomorphic to C. Then X_n satisfies the hypotheses of Lemmas 3.4 and 3.6, and so we have that X_n has an $n+3$-star if and only if $n \in R$ if and only if C does not have an $n+3$-star. So X_n and C cannot be homeomorphic. $\qquad\square$

4 Δ^0_2 Polish spaces with no computable homeomorphic copies

4.1 Totally disconnected spaces

Recall that a Polish space is profinite if it is compact and totally disconnected. The well-known Stone duality states that a countable discrete Boolean algebra B is dual to the profinite space $\widehat{B}$ of its ultrafilters, in the sense that $B \cong_{iso} C$ iff $\widehat{B} \cong_{hom} \widehat{C}$; that is, the isomorphism type of a countable discrete Boolean algebra is uniquely determined by the homeomorphism type of its dual profinite space.

Theorem 4.1. *Let B be a (countable, discrete) Boolean algebra. Then the following are equivalent:*

(1) *B has a computable copy;*

(2) *the Stone space $\widehat{B}$ of B has a computable Polish presentation.*

Proof. Given a computable Boolean algebra B, produce a computable tree-basis of B [Gon97]. Recall that a tree basis is the set of generators of B which form a tree under the standard $\leq$ with root $\mathbf{1}$, such that every generator y in the tree has either no children (and then it is an atom) or exactly two children whose disjoint union is equal to y. Interpret the tree basis as a dense subset of closed subspace of 2^ω under the usual ultrametric. This gives a computable Polish presentation of $\widehat{B}$.

Now suppose the Stone space $\widehat{B}$ of B has a computable Polish presentation.

Lemma 4.2. *Suppose M is a computable compact Polish metric space. If M is not connected then $0'$ can produce a splitting of M into two disjoint clopen components. Furthermore, $0'$ can compute two representations for each of the two components: one via a finite union of basic* open *balls, and the other via a finite union of basic* closed *balls.*

Proof. Suppose M_0 and M_1 are two non-intersecting clopen components of M such that $M_0 \cup M_1 = M$. Since M is compact and each M_i is closed, M_i is compact. Since M_i is open and compact it is equal to a finite union of open balls, say $M_0 = \bigcup_{i=1,\ldots,k} B_i$ and $M_1 = \bigcup_{j=1,\ldots,n} D_i$, where B_i, D_j are basic open balls. If cl stands for the closure operator then $M_0 = cl(M_0) = \bigcup_{i=1,\ldots,k} cl(B_i)$, and similarly for M_1. Identify M_i with the respective finite cover by *closed* balls.

Now we will show that $0'$ can search for M_1 and M_2, as represented above. Suppose we have two subsets M_1 and M_2 of M, represented by finite unions of balls B_i and D_j respectively, but we do not know if M_1 and M_2 are disjoint and we do not know whether they cover M.

We first claim that the property $M_0 \cup M_1 = M$ becomes Π_1^0. Indeed, to see if $M_0 \cup M_1 = M$ it is sufficient to search for a special point in the open set $M \smallsetminus (M_0 \cup M_1)$, i.e., outside of all the finitely many closed balls. This is a Σ_1^0 process which is of course uniform in the finite tuple describing of the balls B_i and D_j.

To guarantee that M_0 and M_1 are also disjoint we must check whether

$$\bigcup_{i=1,\ldots,k} cl(B_i) \cap \bigcup_{j=1,\ldots,n} cl(D_j) = \varnothing$$

which is reduced to verifying finitely many statements of the form $cl(B_i) \cap cl(D_j) = \varnothing$. We suppress i and j. Let B^ϵ be the basic *open* ball with the same centre as B but having radius $r(B) + \epsilon$, where ϵ is a positive rational number. Define D^ϵ similarly. We claim that $cl(B) \cap cl(D) \neq \varnothing$ is equivalent to

$$(\forall \epsilon > 0)\, B^\epsilon \cap D^\epsilon \neq \varnothing.$$

One implication is trivial. For the other implication, assume x_ϵ is a point witnessing non-emptiness for ϵ. By compactness, $(x_{2^{-m}})_{m \in \mathbb{N}}$ has a converging subsequence. The limit of this sequence will be a point witnessing $cl(B) \cap cl(D) \neq \varnothing$. We have just shown that $cl(B) \cap cl(D) \neq \varnothing$ is a Π_2^0-property, which makes $M_0 \cap M_1 = \varnothing$ a Σ_2^0-property.

