There is a no simple characterization of the relatively decidable theories

Matthew Harrison-Trainor

September 3, 2019

Abstract

Given a complete decidable theory T, say that T is relatively decidable if for every countable model $\mathcal{A}$ of T, the atomic diagram of $\mathcal{A}$ can compute the elementary diagram of $\mathcal{A}$. We say that T is uniformly relatively decidable if there is a single Turing functional witnessing all of these computations. Chubb, Miller, and Solomon showed that T is uniformly relatively decidable if and only if it is model complete. They conjectured that T is relatively decidable if and only if there is a conservative extension of T naming new constants which is model complete. We show that not only is this not true, there is no simple classification of the relatively decidable theories. Formally, we show that the index set of the relatively decidable theories is Π^1_1 m-complete.

1 Introduction

Given a theory T, we say that T is *model complete* if whenever $\mathcal{A}$ and $\mathcal{B}$ are models of T, and $\mathcal{A}$ is a substructure of $\mathcal{B}$, $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. Equivalently, T is model complete if every formula is equivalent modulo T to an existential formula. This paper follows up on work of Chubb, Miller, and Solomon [CMS] in exploring computability-theoretic consequences of model completeness. Throughout, we generally assume that all structures are countable with computable domains, and write $\Delta(\mathcal{A})$ for the atomic diagram of $\mathcal{A}$, and $E(\mathcal{A})$ for the elementary diagram of $\mathcal{A}$.

Suppose that T is model complete and c.e. Then given a formula $\varphi(\bar{x})$, we can computably search for a quantifier-free formula $\psi(\bar{x}, \bar{y})$ and a proof from T that

$$T \vDash \varphi(\bar{x}) \longleftrightarrow \exists \bar{y}\ \psi(\bar{x}, \bar{y}).$$

If $\mathcal{A}$ is a countable model of T, and $\bar{a}$ is a tuple from $\mathcal{A}$, then we can decide, using just the atomic diagram of $\mathcal{A}$, whether $\mathcal{A} \vDash \varphi(a)$: Search as above for quantifier-free formulas $\psi_\varphi(\bar{x}, \bar{y})$ and $\psi_{\neg\varphi}(\bar{x}, \bar{z})$ such that

$$T \vDash \varphi(\bar{x}) \longleftrightarrow \exists \bar{y}\ \psi_\varphi(\bar{x}, \bar{y})$$

and

$$T \vDash \neg\varphi(\bar{x}) \longleftrightarrow \exists \bar{y}\ \psi_{\neg\varphi}(\bar{x}, \bar{y}),$$

and then simultaneously search for a tuple $\bar{b}$ such that $\mathcal{A} \models \psi_\varphi(\bar{a}, \bar{b})$ and a tuple $\bar{c}$ such that $\mathcal{A} \models \psi_{\neg\varphi}(\bar{a}, \bar{c})$. In the former case, we have $\mathcal{A} \models \varphi(\bar{a})$, and in the latter case we have $\mathcal{A} \models \neg\varphi(\bar{a})$. Since ψ_φ and $\psi_{\neg\varphi}$ are quantifier-free, we can recognize such a tuple $\bar{b}$ or $\bar{c}$ computably using the atomic diagram, and this search always terminates. So for any model $\mathcal{A}$ of T, the atomic diagram $\Delta(\mathcal{A})$ of $\mathcal{A}$ computes the elementary diagram $E(\mathcal{A})$ of $\mathcal{A}$. (In particular, every computable model of T is decidable.)

We recall the following definitions from [CMS]:

Definition 1.1.

- A theory T is relatively decidable if every $\mathcal{A} \models T$ has $E(\mathcal{A}) \leq_T \Delta(\mathcal{A})$.
- A theory T is uniformly relatively decidable if there is a single Turing function Γ such that every $\mathcal{A} \models T$ has $E(\mathcal{A}) \leq_T \Delta(\mathcal{A})$ via Γ.

Any c.e. model complete theory T is uniformly relatively decidable because the algorithm described above is the same for each model of T. Chubb, Miller, and Solomon show that this is an exact characterization.

Theorem 1.2 (Chubb, Miller, and Solomon [CMS]). *Let T be a c.e. theory. Then T is model complete if and only if it is uniformly relatively decidable.*

Now consider the relatively decidable theories. An excellent motivating example from [CMS] is the theory of ω with a unary successor relation S. $\mathrm{Th}(\omega, S)$ is not model complete, because the formula $(\forall y)\ S(y) \neq x$ describing the initial element is not equivalent to an existential formula. However, $\mathrm{Th}(\omega, S)$ proves that there is a unique element satisfying this formula:

$$\mathrm{Th}(\omega, S) \models (\exists! x)(\forall y)\ S(y) \neq x.$$

Moreover, after naming such an element with a new constant symbol c, the theory $\mathrm{Th}(\omega, S) \cup \{(\forall y)\ S(y) \neq c\}$ is model complete. For any model $\mathcal{A} \models \mathrm{Th}(\omega, S)$, we can find the unique element a with $\mathcal{A} \models (\forall y)\ S(y) \neq a$, and then use the model completeness of $\mathrm{Th}(\omega, S) \cup \{(\forall y)\ S(y) \neq c\}$ to compute the elementary diagram of $\mathcal{A}$ from the atomic diagram of $\mathcal{A}$. This is not a uniform procedure, because it depends on the choice of the element a in the model $\mathcal{A}$. So $\mathrm{Th}(\omega, S)$ is relatively decidable but not uniformly relatively decidable. Chubb, Miller, and Solomon proved:

Theorem 1.3 (Chubb, Miller, and Solomon [CMS]). *Let T be a c.e. theory. Then T is relatively decidable if and only if for each $\mathcal{A} \models T$, there is $\bar{a} \in \mathcal{A}$ such that $\mathrm{Th}(\mathcal{A}, \bar{a})$ is model complete.*

However this is in many ways not a satisfactory characterization because it requires quantification over all models of T, and it does not say anything about how the tuple $\bar{a}$ should be chosen. We would like a characterization that looks only at the theory T, and not at models of T.

In the example of $\mathrm{Th}(\omega, S)$, there was more going on: the choice of element to make the theory model complete was uniform across the models in the sense that we wanted to choose an element satisfying a particular formula. Chubb, Miller, and Solomon suggested that there might be a characterization along the following lines. Let us restrict our attention to complete theories; recall that a c.e. theory which is also complete is decidable. Suppose that T is complete decidable theory, that $T \vDash \exists \bar{x}\, \varphi(\bar{x})$, and that $T \cup \{\varphi(\bar{c})\}$ is model complete where $\bar{c}$ is a new tuple of constant symbols. Then T is relatively decidable. Could this—or something in a similar vein—be a characterization of the relatively decidable theories? If so, it would be a satisfactory characterization as it involves only looking at the theory T without quantifying over models of T.

We prove that a characterization along these lines is too much to hope for: there is no characterization of relative decidability that is simpler than the definition given above. (The characterization in Theorem 1.3 is the same complexity as the definition.) More precisely:

Theorem 1.4. *The index set*

$$I_{RelDec} = \{i : \textit{the } i\textit{th complete decidable theory is relatively decidable}\}$$

of the complete decidable theories which are relatively decidable is Π^1_1 *m-complete.*

What this means is that any characterization of the relatively decidable theories must involve at least one universal quantifier over subsets of ω (e.g. a universal quantifier over models or types). The suggested characterization given above—that T is relatively decidable if and only if there is a formula φ such that $T \vDash \exists \bar{x}\ \varphi(\bar{x})$ and $T \cup \{\varphi(\bar{c})\}$ is model complete—is arithmetic, and hence there must be relatively decidable theories T without this property.

This technique of using index sets was first introduced by Goncharov and Knight [GK02] and has also been used in [DKL+15], where it was shown that there is no reasonable characterization of computable categoricity, and in [DM08], where it was shown that there is no reasonable classification of torsion-free abelian groups. See also [LS07, Fok07, CFG+07, FGK+15, GBM15a, GBM15b, HT18, BHTK+].

Finally, we would like to highlight the following related open problem of Goncharov:

Question (Goncharov)**.** Characterize the decidable theories T such that every computable model of T is decidable.

The corresponding index set is $\Sigma_{\omega+2}$; we conjecture that it is $\Sigma_{\omega+2}$ m-complete. However, the methods from this paper are not applicable to this problem, as (by index set complexity calculations) to prove that the index set is $\Sigma_{\omega+2}$ m-complete one would have to build decidable models $\mathcal{A}$ of T such that for no $\bar{a} \in \mathcal{A}$ is $\mathrm{Th}(\mathcal{A}, \bar{a})$ model complete.

2 Marker Extensions

We will have to use several Marker extensions. In this section, we describe the particular kinds of Marker extension that we will use and prove several results about them.

Let $\mathcal{L}$ be a relational language including relation symbols $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ of arity $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ respectively. We will define the Marker extension making $U_1, U_2, \ldots$ into Σ_1 relations and $V_1, V_2, \ldots$ into Σ_2 relations. (When we take a Marker extension, we will always have relations $U_1, U_2, \ldots$, but sometimes we will not have relations $V_1, V_2, \ldots$. We can either make small modifications to all of the proofs, or just assume that we have relations

$V_1, V_2, \ldots$ which are always trivial.) This will transform the language $\mathcal{L}$ into a language $\mathcal{L}^*$, each $\mathcal{L}$-structure $\mathcal{A}$ into an $\mathcal{L}^*$ structure $\mathcal{A}^*$, and each $\mathcal{L}$-theory T into an $\mathcal{L}^*$-theory T^*.