It follows that $0'$ can search for finitely many basic open B_i and D_j as above. If *some* decomposition of M exists then we will eventually find (perhaps, some other) decomposition of M. Furthermore, for this fixed decomposition $0'$ will be able to see the first found ϵ for which the property $(\forall \epsilon > 0)\, B^\epsilon \cap D^\epsilon \neq \varnothing$ fails. Then the clopen components can be represented as unions of B_i^ϵ and D_j^ϵ rather than B_i and D_j. $\qquad\square$

The idea is to iterate the lemma above to get a $0'$-computable presentation of B with a $0'$-computable set of atoms. This is done as follows. Given a computable Polish presentation of $\widehat{B}$ and using $0'$, initiate the procedure of splitting $\widehat{B}$ into clopen disjoint subsets. At every stage we will have a finite collection of clopen subsets of $\widehat{B}$, and each of these sets will be represented as a union of finitely many open balls as well as a finite union of closed balls. In particular, although the procedure of splitting the space is merely $0'$-computable, at every stage each of the components will naturally be a computable Polish space. To see why, list all special points of the ambient space which belong to the finitely many open balls describing the space. Since the space is also closed, the completion of this set of points will be equal to the whole component. Therefore, we can apply the lemma again to each of the components.

The iterated procedure ensures that the set-theoretic inclusion between the produced clopen components is decidable relative to $0'$. Since the space $\mathcal{B}$ is totally disconnected, each pair of points must belong to disjoint clopen sets. In particular, whenever we are given a clopen subspace represented as a finite union of basic balls, X is an isolated point iff it contains exactly one special point. This property can be decided using $0'$.

It follows that we can produce a Δ_2^0 copy of the Boolean algebra of the clopen subsets of $\widehat{B}$, which furthermore has the atom relation of complexity Δ_2^0. It remains to apply the well-known theorem of Downey and Jockusch [DJ94] who showed that every Δ_2^0 Boolean algebra with Δ_2^0 atom relation is isomorphic to a computable one. □

Thus we obtain:

Corollary 4.3. *There exists a Δ_2^0-presented profinite Polish space not homeomorphic to any computable Polish space.*

Proof. By Feiner [Fei70b], there is a Δ_2^0 Boolean algebra B with no computable copy. By the previous theorem, the Stone space $\hat{B}$ of B is a Δ_2^0-presented Polish space not homeomorphic to any computable Polish space. □

4.2 The locally path-connected case

We say that a Polish space is an LCPC space if it is both locally compact and locally path-connected. Such spaces are more reflective of physical geometry and are in some sense the opposite of totally disconnected spaces. We prove:

Theorem 4.4. *There is a Δ_2^0 LCPC space which is not homeomorphic to any computable Polish space.*

We begin by proving a computational lemma about building computable spaces depending on the answer to a Π_3^0 question.

Lemma 4.5. *Let R and S be Π_3^0 sets and $(k_n)_{n\in\omega}$ a computable sequence of natural numbers ≥ 3. Then there is, uniformly in n, a Δ_2^0 separable metric space (M_n, d_n) such that:*

- *if $n \notin R$, then (M_n, d_n) is the disjoint union of a point and $k_n + 1$ line segments;*

- *if $n \in R$ and $n \notin S$, then (M_n, d_n) is the disjoint union of a k_n-star and a line segment;*

- *if $n \in R$ and $n \in S$, then (M_n, d_n) is a $(k_n + 1)$-star.*

Proof. Let
$$n \in R \iff \forall x \exists y \forall z R^*(x, y, z, n).$$
We may assume that for each x, there is at most one y with $\forall z R^*(x, y, z, n)$, and that if $x' < x$, and there is no witness y for x', then there is no witness y for x (folklore). Similarly, let
$$n \in R \cap S \iff \forall x \exists y \forall z S^*(x, y, z, n).$$
Fix n and let $k = k_n$. We will define $M = M_n$ as a subspace of a $(k + 1)$-star. Let the $(k + 1)$-star have a center point c and arcs $A_1, \ldots, A_{k+1}$ with $A_i = f_i[0, 1]$ and $f_i(0) = c$. Then we let

$$M = \{c\} \cup \bigcup \{f_i(1/x, 1] : 1 \le i \le k, \ x \in \mathbb{N}, \text{ and } \exists y \forall z R^*(x, y, z, n)\}$$
$$\cup \bigcup \{f_{k+1}(1/x, 1] : x \in \mathbb{N} \text{ and } \exists y \forall z S^*(x, y, z, n)\}.$$

More formally, this description of M gives a Σ_2^0 way of deciding which special points of the $(k + 1)$-star to include in M; then as M is Δ_2^0, we can build a computable copy of this subspace of $(k + 1)$-star. $\square$

Now we prove the theorem.