The language $\mathcal{L}^*$ will consist of the symbols $\mathcal{L} - \{U_1, U_2, \ldots, V_1, V_2, \ldots\}$, a new unary relation symbol W, for each i a p_i-tuple of unary function symbols $\bar{f}_i$, and for each i a q_i-tuple $\bar{g}_i$ of unary function symbols and a unary function symbol h_i.

Given an $\mathcal{L}$-structure $\mathcal{A}$, we define an $\mathcal{L}^*$ structure $\mathcal{A}^*$ as follows. The domain of $\mathcal{A}^*$ will consist of the disjoint union of the domain of $\mathcal{A}$, satisfying the unary relation W, together with new elements not satisfying the relation W:

- for each $\bar{a} \in \mathcal{A}$ with $\mathcal{A} \models U_i(\bar{a})$, an element $b_{i,\bar{a}}$ with $\bar{f}_i(b_{i,\bar{a}}) = \bar{a}$;
- for each $\bar{a} \in \mathcal{A}$ of arity q_i, infinitely many elements $c^n_{i,\bar{a}}$ and $d^n_{i,\bar{a}}$ with $\bar{g}_i(c^n_{i,\bar{a}}) = \bar{a}$ and $h_i(d^n_{i,\bar{a}}) = c^n_{i,\bar{a}}$;
- for each $\bar{a} \in \mathcal{A}$ of arity q_i with $\mathcal{A} \models V_i(\bar{a})$, an element $c^*_{i,\bar{a}}$ with $\bar{g}_i(c^*_{i,\bar{a}}) = \bar{a}$.

Whenever we have not defined it otherwise, the functions f, g, and h map an element to itself. Essentially we want the functions to all be partial functions, but we code in partiality by having the function be the identity.

Lemma 2.1. *For each $i \in \omega$:*

(1) there is an $\exists$ $\mathcal{L}^*$-formula $\varphi_i(\bar{x})$ such that given an $\mathcal{L}$-structure $\mathcal{A}$ and $\bar{a} \in \mathcal{A}$,

$$\mathcal{A} \models U_i(\bar{a}) \iff \mathcal{A}^* \models \varphi_i(\bar{a}).$$

(2) there is an $\exists\forall$ $\mathcal{L}^*$-formula $\psi_i(\bar{x})$ such that given an $\mathcal{L}$-structure $\mathcal{A}$ and $\bar{a} \in \mathcal{A}$,

$$\mathcal{A} \models V_i(\bar{a}) \iff \mathcal{A}^* \models \varphi_i(\bar{a}).$$

Proof. We have

$$\mathcal{A} \models U_i(\bar{a}) \iff \mathcal{A}^* \models \bar{a} \in W \wedge (\exists x \notin W) \bar{f}_i(x) = \bar{a}$$

and

$$\mathcal{A} \models V_i(\bar{a}) \iff \mathcal{A}^* \models \bar{a} \in W \wedge (\exists x \notin W)(\forall y \notin W, y \neq x)[\bar{g}_i(x) = \bar{a} \wedge h_i(y) \neq x]. \qquad \square$$

Now given an $\mathcal{L}$-theory T, we will define an $\mathcal{L}^*$-theory T^*. For each $\mathcal{L}$-formula φ, let φ^* be φ with each instance of U_i or V_i in that formula replaced by the corresponding $\mathcal{L}^*$-formula from Lemma 2.1, and each quantifier restricted to W. T^* consists of:

- φ^* for each $\varphi \in T$;
- each relation from $\mathcal{L} - \{U_1, U_2, \ldots, V_1, V_2, \ldots\}$ holds only of elements from W;
- if $x \in W$, then $f^j_i(x) = g^j_i(x) = h_i(x) = x$ for all i, j;
- if $x \notin W$, and $f^j_i(x) \neq x$, then:
    - $\bar{f}_i(x) \in W^{p_i}$;

- $f_{i'}^{j'}(x) = x$ for all $i' \neq i$ and j';
- $g_{i'}^{j'}(x) = x$ for all i' and j';
- $h_{i'}(x) = x$ for all i';

- if $x, y \notin W$ and $x \neq y$ then $\bar{f}_i(x) \neq \bar{f}_i(y)$.
- if $x \notin W$, and $g_i^j(x) \neq x$, then:
  - $\bar{g}_i(x) \in W^{q_i}$;
  - $g_{i'}^{j'}(x) = x$ for all $i' \neq i$ and j';
  - $f_{i'}^{j'}(x) = x$ for all i' and j';
  - $h_{i'}(x) = x$ for all i';
- given $\bar{x} \in W$ of arity q_i, there are infinitely many $y \notin W$ with $\bar{g}_i(y) = \bar{x}$.
- if $x \notin W$, and $h_i(x) \neq x$, then:
  - $h_i(x) \notin W$;
  - $\bar{g}_i(h_i(x)) \in W^{q_i}$;
  - $g_{i'}^{j'}(x) = x$ for all i' and j';
  - $f_{i'}^{j'}(x) = x$ for all i' and j';
  - $h_{i'}(x) = x$ for all $i' \neq i$.
- if $x, y \notin W$ and $h_i(x) \neq x$, $h_i(y) \neq y$, then $h_i(x) \neq h_i(y)$.
- given $\bar{x} \in W$ of arity q_i, there is at most one element $y \notin W$ with $\bar{g}_i(y) = \bar{x}$ such that there is no element $z \neq y$ with $h_i(z) = y$.

Lemma 2.2. *Given $\mathcal{A} \models T$, $\mathcal{A}^* \models T^*$.*

Proof. Given $\varphi \in T$, $\mathcal{A}^* \models \varphi^*$ by Lemma 2.1 and a simple induction argument. The other sentences in T^* are immediate by definition of $\mathcal{A}^*$. □

We would like to prove that if $\mathcal{B} \models T^*$, there is $\mathcal{A} \models T$ such that $\mathcal{B} \cong \mathcal{A}^*$. This is not quite true because of compactness and the fact that we have taken the Marker extension with respect to infinitely many relations. Instead, given $\mathcal{A}$ and $\mathcal{L}$-structure, define $\mathcal{A}_n^*$, for $n \in \omega \cup \{\omega\}$, to be $\mathcal{A}^*$ together with n new elements not in W, with $f_i^j(x) = g_i^j(x) = h_i(x) = x$ for all i, j.

Lemma 2.3. *Given $\mathcal{B} \models T^*$, there is $\mathcal{A} \models T$ and $n \in \omega \cup \{\omega\}$ such that $\mathcal{B} \cong \mathcal{A}_n^*$.*

Proof. Let $\mathcal{A}$ be the structure with domain $W^{\mathcal{B}}$, and define the relation U_i and V_i on $\mathcal{A}$ using the formulas in Lemma 2.1. $\mathcal{A}$ inherits the other relations in $\mathcal{L}$ from $\mathcal{B}$. Since for each $\varphi \in T$, $\mathcal{B} \models \varphi^*$, we have that $\mathcal{A} \models \varphi$. Thus $\mathcal{B} \models T$.

Now we need to show that there is $n \in \omega \cup \{\omega\}$ such that $\mathcal{B} \cong \mathcal{A}_n^*$. Let X be the set of elements $x \notin W^{\mathcal{B}}$ with $f_i^j(x) = x$, $g_i^j(x) = x$, and $h_i(x) = x$ for all i, j. Let $n = |X|$. Let $\mathcal{B}^-$ be $\mathcal{B}$ with these elements removed. The axioms of T^* and the fact that we defined $\mathcal{A}$ using the definitions in Lemma 2.1 imply that $\mathcal{B}^- \cong \mathcal{A}^*$ so that $\mathcal{B} \cong \mathcal{A}_n^*$. □

We cannot distinguish $\mathcal{A}_n^*$ from $\mathcal{A}^*$ by first-order sentences.

Lemma 2.4. *For each $n \in \omega \cup \{\omega\}$, $\mathcal{A}_n^* \equiv \mathcal{A}^*$.*

Proof. For every k, if we restrict $\mathcal{A}^*$ to the first k symbols, this is isomorphic to $\mathcal{A}_n^*$ restricted to the first k symbols; both have infinitely many elements $x \notin W$ with $f_i^j(x) = x$, $g_i^j(x) = x$, and $h_i(x)$ for $i \le k$, and otherwise they are isomorphic. ☐

Next we want to show that the elementary diagram of $\mathcal{A}^*$ does not become too much more complicated than the elementary diagram of $\mathcal{A}$. For each i, define a unary relation $h_i^\exists(x)$ if and only if $(\exists y \neq x) h_i(y) = x$.

Lemma 2.5. *Let T^* be the Marker extension of T. T^* has quantifier elimination in the language*

$$\mathcal{L}^* \cup \{h_i^\exists : i \in \omega\} \cup \{\varphi^* : \varphi \text{ an } \mathcal{L}\text{-formula}\}.$$

Moreover, a formula φ involving only the symbols associated with $U, U_1, \ldots, U_k$ and $V, V_1, \ldots, V_k$ is equivalent to a quantifier-free formula involving only symbols associated with $U, U_1, \ldots, U_k$ and $V, V_1, \ldots, V_k$.

Note that if φ is a sentence, then φ^* is a 0-ary relation. So even though we have no constant symbols, there are quantifier-free $\mathcal{L}^+$-sentences.

Proof. Let

$$\mathcal{L}^+ = \mathcal{L}^* \cup \{h_i^\exists : i \in \omega\} \cup \{\varphi^* : \varphi \text{ an } \mathcal{L}\text{-formula}\}.$$

We use the following quantifier elimination test: Let $\mathcal{B}$ be an $\mathcal{L}^+$-substructure of both $\mathcal{M} \models T^*$ and $\mathcal{N} \models T^*$, $\bar{a} \in \mathcal{B}$, and $b \in \mathcal{M}$ be such that $\mathcal{M} \models \varphi(\bar{a}, b)$ for φ a quantifier-free $\mathcal{L}^+$-formula. We want to show that there is $b' \in \mathcal{N}$ such that $\mathcal{N} \models \varphi(\bar{a}, b')$.