Proof of Theorem 4.4. Let $(M_n, d_n)_{n \ge 1}$ be a list of all the (possible partial) computable Polish spaces. We will construct a Δ_2^0 Polish space (M^*, d^*) while diagonalizing against each $(M_n, d_n)_{n \in \omega}$. The space (M^*, d^*) will be the disjoint union of infinitely many stars, points, and line segments, each separated from the others by open sets. (For example, for any star, there will be an open set containing it and nothing else.) We will make sure that (M^*, d^*) is not isomorphic to (M_n, d_n) by either having, for some k, a k-star in (M^*, d^*) when there is no k-star in (M_n, d_n), or vice versa.

The space (M^*, d^*) will be built entirely by constructing Π_3^0 sets R and S, and a computable sequence k_n, and then letting (M^*, d^*) be the disjoint union of the sequence obtained by Lemma 4.5, with each space in the union set at distance one from the others. The resulting space will be LCPC since each component is.

Recall from Lemma 3.6 that, in a space homeomorphic to a disjoint union of stars, points, and line segments, each of which is compact and open, a special point r is contained within an n-star with $n \ge \ell$ if and only if:

(†) there are distinct points $p_1, p_2, \ldots, p_\ell$ in the same path-component as r such that for each i, j, k $(i, j \ne k)$, there is a δ such that for every $\epsilon < \delta$ there is an ϵ-path $p_i = u_0, u_1, \ldots, u_n = p_j$ from p_i to p_j such that $u_0, \ldots, u_n \notin B_\delta(p_k)$.

Thus for a fixed ℓ, asking whether a point r is contained within an n-star with $n \ge \ell$ is Σ_3^0; and given $p_1, \ldots, p_\ell$, is it Π_2^0 to ask whether $p_1, \ldots, p_\ell$ are points on ℓ different arms of an n-star, $n \ge \ell$. For simplicity, we will write, for example, that an element x is contained in a $(\ge \ell)$-star to mean that it is contained in an n-star for some $n \ge \ell$. Often when talking about one of the spaces (M_n, d_n), which may not be a disjoint union of stars, points, and line segments, we will define a formula using a subformula which says that an element is

contained in a $(\geq \ell)$-star; what we mean is that the subformula expresses this as in $(*)$ even though in the space (M_n, d_n) this might not actually mean that the element is contained in a $(\geq \ell)$-star. To remind the reader of this, we will write e.g. "contained† in a $(\geq \ell)$-star" when we have this formal syntactic meaning in mind.

To begin, we will describe how to make (M^*, d^*) non-homeomorphic to a single computable metric space (M, d). First, ask whether (M, d) has a (≥ 3)-star at all, as expressed using $(\dagger)$. It is Π^0_3 to say that (M, d) has no such star, and in this case (using Lemma 4.5) we can build a 3-star in (M^*, d^*), and so (M^*, d^*) is not homeomorphic to (M, d); if (M, d) does have such a star, then Lemma 4.5 builds a point and three line segments. This is enough to make (M^*, d^*) non-homeomorphic to (M, d), but we want our diagonalization to be more robust; for, if (M, d) has a (≥ 3)-star, it would be very limiting to say that we can never build any star in (M^*, d^*).

So now list the triples $(x^1_i, x^2_i, x^3_i)_{i \in \omega}$ of points from (M, d). If x^1_1, x^2_1, x^3_1 are part of a (≥ 3)-star, in the sense that they are witnesses to $(\dagger)$, then build a 3-star (and a line segment) if there is a fourth element y such that x^1_1, x^2_1, x^3_1, y are part of a (≥ 4)-star in the sense of $(\dagger)$. Build a 4-star if there is no such y, and x^1_1, x^2_1, x^3_1 are just part of a 3-star. Asking whether x^1_1, x^2_1, x^3_1 are part of a (≥ 3)-star is Σ^0_2, and asking whether such a y exists is Σ^0_3; so Lemma 4.5 allows us to build a 3-star or a 4-star as desired. (If x^1_1, x^2_1, x^3_1, y are not part of a (≥ 3)-star, then Lemma 4.5 builds a point and 4 line segments.) So if x^1_1, x^2_1, x^3_1 are part of a (≥ 3)-star in (M, d), then if they are part of a 3-star then (M^*, d^*) has no 3-star, and if they are part of a (≥ 4)-star then (M^*, d^*) has no (≥ 4)-star.

Now we want to do the same thing for x^1_2, x^2_2, x^3_2, except that we do not want to build a 3-star or 4-star if we already built one for x^1_1, x^2_1, x^3_1; if we had built a 4-star because (M, d) had a 3-star, we must make sure that we never build a 3-star in (M^*, d^*). So if x^1_1, x^2_1, x^3_1 are part of a (≥ 3)-star in (M, d), then when acting for the sake of x^1_2, x^2_2, x^3_2 we build a point and 4 line segments; and otherwise, do the same for x^1_2, x^2_2, x^3_2 as we did above for x^1_1, x^2_1, x^3_1.

In the end, either (M, d) has no (≥ 3)-star, in which case the very first action we took built a 3-star in (M^*, d^*); or, for some least i, x^1_i, x^2_i, x^3_i is part of a 3-star in (M, d), and then we built no 3-star in (M^*, d^*); or x^1_i, x^2_i, x^3_i is part of a (≥ 4)-star in (M, d), and we built only a 3-star in (M^*, d^*). So (M^*, d^*) is not homeomorphic to (M, d).

With more computable metric spaces to diagonalize against, we must make sure that they do not add k-stars to (M^*, d^*) that interfere with each other. For example, if x^1_i, x^2_i, x^3_i from above was part of a (≥ 4)-star in (M, d) (say a k-star), and we built a 3-star in (M^*, d^*), we must make sure that we never build a k-star in (M^*, d^*) for any other reason. So other parts of the construction will have to guess at the value of k.

Now we are ready to describe the entire construction. To organize the Π^0_3 sets we feed into Lemma 4.5, we will build a tree. The nth level of the tree will contain attempts to diagonalize against (M_n, d_n). At the nth level of the tree, a node σ with predecessors $\sigma_1, \ldots, \sigma_{n-1}$, consists of the following:

(1) a label which is either ∞ or f (for finite);

(2) for each $i < n$ with σ_i labeled f, a value $k_\sigma[i]$ which is either the symbol "$\geq 4n$" or a natural number $3 \leq k_\sigma[i] < 4n$; if $k_\sigma[i]$ is "$\geq 4n$" then there are also elements $y^1_\sigma[i], \ldots, y^{4n}_\sigma[i] \in M_i$, and if $k_\sigma[i] \leq 4n$ then there are elements $y^1_\sigma[i], \ldots, y^{k_\sigma[i]}_\sigma[i] \in M_i$;

(3) a number ℓ_σ which is the least odd number $< 4n$ such that $\{\ell, \ell+1\}$ is disjoint from

$$\{\ell_\tau, \ell_\tau + 1 : \tau \text{ is a predecessor of } \sigma\} \cup \{k_\sigma[i] : i < n\};$$

(4) if the label is f, elements $x_\sigma^1, \ldots, x_\sigma^{\ell_\sigma} \in M_n$.

(We suppress σ in ℓ_σ and $k_\sigma[i]$.) The number ℓ is the size of star that σ will be building; σ will try to build either an ℓ-star or an $(\ell+1)$-star, just as in the procedure for a single diagonalization described previously we built either a 3-star or a 4-star. The values $k[i]$ are the guesses at the sizes of the stars used to diagonalize against (M_i, d_i); and we must choose ℓ so that building an ℓ-star or an $(\ell+1)$-star for σ would not interfere with the diagonalization against (M_i, d_i). The label is a guess at whether (M_n, d_n) will have a star of size $\geq \ell$; the label ∞ corresponds to having no such star, and the label f corresponds to having such a star. The value $< 4n$ is simply chosen so that there will be some odd ℓ with $\ell < 4n$ such that $\{\ell, \ell+1\}$ is disjoint from

$$\{\ell_\tau, \ell_\tau + 1 : \tau \text{ is a predecessor of } \sigma\} \cup \{k_\sigma[i] : i < n\}.$$

For each node σ of the tree at level n, and each possible choice of these parameters at level $n+1$, there is a *single* child τ of σ at level $n+1$ with those parameters. At each level, order the children of each node from left to right, with order type ω (so that each node has finitely many other nodes to its left).

To each node σ^* which is a child of σ, we associate Π_3^0 predicates R_{σ^*} and S_{σ^*}; the definitions of these will depend on the label, from $\{\infty, f\}$, of σ^*. If σ^* has label ∞, then R_{σ^*} will be of the form $R_\sigma \wedge P_{\sigma^*} \wedge T_{\sigma^*}$ where P_{σ^*} and T_{σ^*} are Π_3^0 and R_σ is the predicate associated to the parent σ of σ^*. If σ^* has label f, then R_{σ^*} will be of the form

$$R_\sigma \wedge P_{\sigma^*} \wedge (\neg Q_{\tau_1^*}) \wedge \cdots \wedge (\neg Q_{\tau_t^*}) \wedge Q_{\sigma^*}$$

where P_{σ^*} is Π_3^0, Q_{σ^*} is Σ_2^0, and $Q_{\tau_1^*}, \ldots, Q_{\tau_t^*}$ are the Σ_2^0 predicates associated to the other children $\tau_1^*, \ldots, \tau_t^*$ of σ which are to the left of σ^* and which have the same parameters, except that they might have different values of $x^1, \ldots, x^\ell$, as σ^*. The predicate P_{σ^*} is defined in the same way for nodes labeled ∞ and f, and does not depend on the values of $x_{\sigma^*}^1, \ldots, x_{\sigma^*}^{\ell_{\sigma^*}}$,