By writing it in disjunctive normal form, we may assume that φ is a conjunction of atomic and negated atomic formulas. We may also assume that there is only a single conjunct of the form ψ^* for ψ an $\mathcal{L}$-formula. Then write

$$\varphi(\bar{a}, x) \equiv \psi^*(\bar{s}(a), \bar{t}(x)) \wedge \cdots$$

where $\bar{s}$ and $\bar{t}$ are terms and the $\cdots$ is a conjunction of atomic and negated atomic formulas in the language $\mathcal{L}^* \cup \{h_i^\exists : i \in \omega\}$. We may also assume that $\cdots$ does not include any equalities or inequalities between elements of W, as these can be included in ψ^*. Suppose that only f_i, g_i, h_i, and $h_i^\exists$ for $i \le n$ appear in φ. Note that since ψ^* holds only of elements in W, each term appearing in $\bar{s}$ or $\bar{t}$ can be assumed to be either the identity, f_i^j, g_i^j, or $g_i^j \circ h_i$.

First, suppose that $b \in W^{\mathcal{M}}$ so that $t(b) = b$. We may assume that $b \notin \mathcal{A}$ as this case is easy. If b is the image of some element of $\bar{a}$ under a function f, g, or h, then $b \in \mathcal{A}$. So we may assume that b is not in the image of some such element. Then since $\mathcal{M} \models \psi^*(\bar{s}(\bar{a}), b)$ and $\mathcal{M} \models \mathrm{Th}(\mathcal{A}^*)$, $\mathcal{M} \models (\exists y\ \psi(\bar{x}, y))^*(\bar{s}(\bar{a}))$. So $\mathcal{N} \models (\exists y\ \psi(\bar{x}, y))^*(\bar{s}(\bar{a}))$ and $\mathcal{N} \models (\exists y \in W)\ \psi^*(\bar{s}(\bar{a}), y)$. Let $b' \in \mathcal{N}$ witness this, so $\mathcal{N} \models \psi^*(\bar{s}(\bar{a}), b')$. Since b and b' are in W, the rest of the atomic and negated atomic formulas in "$\cdots$" above are trivially satisfied.

Now suppose that $b \notin W^{\mathcal{M}}$. Once again, we can assume that b is not equal to the image of any element of $\bar{a}$ under some term. Write $\bar{a}' = \bar{s}(\bar{a})$. We have a number of cases, in each of which we choose b' such that $\mathcal{N} \vDash \varphi(\bar{a}, b')$. Essentially we need to make sure that $\mathcal{N} \vDash \psi^*(\bar{s}(a), \bar{t}(b'))$, and that if $b' \notin W$, then b' and $h_i(b')$ satisfy the same equalities and inequalities with $\bar{a}$ and $h_i(a) \notin W$ for $a \in \bar{a}$. (Other terms applied to b' or elements of $\bar{a}$ are either the identity, or are in W, and inequalities and inequalities in W are expressed in ψ^*.) In the following, $i \leq n$:

- If $\bar{f}_i(b) \neq b$, let $\bar{c} \in \mathcal{M}$ be $\bar{f}_i(b)$. Then we may replace $\bar{t}(b)$ with $\bar{c}$. Then
$$\mathcal{M} \vDash \big(\exists \bar{y} \;\; \psi(\bar{x}, \bar{y}) \wedge U_i(\bar{y})\big)^*(\bar{a}')$$
as witnessed by $\bar{c}$ and so
$$\mathcal{N} \vDash \big(\exists \bar{y} \;\; \psi(\bar{x}, \bar{y}) \wedge U_i(\bar{y})\big)^*(\bar{a}').$$
Thus there is $\bar{d} \in \mathcal{N}$ with $\mathcal{N} \vDash \psi^*(\bar{a}', \bar{d}) \wedge U_i^*(\bar{d})$. Choose $b' \in \mathcal{N}$ with $\bar{f}_i(b') = \bar{d}$. This b' is unique, and since $b \notin \bar{a}$, we can choose $b' \notin \bar{a}$.

- If $\bar{g}_i(b) \neq b$, let $\bar{c} \in \mathcal{M}$ be $\bar{g}_i(b)$. Then we may replace $\bar{t}(b)$ with $\bar{c}$.
If $\mathcal{M} \vDash h_i^{\exists}(b)$, then
$$\mathcal{M} \vDash \big(\exists \bar{y} \; \psi(\bar{x}, \bar{y})\big)^*(\bar{a}')$$
and so
$$\mathcal{M} \vDash \big(\exists \bar{y} \; \psi(\bar{x}, \bar{y})\big)^*(\bar{a}').$$
Thus there is $\bar{d} \in \mathcal{N}$ with $\mathcal{N} \vDash \psi^*(\bar{a}', \bar{d})$. Choose $b' \in \mathcal{N}$ with $\bar{g}_i(b') = \bar{d}$ and such that $\mathcal{N} \vDash h_i^{\exists}(b')$. We may choose $b' \notin \bar{a}$ since there are infinitely many choices for b'.
On the other hand, if $\mathcal{M} \vDash \neg h_i^{\exists}(b)$ then
$$\mathcal{M} \vDash \big(\exists \bar{y} \;\; \psi(\bar{x}, \bar{y}) \wedge V_i(\bar{y})\big)^*(\bar{a}')$$
and so
$$\mathcal{N} \vDash \big(\exists \bar{y} \;\; \psi(\bar{x}, \bar{y}) \wedge V_i(\bar{y})\big)^*(\bar{a}')$$
Thus there is $\bar{d} \in \mathcal{N}$ with $\mathcal{N} \vDash \psi(\bar{a}', \bar{d}) \wedge V_i(\bar{d})$. Choose $b' \in \mathcal{N}$ with $\bar{g}_i(b') = \bar{d}$ and such that $\mathcal{N} \vDash \neg h_i^{\exists}(b')$. This b' is unique, so since $b \notin \bar{a}$, we can choose $b' \notin \bar{a}$.

- If $h_i(b) \neq b$, $h_i(b) = a \in \bar{a}$, then let $b' \in \mathcal{N}$ be the unique b' with $\mathcal{N} \vDash h_i(b') = a$.

- If $h_i(b) \neq b$, $h_i(b) \notin \bar{a}$, then $\bar{g}_i(h_i(b)) \neq b$. Let $\bar{c} \in \mathcal{M}$ be $\bar{g}_i(h_i(b))$. We may replace $\bar{t}(b)$ with $\bar{c}$. We have
$$\mathcal{M} \vDash \big(\exists \bar{y} \; \psi(\bar{x}, \bar{y})\big)^*(\bar{a}')$$
and so
$$\mathcal{M} \vDash \big(\exists \bar{y} \; \psi(\bar{x}, \bar{y})\big)^*(\bar{a}').$$
Thus there is $\bar{d} \in \mathcal{M}$ with $\mathcal{N} \vDash \psi(\bar{a}', \bar{d})$. Choose $b' \in \mathcal{N}$, $b' \notin \bar{a}$, with $\bar{g}_i(h_i(b')) = \bar{d}$ and $h_i(b') \notin \bar{a}$. We can do this as there are infinitely many b' with $\bar{g}_i(h_i(b')) = \bar{d}$.

- Otherwise, choose $b' \notin \bar{a}$ to be some element with $\mathcal{N} \vDash b' \notin W$ with $f_i^j(b') = g_i^j(b') = h_i(b') = b'$ for all $i \leq n$ and j.

The moreover clause is not hard to see using the same proof but restricting the language. ◻

Corollary 2.6. *Let T^* be the Marker extension of T making $U_1, U_2, \ldots$ into Σ_1 relations (but with no relations $V_1, V_2, \ldots$). T has quantifier elimination in the language*

$$\mathcal{L}^* \cup \{\varphi^* : \varphi \text{ an } \mathcal{L}\text{-formula}\}.$$

Corollary 2.7. *If T is complete, then so is T^*.*

Proof. Given an $\mathcal{L}^*$-sentence φ, we can write φ as a boolean combination of atomic and negated atomic formulas from

$$\mathcal{L}^* \cup \{h_i^{\exists} : i \in \omega\} \cup \{\varphi^* : \varphi \text{ an } \mathcal{L}\text{-formula}\}.$$

with no free variables. The only atomic or negated atomic formulas with no free variables are of the form ψ^* or $\neg\psi^*$ for ψ and $\mathcal{L}$-sentence. Since T is complete, T decides ψ, and so T^* decides ψ^*. Thus T^* decides φ. ◻

Now we want to define a notion of substructure for $\mathcal{L}$-structures which is more relaxed than the standard notion of substructure. The idea is that we want to view the relations $U_1, U_2, \ldots$ as already being Σ_1 in $\mathcal{L}$-structures $\mathcal{A}$, so that we can have $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \vDash \neg U_i(\bar{a})$, and $\mathcal{B} \vDash U_i(\bar{b})$ because the Σ_1 witness only appears in $\mathcal{B}$.

Definition 2.8. For $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, define $\mathcal{A} \subseteq \mathcal{B}$ if and only if:

- $A \subseteq B$;
- For each U_i, if $\mathcal{A} \vDash U_i(\bar{x})$ then $\mathcal{B} \vDash U_i(\bar{x})$;
- For each $R \in \mathcal{L} - \{U_1, U_2, \ldots, V_1, V_2, \ldots\}$, $\mathcal{A} \vDash R(\bar{x})$ if and only if $\mathcal{B} \vDash R(\bar{x})$.

There is no requirement on the relations $V_1, V_2, \ldots$.