We must define these predicates inductively, defining the predicates associated to a parent before its children, and to nodes at a single level from left to right. Let σ^* be a node at level $n+1$, with predecessors $\sigma_1, \sigma_2, \ldots, \sigma_n$ at levels $1, \ldots, n$. We define, if σ^* has label ∞:

- P_{σ^*} says that:

  (1) for $i \leq n$ with σ_i labeled f, if $k_\sigma[i]$ is "$\geq 4n$" then the $x_{\sigma_i}^1, \ldots, x_{\sigma_i}^{\ell_{\sigma_i}}$ from σ_i are part[†] of a $(\geq 4n)$-star as witnessed by $y_{\sigma^*}^1[i], \ldots, y_{\sigma^*}^{4n}[i]$;

  (2) for $i \leq n$ with σ_i labeled f, if $k_\sigma[i] < 4n$ then the $x_{\sigma_i}^1, \ldots, x_{\sigma_i}^{\ell_{\sigma_i}}$ from σ_i are part[†] of a $(\geq k_\sigma[i])$-star as witnessed by $y_{\sigma^*}^1[i], \ldots, y_{\sigma^*}^{k_{\sigma^*}[i]}[i]$, but are not part[†] of a $(> k_\sigma[i])$-star.

- T_{σ^*} says that M_{n+1} does not have[†] an r-star, $r \geq \ell_{\sigma^*}$.

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- S_{σ^*} is $\bot$, i.e. always false.

If σ^* has label f, we define:

- P_{σ^*} is defined in the same way as above.

- Q_{σ^*} says that $x_{\sigma^*}^1, \ldots, x_{\sigma^*}^{\ell_{\sigma^*}}$ are part† of a $(\geq \ell_{\sigma^*})$-star.

- S_{σ^*} says that $x_{\sigma^*}^1, \ldots, x_{\sigma^*}^{\ell_{\sigma^*}}$ are not part† of a $(> \ell_{\sigma^*})$-star.

Note that P_{σ^*} does not depend on the values of $x_{\sigma^*}^1, \ldots, x_{\sigma^*}^{\ell_{\sigma^*}}$. The predicates are all expressed using the formula (†) and have their intended meaning in spaces which are disjoint unions of stars, points, and line segments.

Now let (M^*, d^*) be obtained from Lemma 4.5 using the Π_3^0 predicates R_σ and S_σ, and the sequence (ℓ_σ). Write M_σ^* for the component built for σ, so that:

- if R_σ is false, then M_σ^* is the disjoint union of a point and ℓ_σ line segments;

- if R_σ is true and S_σ is false, then M_σ^* is the disjoint union of an ℓ_σ-star and a line segment;

- if R_σ and S_σ are both true, then M_σ^* is an $(\ell_\sigma + 1)$-star.

We prove that M^* is not homeomorphic to any M_n through the following sequence of claims.

Claim 4.6. *For each σ with R_σ true, R_{σ^*} is true for exactly one child σ^* of σ.*

Proof. Let n be the height of σ, and $\sigma_1, \ldots, \sigma_{n-1}$ the predecessors of $\sigma = \sigma_n$.

First we argue that R_{σ^*} cannot be true for two different children σ^* of σ. First, if two children σ^* and σ^{**} disagree about the value of $k_{\sigma_i}[i]$ for some σ_i labeled f, then it cannot be that R is true for both of them; indeed, (1) of P is incompatible with (2) of P, and (2) cannot be true for two different values of k. So if R is true for both σ^* and σ^{**}, then they agree on the values of $k_{\sigma_i}[i]$ and hence also on ℓ. If both σ^* and σ^{**} are labeled f, then R can be true of only the one which is to the left. They cannot both be labeled ∞, as then they would have the same parameters, and there is only one child of σ with each set of parameters. Finally, if σ^* is labeled f and σ^{**} is labeled ∞, then Q_{σ^*} being true is incompatible with $T_{\sigma^{**}}$.

Now we will show that R_{σ^*} is true for at least one child σ^* of σ. For each i with σ_i labeled f, since R_{σ_i} is true, $x_{\sigma_i}^1, \ldots, x_{\sigma_i}^{\ell_{\sigma_i}}$ are part† of a $(\geq \ell_{\sigma_i})$-star; let $k[i] < 4n$ be least such that $x_{\sigma_i}^1, \ldots, x_{\sigma_i}^{\ell_{\sigma_i}}$ are part† of a $(\geq k[i])$-star, or $k[i]$ is "$\geq 4n$" if $x_{\sigma_i}^1, \ldots, x_{\sigma_i}^{\ell_{\sigma_i}}$ are part† of a $(\geq 4n)$-star. Let $y^1[i], \ldots, y^{k[i]}[i]$ witness this. Then (1) and (2) of P_{σ^*} are true of any σ^* with these parameters.