Lemma 2.9. *Given $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$, and a Marker extension $\mathcal{A}^*$ of $\mathcal{A}$, there is a Marker extension $\mathcal{B}^*$ of $\mathcal{B}$ with $\mathcal{A}^* \subseteq \mathcal{B}^*$. Moreover, suppose that $\bar{a} = \bar{a}_1\bar{a}_2 \in \mathcal{A}^*$, with $\bar{a}_1 \in \mathcal{A}$ and $\bar{a}_2 \in \mathcal{A}^* - \mathcal{A}$; and suppose that $\bar{a}$ is closed under the functions f, g, and h. If $\mathcal{B}, \bar{a}_1 \vDash \mathrm{Th}(\mathcal{A}, \bar{a}_1)$ then $\mathcal{B}^*, \bar{a} \vDash \mathrm{Th}(\mathcal{A}^*, \bar{a})$.*

Proof. Given a copy of $\mathcal{B}^*$, we will define an embedding $\mathcal{A}^* \to \mathcal{B}^*$ making $\mathcal{A}^*$ a substructure of $\mathcal{B}^*$ as follows. Identify $W^{\mathcal{A}^*}$ with $\mathcal{A}$ and $W^{\mathcal{B}^*}$ with $\mathcal{B}$, and use the inclusion $\mathcal{A} \subseteq \mathcal{B}$. Given $v \in \mathcal{A}^*$ with $\bar{f}_i(v) = \bar{u}$, we have $\mathcal{A} \vDash U_i(\bar{u})$, and so $\mathcal{B} \vDash U_i(\bar{u})$; then there is a unique $v' \in \mathcal{B}^*$ with $\bar{f}_i(v') = \bar{u}$. Map v to v'.

Given $\bar{u} \in \mathcal{A}$ with $|\bar{u}| = q_i$, there are four possibilities:

(1) $\mathcal{A} \vDash V_i(\bar{u})$ and $\mathcal{B} \vDash V_i(\bar{u})$. Each of $\mathcal{A}^*$ and $\mathcal{B}^*$ have infinitely many elements v with $\bar{g}_i(v) = \bar{u}$, and each of these v has a w with $h_i(w) = v$. Map these to each other in a one-to-one way.

(2) $\mathcal{A} \models V_i(\bar{u})$ and $\mathcal{B} \models \neg V_i(\bar{u})$. Each of $\mathcal{A}^*$ and $\mathcal{B}^*$ have infinitely many elements v with $\bar{g}_i(v) = \bar{u}$, and in $\mathcal{A}^*$, each such v has a w with $h_i(w) = v$. In $\mathcal{B}^*$ there is a unique v with no such w. So map each pair v, w in $\mathcal{A}^*$ to a corresponding pair in $\mathcal{B}^*$, with this unique v in $\mathcal{B}^*$ being the only element not in the image of the map.

(3) $\mathcal{A} \models \neg V_i(\bar{u})$ and $\mathcal{B} \models V_i(\bar{u})$. Each of $\mathcal{A}^*$ and $\mathcal{B}^*$ have infinitely many elements v with $\bar{g}_i(v) = \bar{u}$; in $\mathcal{A}^*$ all but one such v have a w with $h_i(w) = v$, and in $\mathcal{B}^*$ every v has such a w. So map the v's to each other, so that there is a single pair v, w in $\mathcal{B}^*$ with v in the image of the map, but w is not.

(4) $\mathcal{A} \models \neg V_i(\bar{u})$ and $\mathcal{B} \models \neg V_i(\bar{u})$. Each of $\mathcal{A}^*$ and $\mathcal{B}^*$ have infinitely many elements v with $\bar{g}_i(v) = \bar{u}$, and all but one of these v has a w with $h_i(w) = v$. Map these to each other in a one-to-one way, with the unique elements v with no w with $h_i(w) = v$ mapped to each other.

Since $\mathcal{A}^* \models \mathrm{Th}(\mathcal{A})^*$ and $\mathrm{Th}(\mathcal{A})^*$ is complete, we have that $\mathrm{Th}(\mathcal{A}^*) = \mathrm{Th}(\mathcal{A})^*$. Similarly, $\mathrm{Th}(\mathcal{B}^*) = \mathrm{Th}(\mathcal{B})^*$. Since $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$, we have $\mathrm{Th}(\mathcal{A}^*) = \mathrm{Th}(\mathcal{B}^*)$. This theory also has quantifier elimination to the language

$$\mathcal{L}^+ = \mathcal{L}^* \cup \{h_i^\exists : i \in \omega\} \cup \{\varphi^* : \varphi \text{ an } \mathcal{L}\text{-formula}\}.$$

So to show that $\mathcal{B}^*, \bar{a} \models \mathrm{Th}(\mathcal{A}^*, \bar{a})$, it suffices to show that $\bar{a}$ has the same atomic $\mathcal{L}^+$-type in both $\mathcal{A}^*$ and $\mathcal{B}^*$.

Since $\mathcal{B}, \bar{a}_1 \models \mathrm{Th}(\mathcal{A}, \bar{a}_1)$, for each $\mathcal{L}$-formula $\varphi(\bar{x})$,

$$\mathcal{A}^* \models \varphi^*(\bar{a}_1) \iff \mathcal{B}^* \models \varphi^*(\bar{a}_1).$$

Since $\bar{a}$ is closed under the applications of the functions f, g, and h, for every tuple of terms $\bar{s}$,

$$\mathcal{A}^* \models \varphi^*(\bar{s}(\bar{a})) \iff \mathcal{B}^* \models \varphi^*(\bar{s}(\bar{a})).$$

We define the embedding $\mathcal{A}^* \to \mathcal{B}^*$ specifically so that $\mathcal{A}^*$ is an $\mathcal{L}^*$-substructure of $\mathcal{B}^*$. Finally, we must argue that if $b \in \bar{a}_2$, then

$$\mathcal{A}^* \models h_i^\exists(b) \iff \mathcal{B}^* \models h_i^\exists(b).$$

If $\mathcal{A}^* \models h_i^\exists(b)$, then as $h_i^\exists$ is defined by an existential $\mathcal{L}^*$-formula, $\mathcal{B}^* \models h_i^\exists(b)$. Suppose that $\mathcal{A}^* \models \neg h_i^\exists(b)$. Let $\bar{u} \in \mathcal{A}$ be such that $\mathcal{A}^* \models \bar{f}_i(b) = \bar{u}$. We have $\mathcal{A} \models \neg V_i(\bar{u})$. Since $b \in \bar{a}$ and $\bar{a}$ is closed under the application of the functions f, $\bar{u} \subseteq \bar{a}$. So $\mathcal{B} \models \neg V_i(\bar{u})$, and we define the image of b in $\mathcal{B}^*$ using (4) above; we see that $\mathcal{B}^* \models \neg h_i^\exists(b)$. □

3 The Idea of the Construction

Fix a Π^1_1 m-complete set S. We want to build a computable sequence of complete decidable theories $(T_n)_{n\in\omega}$ such that

$$n \in S \iff T_n \text{ is relatively decidable.}$$

Recall from Theorem 1.3 that T_n is relatively decidable if and only if for each model $\mathcal{A} \models T_n$, there is $\bar{a} \in \mathcal{A}$ such that $\mathrm{Th}(\mathcal{A}, \bar{a})$ is model complete. Whether or not $\mathrm{Th}(\mathcal{A}, \bar{a})$ is model complete is reflected in the type of $\bar{a}$: $\mathrm{Th}(\mathcal{A}, \bar{a})$ is model complete if for every formula φ, there is an existential formula ψ such that the type of $\bar{a}$ says that φ and ψ are equivalent.

Consider the Stone space $S_1(T_n)$ of 1-types of T_n. This is compact, and we can think of it as being isomorphic to Cantor space 2^ω. We really want to think of it as being isomorphic to Baire space ω^ω. Consider the following embedding of Baire space into Cantor space.

Definition 3.1. There is an embedding Γ of Baire space ω^ω into Cantor space 2^ω:

$$\Gamma(\pi) = 0^{\pi(0)}10^{\pi(1)}10^{\pi(2)}1\cdots.$$

The image of Γ is the strings in 2^ω which have infinitely many 1's. We can also think of Γ as a map $\omega^{<\omega} \to 2^{<\omega}$:

$$\Gamma(\langle n_0, \ldots, n_\ell \rangle) = 0^{n_0}10^{n_1}1\cdots1\cdots0^{n_\ell}.$$

Heavily abusing notation, we write $S_1(T_n) = 2^\omega$ and write $\Gamma(\omega^\omega) \subseteq 2^\omega$ for the image of Baire space under this embedding. Given a 1-type $p(x) \in 2^\omega - \Gamma(\omega^\omega)$, we will ensure that $p(x)$ has an extension to a 2-type $q(x,y)$ which is isolated over $p(x)$ and such that $T \cup \{q(c,d)\}$ is model complete. Thus any model $\mathcal{A}$ containing element a realizing p also contains an element b with ab realizing q, and so $\mathrm{Th}(\mathcal{A}, ab)$ is model complete.

Now consider the remaining 1-types from $\Gamma(\omega^\omega) \cong \omega^\omega$. The set P_n of all types $p \in \Gamma(\omega^\omega)$ such that $T_n \cup \{p(c)\}$ is model complete is a Π^0_2 subset of $\Gamma(\omega^\omega) \cong \omega^\omega$. If the compliment of P_n is dense, then we can hope to make a model $\mathcal{A}$ of T_n that realizes only 1-types in $\Gamma(\omega^\omega) - P_n$, and T_n will not be relatively decidable. On the other hand, if the compliment of P_n is not dense in $\Gamma(\omega^\omega)$, then there is a formula $\varphi(x)$ such that $T_n \models \exists x\, \varphi(x)$ and for every type $p \in \Gamma(\omega^\omega)$ extending $\varphi(x)$, $T_n \cup \{p(c)\}$ is model complete. Then T_n is relatively decidable, because every $\mathcal{A} \models T_n$ contains an element a with $\mathcal{A} \models \varphi(a)$, and either the type of a is some $p \in P_n$ and $\mathrm{Th}(\mathcal{A}, a)$ is model complete, or the type of a is some $p \in 2^\omega - \Gamma(\omega^\omega)$ in which case $\mathcal{A}$ contains an element b such that $\mathrm{Th}(\mathcal{A}, ab)$ is model complete.

We will prove that there is a computable sequence of Π^0_2 sets C_n such that

$$n \in S \Longrightarrow C_n = \omega^\omega$$

and

$$n \notin S \Longrightarrow \omega^\omega - C_n \text{ is dense in } \omega^\omega.$$

Then we will construct T_n such that P_n corresponds to C_n, and prove that

$$n \in S \Longleftrightarrow T_n \text{ is relatively decidable.}$$

This is the idea of the construction, the details of which will follow in the next section.

The following lemma gives the construction of the sets C_n, which must also have several additional properties that we will use in the formal construction.

Lemma 3.2. *Let S be a Π^1_1 set. There is a computable sequence of Π^0_2 sets C_n such that*

$$n \in S \Longrightarrow C_n = \omega^\omega$$

and

$$n \notin S \Longrightarrow \omega^\omega - C_n \text{ is dense in } \omega^\omega.$$

Moreover,

$$C_n = \bigcap_i U^n_i$$

with the U^n_i being open sets $U^n_i = \bigcup_{\sigma \in W^n_i} [\sigma]$. The U_i are nested ($U^n_0 \supseteq U^n_1 \supseteq U^n_2 \supseteq \cdots$), the W_i are uniformly computable, and there are no $\sigma, \sigma' \in W_i$ with $\sigma \preceq \sigma'$. We also have the following properties:

(P1) for each $\sigma \in \omega^{<\omega}$, either an initial segment of σ is in W^n_i, or there is $\tau \succeq \sigma$ with $\tau \in W^n_i$;

(P2) if $n \notin S$, there is $\pi \in \omega^\omega$ such that:

- *for each $\sigma \in \omega^{<\omega}$, $\sigma\frown\pi \notin C_n$;*
- *for each i and $\sigma \in [i+2]^{\leq i+2}$, $\sigma\frown\pi \notin U^n_i$;*
- *for each i and $\sigma \in W^n_i$, $\sigma\frown\pi \notin U^n_{i+1}$.*

(P3) There is no $\sigma \in W^n_i$ with $\sigma \in [i+2]^{\leq i+2}$.

(P4) If $\sigma\frown k \in W^n_i$, then $\sigma\frown k' \in W^n_i$ for all k'.

Proof. Let T_n be a computable sequence of trees such that

$$n \notin S \Longleftrightarrow T_n \text{ has a path.}$$

Fix n for which we will define

$$C = C_n = \bigcap_i U_i$$

using $T = T_n$. We may assume that $T \subseteq \{1, 2, 3, \ldots\}^{<\omega}$, i.e., that 0 does not appear as an entry of any node on T. Define computable trees V_i inductively, starting with $V_0 = \{\varnothing\}$. Let

$$\hat{V}_{i+1} = V_i \cup [i+3]^{\leq i+3} \cup \bigcup_{\sigma \in [i+3]^{\leq i+3}} \{\sigma\frown\tau : \tau \in T\} \cup \bigcup_{\sigma \in V_i, \sigma\frown k \notin V_i} \{\sigma\frown k\frown\tau : \tau \in T\}$$

and let

$$V_{i+1} = \{\sigma\frown k : \sigma \in \hat{V}_{i+1}, k \in \omega\}.$$

Let $W_i = \{\sigma\frown k \in \omega^{<\omega} : \sigma \in V_i, \sigma\frown k \notin V_i\}$ and let

$$U_i = \omega^\omega - [V_n] = \bigcup_{\sigma \in W_i} [\sigma].$$

It is easy to see that the U_i are nested ($U^n_0 \supseteq U^n_1 \supseteq U^n_2 \supseteq \cdots$), the W_i are uniformly computable, and there are no $\sigma, \sigma' \in W_i$ with $\sigma \preceq \sigma'$.

If $n \in S$ then T does not have a path, and we can argue inductively that for each i, $[V_i] = \varnothing$. So $U_i = \omega^\omega$ and $C_n = \bigcap_i U_i = \omega^\omega$.

To see (P1), suppose that no initial segment of σ is in W_i; then $\sigma \in V_i$. Since $T \subseteq \{1,2,3,\dots\}^{<\omega}$, for some k, $\sigma^\frown 0^k \notin V_i$, and so for some k we have $\sigma^\frown 0^k \in W_i$.

If $n \notin S$, then T has a path π. To see (P2):

- for each $\sigma \in \omega^{<\omega}$, let i be sufficiently large that $\sigma \in [i]^{<i}$. Then $\sigma^\frown \pi \in [V_i]$ and so $\sigma^\frown \pi \notin C_n$;
- for each $\sigma \in [i+2]^{\leq i+2}$, $\sigma^\frown \pi \in [V_i]$ and so $\sigma^\frown \pi \notin U_i$;
- if $\sigma \in W_i$, then $\sigma \notin V_i$. Then we can write $\sigma = \tau^\frown k$ with $\tau \in V_i$ but $\tau^\frown k \notin V_i$. Then $\tau^\frown k^\frown \pi \in [V_{i+1}]$, and hence $\tau^\frown k^\frown \pi \notin U_{i+1}$;

In particular, $\omega^\omega - C_n$ is dense.

For (P3), note that $[i+3]^{\leq i+2} \subseteq V_2$. Thus such a $\sigma \in [i+2]^{\leq i+2}$ cannot be in W_i.

For (P4), if $\sigma^\frown k \in W_i$ but $\sigma^\frown k' \notin W_i$, then $\sigma \in V_i$ and so $\sigma^\frown k' \in V_i$. But then $\sigma^\frown k \in V_i$, a contradiction. $\square$

4 The Main Construction

We will now prove Theorem 1.4. Let S be a Π^1_1 m-complete set. Fix a sequence of Π^0_2 sets C_n such that

$$n \in S \Longrightarrow C_n = \omega^\omega$$

and

$$n \notin S \Longrightarrow \omega^\omega - C_n \text{ is dense in } \omega^\omega$$

as in Lemma 3.2. We will define a sequence of complete first-order theories T_n such that

$$C_n = \omega^\omega \Longleftrightarrow T_n \text{ is relatively decidable.}$$

Fix n for which we will define $T = T_n$ using $C = C_n$. (In general we drop the subscript n everywhere.) Write

$$C = \bigcap_i \bigcup_{\sigma \in W_i} [\sigma].$$

We have all the properties from Lemma 3.2.

We will define some intermediate theories and languages. We will begin by defining a quite simple theory T_0 in a language $\mathcal{L}_0$, and then we will take a Marker extension T_0^* of T_0. Then we will make a definitional expansion T_1 of T_0^*, and our final theory T will be a Marker extension T_1^* of T_1.

Let $\mathcal{L}_0$ be the language containing the following symbols:

- unary relations U and Q;
- for each $n \in \omega$, unary relations U_n and Q_n.

Let T_0 be the theory which says that:

(1) U and Q partition the universe, and the relations U_n can only hold of elements of U, and Q_n of elements of Q;

(2) every possible finite combination of the U_n occurs;

(3) the Q_n are disjoint, and there are infinitely many elements satisfying each of them.

Claim 4.1. *T_0 is complete, decidable, and has quantifier elimination.*[1] *Moreover, an $\mathcal{L}_0$-formula involving the symbols Q, U, $U_0, \dots, U_i$, and $Q_0, \dots, Q_i$, is equivalent to a quantifier-free formula with the same symbols.*

Proof. The proof is simple using standard techniques. □

Given a model $\mathcal{A}$ of T_0 and $u \in U^{\mathcal{A}}$, we can think of having a binary string $\rho_{2^\omega}[u] \in 2^\omega$ associated to u: $\rho_{2^\omega}[u](n) = 1$ if $\mathcal{A} \models U_n(u)$ and $\rho_{2^\omega}[u](n) = 0$ if $\mathcal{A} \models \neg U_n(u)$. Recall that we defined an embedding Γ of Baire space ω^ω into Cantor space 2^ω by

$$\Gamma(\pi) = 0^{\pi(0)}10^{\pi(1)}10^{\pi(2)}1\cdots.$$

The image of Γ is the set of strings which have infinitely many 1's. If $\rho_{2^\omega}[u]$ is in the image of Γ, define $\rho_{\omega^\omega}[u]$ to be the pre-image of $\rho_{2^\omega}[u]$ under Γ; that is, if

$$\rho_{2^\omega}[u] = 0^{n_0}10^{n_1}10^{n_2}1\cdots$$

then $\rho_{\omega^\omega}[u] = \langle n_0, n_1, n_2, \dots \rangle$.

Consider the Marker extension T_0^* of T_0 making the U_n into Σ_2 relations and the Q_n into Π_1 relations (i.e., making $\neg Q_n$ into Σ_1 relations). Let $\mathcal{L}_0^*$ be the language of this Marker extension.