Let ℓ be the least odd number $\leq 4n$ such that $\{\ell, \ell+1\}$ is disjoint from

$$\{\ell_\tau, \ell_\tau + 1 : \tau \text{ is a predecessor of } \sigma^*\} \cup \{k[i] : i < n\}.$$

If M_{n+1} does not have an r-star, $r \geq \ell$, then the node σ^* with the parameters described above and labeled ∞ has R_{σ^*} true. Otherwise, M_{n+1} has an r-star, $r \geq \ell$; let $x^1, \ldots, x^\ell$ be a witness to this such that the corresponding child σ^* of σ is the leftmost such child. Then Q_{σ^*} is true, but Q is not true of any child to the left of σ^*. $\qquad\square$

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Let $\sigma_1, \sigma_2, \sigma_3, \ldots$ be the sequence of nodes at level 1, 2, and so on for which R holds. We call σ the true path.

Claim 4.7. *Fix $n < m$. Suppose that M_n is homeomorphic to the disjoint union of stars, points, and line segments, each of which is compact and open. Suppose that σ_n is labeled f, and $m > n$. Then if $k_{\sigma_m}[n]$ is "$\geq 4m$" if and only if $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ is part of a $(\geq 4m)$-star, and otherwise $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ is part of a $k_{\sigma_m}[n]$-star.*

Proof. This is immediate from the definitions of the predicate P. □

Claim 4.8. *Fix n. Suppose that M_n is homeomorphic to the disjoint union of stars, points, and line segments, each of which is compact and open. Then:*

- *if σ_n has label ∞, then M_n has no $\geq \ell$-star but M^* has an ℓ-star.*

- *if σ_n has label f, then:*

    - *if $x_1, \ldots, x_k$ is in an ℓ-star, then M^* has no ℓ-star;*
    - *if $x_1, \ldots, x_k$ is in a $(> \ell)$-star, then M^* has no star of the same size.*

Proof. First suppose that σ_n has label ∞. Then from T_{σ_n} we see that M_n does not have an r-star, $r \geq \ell_{\sigma_n}$. But $M^*_{\sigma_n}$ is an $(\ell_{\sigma_n} + 1)$-star.

Now suppose that σ_n has label f. The stars present in M^* are as follows, and no more: for each m, either an ℓ_{σ_m}-star or an $(\ell_{\sigma_m} + 1)$-star, where ℓ_{σ_m} is the least odd number $< 4m$ such that $\{\ell_{\sigma_m}, \ell_{\sigma_m} + 1\}$ is disjoint from

$$\{\ell_{\sigma_i}, \ell_{\sigma_i} + 1 : i < m\} \cup \{k_{\sigma_m}[i] : i < m\}.$$

Since Q_{σ_n} is true, $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ are part of a $(\geq \ell_{\sigma_n})$-star in M_n.

If they are part of an ℓ_{σ_n}-star, then S_{σ_n} is true, and $M^*_{\sigma_n}$ is an $\ell_{\sigma_n} + 1$-star; for any i, j, $\{\ell_{\sigma_i}, \ell_{\sigma_i} + 1\}$ and $\{\ell_{\sigma_j}, \ell_{\sigma_j} + 1\}$ are disjoint, and so M^* does not have an ℓ_{σ_n}-star.

If $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ are part of a t-star for some $t > \ell_{\sigma_n}$, then S_{σ_n} is false, and $M^*_{\sigma_n}$ is an ℓ_{σ_n}-star. We claim that $t \notin \{\ell_{\sigma_i}, \ell_{\sigma_i} + 1\}$ for any $i \neq n$, so that M^* does not have a t-star. For $i < n$, $\ell_{\sigma_i} + 1 < \ell_{\sigma_n} < t$. For $i > n$, $\ell_{\sigma_i} < 4i$ is chosen so that if $k_{\sigma_m}[i] < 4i$, then $k_{\sigma_m}[i] \notin \{\ell_{\sigma_i}, \ell_{\sigma_i} + 1\}$. □

It follows from this claim that M^* is not homeomorphic to M_n for any n; indeed, if M^* was homeomorphic to M_n, then M_n would be the disjoint union of stars, points, and line segments, each of which is compact and open. Then either M_n has no $(\geq \ell)$-star but M^* does, or M_n has an ℓ-star but M^* does not, or M_n has a t-star for some $t > \ell$ but M^* does not have a t-star. □

5   Groups

Theorem 5.1. *There exists a $0'$-computable compact Polish group not topologically isomorphic to any computable Polish group.*

Proof. The following lemma is a consequence of Lemma 4.2 in a compact group.