Claim 4.2. *T_0^* is complete and decidable.*

Proof. T_0^* is complete by Corollary 2.7. It is decidable because it is complete and has a computable list of axioms. □

Let $(\varphi_i)_{i \in \omega}$ list the $\mathcal{L}_0^*$-formulas such that φ_i involves only the symbols U, Q, and the symbols (the functions f, g, and h) associated to $U_0, \dots, U_i$ and $Q_0, \dots, Q_i$ in the Marker extension. For each ℓ, let $(\eta_i^\ell)_{i \in \omega}$ list the elements of W_ℓ. Let $\mathcal{L}_1 \supseteq \mathcal{L}_0^*$ be the extended language which includes:

- for each $i \in \omega$ and $t \in \omega$, a relation $\alpha_{i,t}(x, \bar{y})$ where $|\bar{y}|$ is the arity of φ_i;
- for each $n \in \omega$ and $i \in \omega$, a relation $\beta_{i,n}(x, y, \bar{z})$ where $|\bar{z}|$ is the arity of φ_i.

Let T_1 be the theory extending T_0^* which says:

[1] For technical reasons, because the language is relational, we need to include $\top$ and $\bot$ in the language as otherwise there are no quantifier-free sentences.

(1) the relation $\alpha_{i,t}(x,\bar{y})$ can only hold of $x \in U$;

(2) the relation $\beta_{i,n}(x,y,\bar{z})$ can only hold of $x \in U$ and $y \in Q$;

(3) for each i and t, with $\tau_\ell = \Gamma(\eta^i_\ell) \in 2^\omega$,

$$\bigvee_{\ell=1,\dots,t}\left(\bigwedge_{n<|\tau_\ell|} U_n^{\tau_\ell(n)}(x)\right) \quad\longrightarrow\quad (\forall\bar{y})[\alpha_{i,t}(x,\bar{y}) \longleftrightarrow \varphi_i(\bar{y})]$$

and

$$\bigwedge_{\ell=1,\dots,t}\left(\bigvee_{n<|\tau_\ell|} \neg U_n^{\tau_\ell(n)}(x)\right) \quad\longrightarrow\quad (\forall\bar{y})[\alpha_{i,t}(x,\bar{y})]$$

(4) for each i and $n \in \omega$:

$$\left(Q_n(y) \wedge \bigwedge_{0\le m\le 2i} \neg U_{n+m}(x)\right) \quad\longrightarrow\quad (\forall\bar{z})[\beta_{i,n}(x,y,\bar{z}) \longleftrightarrow \varphi_i(\bar{z})]$$

and

$$\left(\neg Q_n(y) \vee \bigvee_{0\le m\le 2i} U_{n+m}(x)\right) \quad\longrightarrow\quad (\forall\bar{z})[\beta_{i,n}(x,y,\bar{z})]$$

Let T_1^* be the Marker extension of T_1 making the $\alpha_{i,t}(x,\bar{y})$ and $\beta_{i,n}(x,y,\bar{z})$ into Σ_1 relations. Note that T_1 is a definitional extension of T_0^*.

Claim 4.3. *T_1 and T_1^* are also decidable and complete.*

Proof. T_1 is decidable and complete because it is a definitional extension of T_0^*, which is decidable and complete. Then T_1^* is complete by Corollary 2.7 and decidable because it is computably axiomatizable. □

Lemma 4.4. *Every $\mathcal{L}_1^*$-formula is equivalent, modulo T_1^*, to a boolean combination of $\mathcal{L}_0^*$-formulas and quantifier-free $\mathcal{L}_1^*$-formulas. Here, we view an $\mathcal{L}_0^*$-formula as being restricted to the main sort of a model of T_1^*.*

Proof. By Corollary 2.6, each $\mathcal{L}_1^*$-formula is equivalent, modulo T_1^*, to the boolean combination of quantifier-free $\mathcal{L}_1^*$-formulas and $\mathcal{L}_1$-formulas. Each $\mathcal{L}_1$-formula is equivalent, modulo T_1, to an $\mathcal{L}_0^*$-formula as T_1 is a definitional extension of T_0^*. □

We are now ready to show that the theory T_1^* is the theory we wanted to build, i.e., that:

$$C = \omega^\omega \iff T = T_1^* \text{ is relatively decidable.}$$

Recall from Theorem 1.3 that T_1^* is relatively decidable if and only if for each model $\mathcal{M}$ of T_1^*, there is a tuple $\bar{a} \in \mathcal{M}$ such that $\mathrm{Th}(\mathcal{M},\bar{a})$ is model complete.

Lemma 4.5. *If $C = \omega^\omega$, and $\mathcal{M}$ is a model of T_1^*, then there are $a,b \in \mathcal{M}$ such that* $\mathrm{Th}(\mathcal{M},ab)$ *is model complete.*

Proof. Fix $a \in U^{\mathcal{M}}$. We have two possibilities:

Case 1. For some n, for every $m \geq 0$, $\mathcal{M} \vDash \neg U_{n+m}(a)$.

Choose $b \in \mathcal{M}$ such that $\mathcal{M} \vDash Q_n(b)$. We will show that $\mathrm{Th}(\mathcal{M}, ab)$ is model complete. Given an $\mathcal{L}_1^*$-formula φ, by Lemma 4.4 write φ in disjunctive normal form as a disjunction of formulas each of which is the conjunction of an $\mathcal{L}_0^*$-formula φ_i and a quantifier-free $\mathcal{L}_1^*$-formula ψ. To show that φ is equivalent to an existential formula, we just need to show that φ_i is equivalent to an existential formula with parameters a, b. This is the case because

$$\mathcal{M} \vDash (\forall \bar{z})[\beta_{i,n}(a, b, \bar{z}) \longleftrightarrow \varphi_i(\bar{z})].$$

Case 2. For every n, there is $m \geq n$, such that $\mathcal{M} \vDash U_{n+m}(a)$.

We will show that $\mathrm{Th}(\mathcal{M}, a)$ is model complete. Given an $\mathcal{L}_1^*$-formula φ, by Lemma 4.4 write φ in disjunctive normal form as a disjunction of formulas each of which is the conjunction of an $\mathcal{L}_0^*$-formula φ_i and a quantifier-free $\mathcal{L}_1^*$-formula ψ. To show that φ is equivalent to an existential formula, we just need to show that φ_i is equivalent to an existential formula with parameter a. Let $\pi = \rho_{\omega^\omega}[a]$. Since

$$C = \bigcap_i \bigcup_{\sigma \in W_i} [\sigma] = \omega^\omega$$

there is $\sigma \in W_i$ such that $\sigma \leq \pi$. Let $\tau = \Gamma(\sigma)$; then

$$\mathcal{M} \vDash \bigwedge_{n < |\tau|} U_n^{\tau(n)}(a).$$

So

$$\mathcal{M} \vDash (\forall \bar{y})[\alpha_{i,t}(a, \bar{y}) \longleftrightarrow \varphi_i(\bar{y})]. \qquad \square$$

Lemma 4.6. *If $C \neq \omega^\omega$, there is a model $\mathcal{M}$ of T_1^* such that for no $\bar{a} \in \mathcal{M}$ is $\mathrm{Th}(\mathcal{M}, \bar{a})$ model complete.*

Proof. Since $C \neq \omega^\omega$, $\omega^\omega - C$ is dense in ω^ω. Build a model $\mathcal{A}_0$ of T_0 such that for every $x \in U$, $\rho_{2^\omega}[x]$ has infinitely many 1's, so that $\rho_{2^\omega}[x]$ is in the image of the embedding $\Gamma : \omega^\omega \to 2^\omega$, and $\rho_{\omega^\omega}[x]$ is defined. Moreover, build $\mathcal{A}_0$ so that $\rho_{\omega^\omega}[x] \notin C$ for each $x \in U$. We can do this because $\omega^\omega - C$ is dense. More specifically, for each $\sigma \in \omega^{<\omega}$, by (P1) and (P2) choose $\tau_\sigma \geq \sigma$ with $\tau_\sigma \in W_i$. Then $\tau_\sigma \frown \pi \notin \bigcup_{\sigma \in W_{i+1}} [\sigma]$. Have $U^{\mathcal{A}_0}$ consist of an x_σ for each $\sigma \in \omega^{<\omega}$ with $\rho_{\omega^\omega}[x_\sigma] = \tau_\sigma \frown \pi$.

This model $\mathcal{A}_0$ of T_0 gives rise to a unique model $\mathcal{M} = \mathcal{A}_1^*$ of T_1^*: first we take a Marker extension to get a model $\mathcal{A}_0^*$ of T_0^*, then we take a definitional expansion to get a model $\mathcal{A}_1$ of T_1, and then we take a Marker extension to get a model $\mathcal{M} = \mathcal{A}_1^*$ of T_1^*.

Fix $\bar{a} \in \mathcal{M}$. We claim that $\mathrm{Th}(\mathcal{M}, \bar{a})$ is not model complete. We will do this by finding an extension $\mathcal{N} \supseteq \mathcal{M}$ with $\mathcal{N} \vDash \mathrm{Th}(\mathcal{M}, \bar{a})$ that is not an elementary extension of $\mathcal{M}$. We will build $\mathcal{N}$ by finding $\mathcal{B}_0 \supseteq \mathcal{A}_0$, $\mathcal{B}_0^* \supseteq \mathcal{A}_0^*$, $\mathcal{B}_1 \supseteq \mathcal{A}_1$, and finally $\mathcal{N} = \mathcal{B}_1^* \supseteq \mathcal{A}_1^* = \mathcal{M}$. The notion of substructure here is that of Definition 2.8.

Let $\bar{a}$ consist of $a_1, \ldots, a_n \in U^{\mathcal{A}_0}$ and $b_1, \ldots, b_m \in Q^{\mathcal{A}_0}$ and... . We may assume that...

Let k be sufficiently large that:

(1) for each $i = 1, \dots, m$, if $Q_\ell(b_i)$, then $\ell < k$;

(2) for each $i = 1, \dots, n$, $\rho_{\omega^\omega}[a_i] \notin \bigcup_{\sigma \in W_k}[\sigma]$; and

(3) for each $i = 1, \dots, n$, there is $j \in \{k, \dots, 2k\}$ such that $\mathcal{A}_0 \vDash U_j(a_i)$.

To see why we can find k sufficiently large to make (3) true, let $\pi \in \omega^\omega$ as chosen above be such that for each i there is $\sigma_i \in \omega^{<\omega}$ such that $\rho_{\omega^\omega}[a_i] = \sigma_i \frown \pi$. Then

$$\mathcal{A}_0 \vDash U_{\ell+|\Gamma(\sigma_i)|}(a_i) \Longleftrightarrow \Gamma(\pi)(\ell) = 1.$$

Choose k such that $k \geq |\Gamma(\sigma_i)|$ for each i, and such that $\Gamma(\pi)(k) = 1$.

Define $\mathcal{B}_0 \supseteq \mathcal{A}_0$ as follows:

- For each $b \in Q^{\mathcal{A}_0}$:
  - If $\ell < k$, set
  $$\mathcal{B}_0 \vDash Q_\ell(b) \Longleftrightarrow \mathcal{A}_0 \vDash Q_\ell(b).$$
  - If $\ell \geq k$, set
  $$\mathcal{B}_0 \vDash \neg Q_\ell(b).$$

  Note that for $b_i \in \bar{b}$ and any ℓ we have
  $$\mathcal{B}_0 \vDash Q_\ell(b_i) \Longleftrightarrow \mathcal{A}_0 \vDash Q_\ell(b_i).$$

- For each $a \in U^{\mathcal{A}_0}$:
  - If $a = a_1, \dots, a_n$, set
  $$\mathcal{B}_0 \vDash U_\ell(a) \Leftrightarrow \mathcal{A}_0 \vDash U_\ell(a).$$
  - Otherwise, suppose that there is $\ell < k$ and $\sigma \preceq \rho_{\omega^\omega}[a]$ with $\sigma \in W_\ell$. Choose ℓ to be the greatest such. Let $0^{c_0}10^{c_1}10^{c_2}1\cdots10^{c_t}$ be the first k bits of $\rho_{2^\omega}[a]$. Let $c_0c_1\cdots c_{t-1}d$ be the initial segment of $\rho_{\omega^\omega}[a]$ of length $t+1$, so that $0^{c_0}10^{c_1}10^{c_2}1\cdots10^d$ is an initial segment of $\rho_{2^\omega}[a]$.

    Define $\mu \in \omega^\omega$ with $\Gamma(\mu) \succeq 0^{c_0}10^{c_1}10^{c_2}1\cdots10^{c_t}1$ according to the following cases. Then set
    $$\mathcal{B}_0 \vDash U_\ell(a) \Leftrightarrow \Gamma(\mu)(\ell) = 1.$$

    Note that for $\ell < k$,
    $$\mathcal{B}_0 \vDash U_\ell(a) \Leftrightarrow \Gamma(\mu)(\ell) = 1 \Leftrightarrow \mathcal{A}_0 \vDash U_\ell(a).$$

    We have $\rho^{\mathcal{B}_0}_{\omega^\omega}(a) = \mu$.

    * Suppose that $\ell = k-1$. By (P3), since $c_0c_1\cdots c_t \in [k+1]^{\leq k+1}$ we have $c_0c_1\cdots c_t \notin W_{k-1}$. Let t be such that $\sigma = \eta^{k-1}_t \in W_{k-1}$. By (P1) we can choose $s \geq t$ such that $c_0\cdots c_t \preceq \eta^{k-1}_s \in W_{k-1}$ (indeed, for each ℓ, there is $\eta \succeq c_0\cdots c_t\ell$ with $\eta \in W_{k-1}$, so for some ℓ this σ is η^{k-1}_s with $s \geq t$). Then by (P2) there is $\mu \succeq \sigma_s$ with $\mu \notin \bigcup_{\sigma \in W_k}[\sigma]$.

* Suppose that $\ell < k-1$ and $\sigma \le c_0c_1\cdots c_t$. Then by construction of $\mathcal{A}_0$, $\rho_{\omega^\omega}[a] = \sigma\widehat{\ }\pi$. Let $\mu = \sigma\widehat{\ }\pi$.
* Suppose that $\ell < k-1$ and $\sigma = c_0c_1\cdots c_{t-1}d$. Then by (P4), $c_0c_1\cdots c_t \in W_\ell$. By (P2) choose $\mu \ge c_0c_1\cdots c_t$ such that $\mu \notin \bigcup_{\sigma \in W_{\ell+1}}[\sigma]$.
* Suppose that $\ell < k-1$ and $\sigma > c_0c_1\cdots c_{t-1}d$. Then $c_0c_1\cdots c_{t-1}d \notin W_\ell$ and by (P4), $c_0c_1\cdots c_t \notin W_\ell$. Let t be such that $\sigma = \eta_t^\ell \in W_\ell$. There is $s \ge t$ such that $\eta_s^\ell \ge c_0c_1\cdots c_t$. By (P2) choose $\mu \ge \eta_s^\ell$ such that $\mu \notin \bigcup_{\sigma \in W_{\ell+1}}[\sigma]$.

– Otherwise, let $0^{c_0}10^{c_1}10^{c_2}1\cdots 10^{c_t}$ be the first k bits of $\rho_{2^\omega}[a]$. Then $c_0c_1\cdots c_t \in [k+1]^{\le k+1}$. By (P2), we can choose $\mu \ge c_0c_1\cdots c_t$ such that $\mu \notin W_k$. Set

$$\mathcal{B}_0 \vDash U_\ell(a) \Leftrightarrow \Gamma(\mu)(\ell) = 1.$$

Note that for $\ell < k$,

$$\mathcal{B}_0 \vDash U_\ell(a) \Leftrightarrow \Gamma(\mu)(\ell) = 1 \Leftrightarrow \mathcal{A}_0 \vDash U_\ell(a).$$

Then $\rho_{\omega^\omega}^{\mathcal{B}_0}(a) = \mu$. Also, $\mathcal{B}_0 \vDash U_k(a)$.

• Add new elements to $U^{\mathcal{B}_0}$ and $Q^{\mathcal{B}_0}$ to extend $\mathcal{B}_0$ to a model of T_0.

Claim 4.7. $\mathcal{B}_0, \bar{a} \vDash \mathrm{Th}(\mathcal{A}_0, \bar{a})$.

Proof. We have that $\mathcal{B}_0 \vDash T_0$ and T_0 has quantifier elimination. So we just need to note that $\bar{a}$ has the same atomic type in $\mathcal{B}_0$ that it has in $\mathcal{A}_0$. □

Claim 4.8. $\mathcal{B}_0 \supseteq \mathcal{A}_0$.

Proof. We need to see that if $\mathcal{A}_0 \vDash \neg Q_\ell(a)$ then $\mathcal{B}_0 \vDash \neg Q_\ell(a)$; this can be immediately seen from the definition of $\mathcal{B}_0$. There is nothing to check with the relations U_ℓ because we are making these Σ_2^0 in the Marker extension. □

By Lemma 2.9, since $\mathcal{A}_0 \subseteq \mathcal{B}_0$, we can choose a Marker extension $\mathcal{B}_0^* \supseteq \mathcal{A}_0^*$ of $\mathcal{B}_0$. Moreover, we can choose $\mathcal{B}_0^*$ such that $\mathcal{B}_0^*, \bar{a} \vDash \mathrm{Th}(\mathcal{A}_0^*, \bar{a})$.

Claim 4.9. *For $i < k$ and $\bar{a} \in \mathcal{A}_0^*$,*

$$\mathcal{A}_0^* \vDash \varphi_i(\bar{a}) \Longleftrightarrow \mathcal{B}_0^* \vDash \varphi_i(\bar{a})$$

Proof. Recall that φ_i involves only the symbols U, Q, and the symbols (the functions f, g, and h) associated to $U_0, \ldots, U_i$ and $Q_0, \ldots, Q_i$ in the Marker extension. We defined $\mathcal{B}_0$ such that for $x \in \mathcal{A}_0$ and $i < k$

$$\mathcal{A}_0 \vDash U_i(x) \Longleftrightarrow \mathcal{B}_0 \vDash U_i(x)$$

and

$$\mathcal{A}_0 \vDash Q_i(x) \Longleftrightarrow \mathcal{B}_0 \vDash Q_i(x).$$

By Claim 4.1 (quantifier elimination), for any $\mathcal{L}_0$-formula ψ involving only U, Q, $U_0, \ldots, U_i$ and $Q_0, \ldots, Q_i$, and $\bar{x} \in \mathcal{A}_0$,

$$\mathcal{A}_0 \vDash \psi(\bar{x}) \Longleftrightarrow \mathcal{B}_0 \vDash \psi(\bar{x}).$$

Now by Lemma 2.5 φ_i is equivalent to a boolean combination of $\mathcal{L}_0$-formulas (involving only U, Q, $U_0, \ldots, U_i$ and $Q_0, \ldots, Q_i$) and quantifier-free $\mathcal{L}_0^*$-formulas. So, as $\mathcal{A}_0^* \subseteq \mathcal{B}_0^*$,

$$\mathcal{A}_0^* \vDash \varphi_i(\bar{a}) \Longleftrightarrow \mathcal{B}_0^* \vDash \varphi_i(\bar{a}).$$ □

Now let $\mathcal{B}_1$ be the definitional extension of $\mathcal{B}_0^*$.

Claim 4.10. $\mathcal{B}_1 \supseteq \mathcal{A}_1$.

Proof. We have two claims to prove, each of which divides into a number of cases.