Lemma 5.2. *There exists a uniformly $0'$-computable procedure which, on input a computable compact Polish group G enumerates (the Cayley/multiplication table of) all finite groups K of the form G/N, where N ranges over (clopen) normal subgroups of A.*

Proof. The proof is similar to the proof of Cor. 4.8 from [Mel18]. We give details.

By Lemma 4.2, with the help of $0'$ we can enumerate all clopen subsets of G. Furthermore, these clopen subsets will be represented as finite unions of basic open balls. It also follows from the proof of Lemma 4.2 that the same clopen set N will also be described by a finite union of slightly smaller closed basic balls, and $0'$ can uniformly produce both the closed and the open description of N.

Since both operations $\cdot$ and $^{-1}$ are computable in G and N has two descriptions, checking whether N forms a normal subgroup requires merely $0'$. For example, $(\exists x)(x \in N \& x^{-1} \notin N)$ is a Σ_1^0-property because we can use the open description of N to check $x \in N$ and the closed description to verify $x^{-1} \notin N$. Note that we can restrict the quantifier to special points because $x \in N \& x^{-1} \notin N$ describes an open set. Using a similar trick we can see that normality, emptiness, and that N is closed under $\times$ are also $0'$-decidable properties.

Since G is compact, for a fixed clopen N the quotient group G/N is finite. Note that every coset of the form $\tilde{x}N$ is open and thus contains a special point x, thus is of the form xN for some special x. We claim that, given such an N, $0'$ can find finitely many special points $x_0 = e, x_1, \ldots, x_n$ such that $\{x_i N\}$ is a disjoint cover of G. To see why, note that $x_i N \cap x_j N \neq \varnothing$ iff for some special y, $yx_i^{-1} \in N$ and $yx_j^{-1} \in N$, both events are c.e. because N has a finite open description. Also, since left-translation is a self-homeomorphism of G onto itself and N is clopen, each $x_i N$ is clopen as well. Thus, $\{x_i N\}$ is a closed cover iff for every special y there is an i such that $yx_i^{-1} \in N$; if we view the latter as a finite union of *closed* balls then the statement becomes Π_1^0 and thus can be decided using $0'$. Similarly, the group structure upon $\{x_i\}$ mod N can be reconstructed effectively and uniformly, in N. Simply search for an x_k such that $x_i x_j x_k^{-1} \in N$ (this is an effective search in the open name of N) and then declare $x_i x_j =_N x_k$ in G/N. Note that the procedure above is uniform in the description of N. $\qquad\square$

Consider the discrete countable group $G_S = \bigoplus_{p \in S} \mathbb{Z}_p \oplus \bigoplus_{i \in \omega} \mathbb{Z}$, where S is a set of primes (which is not necessarily the set of all primes). We use $\mathbb{Z}_p$ for the cyclic group of order p. The Pontryagin dual of G_S is the compact Polish group $A_S = \prod_{p \in S} \mathbb{Z}_p \times \prod_{i \in \omega} \mathbb{T}$, where $\mathbb{T}$ is the unit circle group. For this proof we only need one fact from Pointryagin duality: the finite quatients of A_S by its clopen subgroups are isomorphic to finite subgroups of G_S. This property can be seen directly for G_S and A_S, and therefore we will not give further details; see book [Pon66] for Pontryagin duality theory.

If we can list all finite factors of A_S, then we can also enumerate the set S. Using Lemma 5.2 above, to prove the theorem is sufficient to produce a $0'$-computable Polish presentation of A_S for a Π_2^0-complete set of primes S; then if A_S had a computable presentation, by Lemma 5.2 and the fact stated about about Pointryagin duality we would be able to use $0'$ to enumerate the set S; but since S is not Σ_2^0, there is no enumeration of S relative to $0'$, and so A_S has no computable presentation.

Let $(p_i)_{i \in \omega}$ be the natural enumeration of all primes. Fix such a Π_2^0-complete set S and a computable predicate R such that $p_i \in S \iff \exists^\infty y R(i, y)$.

The $0'$-presentation of A_S will be a closed subgroup of the natural computable Polish presentation of $\prod_{i\in\omega}\mathbb{T}$. In this presentation, the special points are ω-tuples of rational numbers having finite support (i.e., zero almost everywhere).