(1) Given $x, \bar{y} \in \mathcal{A}_1$, if $\mathcal{A}_1 \models \alpha_{i,t}(x, \bar{y})$ then $\mathcal{B}_1 \models \alpha_{i,t}(x, \bar{y})$.

- Suppose that $x \in \bar{a}$ and $i < k$, then

$$\mathcal{B}_0 \models U_\ell(a) \Leftrightarrow \mathcal{A}_0 \models U_\ell(a)$$

  and by Claim 4.9, $\mathcal{A}_1 \models \varphi_i(\bar{y})$ if and only if $\mathcal{B}_1 \models \varphi_i(\bar{y})$.

- Suppose that $x \notin \bar{a}$ and $i < k$. Then by Claim 4.9, $\mathcal{A}_1 \models \varphi_i(\bar{y})$ if and only if $\mathcal{B}_1 \models \varphi_i(\bar{y})$.

  So the only case we have to worry about is if, with $\tau_\ell = \Gamma(\sigma_\ell) \in 2^\omega$,

$$\mathcal{A}_0 \models \bigwedge_{\ell=1,\ldots,t} \bigvee_{n<|\tau_\ell|} \neg U_n^{\tau_t(n)}(x)$$

  but for some $\ell = 1, \ldots, t$,

$$\mathcal{B}_0 \models \bigwedge_{n<|\tau_\ell|} U_n^{\tau_\ell(n)}(x).$$

  If it was the case that $\rho_{\omega^\omega}(a) \notin W_i$, then we would have set

$$\mathcal{B}_0 \models U_\ell(a) \Leftrightarrow \mathcal{A}_0 \models U_\ell(a).$$

  So the only problem would be if $\rho_{\omega^\omega}(a) \in W_i$, but $\sigma_1, \ldots, \sigma_t \nprec \rho_{\omega^\omega}(a)$. In this case, we ensured that for some $s > t$ we have $\sigma_s \prec \rho_{\omega^\omega}^{\mathcal{B}_0}(a)$. So we cannot have

$$\mathcal{B}_0 \models \bigwedge_{n<|\tau_\ell|} U_n^{\tau_\ell(n)}(x)$$

  for any $\ell = 1, \ldots, t$.

- Suppose that $i \geq k$. then since $\rho_{\omega^\omega}^{\mathcal{B}}(x) \notin W_i$, we have $\mathcal{B}_1 \models (\forall \bar{z})\alpha_{i,t}(x, \bar{z})$.

(2) Given $x, y, \bar{z} \in \mathcal{A}_1$, if $\mathcal{A}_1 \models \beta_{i,n}(x, y, \bar{z})$ then $\mathcal{B}_1 \models \beta_{i,n}(x, y, \bar{z})$.

- Suppose that $x \in \bar{a}$, $n < k$, and $i < k$. Then by Claim 4.9, $\mathcal{A}_1 \models \varphi_i(\bar{z})$ if and only if $\mathcal{B}_1 \models \varphi_i(\bar{z})$. Also $\mathcal{A}_1 \models Q_n(y)$ if and only if $\mathcal{B}_1 \models Q_n(y)$, and $\mathcal{A}_1 \models U_\ell(x)$ if and only if $\mathcal{B}_1 \models U_\ell(x)$.

- Suppose that $x \in \bar{a}$, $n < k$, and $i \geq k$. Then there is $\ell = k, \ldots, 2k$ such that $\mathcal{A}_0 \models U_\ell(x)$. So

$$\mathcal{A}_0 \models \bigvee_{0 \leq m \leq 2i} U_{n+m}(x)$$

  and so, since $\mathcal{A}_1 \models U_\ell(x)$ if and only if $\mathcal{B}_1 \models U_\ell(x)$,

$$\mathcal{B}_0 \models \bigvee_{0 \leq m \leq 2i} U_{n+m}(x).$$

  Thus $\mathcal{B}_1 \models (\forall \bar{u})\beta_{i,n}(x, y, \bar{u})$.

- Suppose that $x \notin \bar{a}$, and $n < k$ but $i \geq k$. Then examining the definition of $\mathcal{B}_0$, we have two possibilities:
  - $\mathcal{B}_1 \models U_k(x)$. So $\mathcal{B}_1 \models \bigvee_{j=n,\dots,n+2i} U_j(x)$. Thus $\mathcal{B}_1 \models (\forall \bar{u})\beta_{i,n}(x,y,\bar{u})$.
  - $\mathcal{A}_1 \models U_\ell(x)$ if and only if $\mathcal{B}_1 \models U_\ell(x)$ and $\rho_{\omega^\omega}[x] = \sigma^\frown\pi$, with $|\sigma| < k$. Then
  $$\mathcal{B}_0 \models \bigvee_{0 \leq m \leq 2i} U_{n+m}(x).$$
  Thus $\mathcal{B}_1 \models (\forall \bar{u})\beta_{i,n}(x,y,\bar{u})$.
- Suppose that $n < k$ and $i < k$. Then by Claim 4.9, $\mathcal{A}_1 \models \varphi_i(\bar{z})$ if and only if $\mathcal{B}_1 \models \varphi_i(\bar{z})$. Also $\mathcal{A}_1 \models Q_n(y)$ if and only if $\mathcal{B}_1 \models Q_n(y)$. So our only problem can occur if
  $$\mathcal{A}_0 \models \bigvee_{0 \leq m \leq 2i} U_{n+m}(x)$$
  but
  $$\mathcal{B}_0 \models \bigwedge_{0 \leq m \leq 2i} \neg U_{n+m}(x).$$
  But for $m < k$, $\mathcal{A}_0 \models U_m(x)$ if and only if $\mathcal{B}_0 \models U_m(x)$; and, examining all the cases in the definition of $\mathcal{B}_0$, we can see that the least $m \geq k$ such that $\mathcal{A}_0 \models U_m(x)$ is $\geq$ the least $m' \geq k$ such that $\mathcal{B}_0 \models U_{m'}(x)$. So this problematic situation never occurs.
- Suppose that $n \geq k$. Then $\mathcal{B}_1 \models \neg Q_n(y)$ and so $\mathcal{B}_1 \models (\forall \bar{u})\beta_{i,n}(x,y,\bar{u})$. □

Claim 4.11. $\mathcal{B}_1, \bar{a} \models \mathrm{Th}(\mathcal{A}_1, \bar{a})$.

Proof. This is because $\mathcal{B}_0^*, \bar{a} \models \mathrm{Th}(\mathcal{A}_0^*, \bar{a})$ and $\mathcal{B}_1$ and $\mathcal{A}_1$ are definitional extensions of $\mathcal{B}_0^*$ and $\mathcal{A}_0^*$ respectively. □

Let $\mathcal{B}_1^* \supseteq \mathcal{A}_1^*$ be the Marker extension of $\mathcal{B}_1$. By Lemma 2.9 we can choose $\mathcal{B}_1^*$ such that $\mathcal{B}_1^*, \bar{a} \models \mathrm{Th}(\mathcal{A}_1^*, \bar{a})$. Note that $\mathcal{B}_1^*$ is not an elementary extension of $\mathcal{A}_1^*$; indeed, given $b \in \mathcal{A}_1^*$ with $\mathcal{A}_1^* \models Q_k(b)$, we have $\mathcal{B}_1^* \models \neg Q_k(b)$. □

References

[BHTK+] N. Bazhenov, M. Harrison-Trainor, I. Kalimullin, A. Melnikov, and K. Meng Ng. Automatic and polynomial-time algebraic structures. *J. Symbolic Logic*. To appear.

[CFG+07] W. Calvert, E. Fokina, S. S. Goncharov, J. F. F. Knight, O. Kudinov, A. S. Morozov, and V. Puzarenko. Index sets for classes of high rank structures. *J. Symbolic Logic*, 72(4):1418–1432, 2007.

[CMS] J. Chubb, R. Miller, and R. Solomon. Model completeness and relative decidability. Preprint.

[DKL⁺15] R. G. Downey, A. M. Kach, S. Lempp, A. E. M. Lewis-Pye, A. Montalbán, and D. D. Turetsky. The complexity of computable categoricity. *Adv. Math.*, 268:423–466, 2015.

[DM08] R. G. Downey and A. Montalbán. The isomorphism problem for torsion-free abelian groups is analytic complete. *J. Algebra*, 320(6):2291–2300, 2008.

[FGK⁺15] E. B. Fokina, S. S. Goncharov, V. Kharizanova, O. V. Kudinov, and D. Turetski. Index sets of n-decidable structures that are categorical with respect to m-decidable representations. *Algebra Logika*, 54(4):520–528, 544–545, 547–548, 2015.

[Fok07] E. B. Fokina. Index sets of decidable models. *Sibirsk. Mat. Zh.*, 48(5):1167–1179, 2007.

[GBM15a] S. S. Goncharov, N. A. Bazhenov, and M. I. Marchuk. The index set of Boolean algebras that are autostable relative to strong constructivizations. *Sibirsk. Mat. Zh.*, 56(3):498–512, 2015.

[GBM15b] S. S. Goncharov, N. A. Bazhenov, and M. I. Marchuk. Index sets of constructive models of natural classes that are autostable with respect to strong constructivizations. *Dokl. Akad. Nauk*, 464(1):12–14, 2015.

[GK02] S. S. Goncharov and J. Knight. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.

[HT18] M. Harrison-Trainor. There is no classification of the decidably presentable structures. *J. Math. Log.*, 18(2):1850010, 41, 2018.

[LS07] S. Lempp and T. A. Slaman. The complexity of the index sets of $\aleph_0$-categorical theories and of Ehrenfeucht theories. In *Advances in logic*, volume 425 of *Contemp. Math.*, pages 43–47. Amer. Math. Soc., Providence, RI, 2007.