For the i-th prime p_i, we reserve the i-th unit circle in $\prod_{i\in\omega}\mathbb{T}$. We intend to make the point $1/p_i$ in this circle isolated iff there are infinitely many y with $R(i,y)$. To do this, start with the set of fractions $\{1/p_j : j \geq i\}$. We will keep $1/p_i$ in the set regardless of the outcome. We will however extract $1/p_{i+1}, \ldots, 1/p_{i+t}$ from the set when we find that there are at least t numbers y with $R(i,y)$. If for i there are infinitely many such y, and thus $p_i \in S$, we will end up with $\{1/p_i\}$. Otherwise we will be left with $\{1/p_i\} \cup \{1/p_j : j \geq t\}$ for some t. Note that in the latter case the completion of the set is equal to the whole circle.

For each fixed j, $0'$ can uniformly decide whether $1/p_j$ will be permanently kept in the set. Therefore, $0'$ can enumerate a dense subset of $\mathbb{T}$ whose completion is equal to $\mathbb{T}$ iff $p_i \notin S$, and furthermore the completion is isomorphic to $\mathbb{Z}_{p_i}$ iff $p_i \in S$. Since S is Π_2^0-complete it must be coinfinite, and therefore the product of the resulting uniformly $0'$-computable closed subgroups (sitting within their respective copies of the unit circle) will be isomorphic to $A_S = \prod_{p\in S}\mathbb{Z}_p \times \prod_{i\in\omega}\mathbb{T}$, as desired. This subgroup has a $0'$-computable Polish presentation which is naturally given by the $0'$-enumerable set of ω-tuples of special points with finite support, and with non-zero components corresponding to those fractions which have already been $0'$-effectively listed in the respective circle. $\qquad\square$

Theorem 5.3. *For every computable α there exists an effectively closed compact (thus, profinite) subgroup of S_∞ not homeomorphic to any Δ_α^0 Polish group.*

Proof. As in the previous theorem, a Δ_α^0-computable Polish presentation of the group $A_S = \prod_{p\in S}\mathbb{Z}_p$ gives rise to a $\Delta_{\alpha+1}^0$-enumeration of S. The theorem would follow if for each computable successor ordinal β we could produce an effectively closed subgroup of S_∞ isomorphic to A_S, where S is Σ_β^0-complete. (To witness the theorem for α take $\beta > \alpha + 1$.)

To do that, we will construct a computable structure $\mathcal{M}$ with domain ω and automorphism group $Aut(\mathcal{M}) \cong A_S$. Since the automorphism groups of such computable algebraic structures are clearly effectively closed subgroups of S_∞, the theorem will follow.

The structure $\mathcal{M}$ will consist of infinitely many disjoint gadget-substructures $\mathcal{M}_i$, each working with the respective prime p_i. The automorphism group of $\mathcal{M}$ will naturally be the Cartesian product of the automorphism groups of all these $\mathcal{M}_i$.

It remains to produce a uniformly effective sequence of computable structures $(\mathcal{M}_i)_{i\in\omega}$ with the property:

$$Aut(\mathcal{M}_i) \cong \mathbb{Z}_{p_i} \iff i \in S,$$

where S is the Σ_β^0-complete set we fixed above, and $\mathcal{M}_i$ is rigid if $i \notin S$.

Using the standard technique due to Ash (see [Ash86] and, for the specific result, see [GK02]) on input i we can uniformly produce a computable ordinal γ_i such that $\gamma_i \cong \delta$ if $i \in S$ and $\gamma_i \cong \delta'$ if $i \notin S$, where $\delta \not\cong \delta'$ are computable ordinals which depend on β. The structure $\mathcal{M}_i$ consists of a loop of size p_i realised using a unary function u:

$$u(x_{i,j}) = x_{i,j+1 \bmod p_i}.$$

In M_i, each point $x_{i,j}$ will be computably associated with a "box"; more formally, using another unary function s we isolate the set $Y_{i,j} = \{y : s(y) = x_{i,j}\}$ which will be disjoint from

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$Y_{i',j'}$ whenever $j \neq j'$. On each such set $Y_{i,0}$, using the aforementioned result of Ash and a special binary relational symbol, we will uniformly construct a computable well-ordering which will be isomorphic to δ if $i \in S$, and δ' if $i \notin S$. On each other set $Y_{i,j}$, $j \neq 0$, we construct a computable copy of δ.

For each fixed i, we will end up with all $x_{i,j}$ being automorphic to each other if and only if $i \in S$; if $i \in S$, then they each have a copy of δ in their associated "box", and if $i \notin S$, then $x_{i,0}$ has a copy of δ' while each other $x_{i,j}$ has a copy of δ. Furthermore, since ordinals are rigid, $Aut(M_i) \cong \mathbb{Z}_{p_i}$ if $i \in S$, and $\mathcal{M}_i$ is rigid otherwise, as desired. $\square$

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