# Relative to any non-arithmetic set

Matthew Harrison-Trainor

May 29, 2025

**Abstract**

Given a countable structure $\mathcal{A}$, the degree spectrum of $\mathcal{A}$ is the set of all Turing degrees which can compute an isomorphic copy of $\mathcal{A}$. One of the major programs in computable structure theory is to determine which (upwards closed, Borel) classes of degrees form a degree spectrum. We resolve one of the major open problems in this area by showing that the non-arithmetic degrees are a degree spectrum. Our main new tool is a new form of unfriendly jump inversions where the back-and-forth types are maximally complicated. This new tool has several other applications.

# 1 Introduction

It is well-known that if a set $A \subseteq \mathbb{N}$ is computable relative to every non-computable set then $A$ is computable [KP54]. Nevertheless the non-computable sets do hold some non-trivial information in common. Answering a question of Lempp, Slaman [Sla98] and Wehner [Weh98] independently showed that there is a mathematical problem which can be solved by exactly the non-computable sets, namely the problem of computing a copy of some particular mathematical structure.

**Theorem 1.1** (Slaman [Sla98] and Wehner [Weh98])**.** *There is a countable structure $\mathcal{M}$ such that for any set $X$, $X$ computes an isomorphic copy of $\mathcal{M}$ if and only if $X$ is not computable.*

By computing a copy of $\mathcal{M}$ we mean that we compute an isomorphic structure with domain $\mathbb{N}$ together with all of the functions, relations, and constants on $\mathbb{N}$.

This result fits into a general program in computable structure theory of studying the *degree spectra* of countable structures. Given a countable structure $\mathcal{A}$, the degree spectrum of $\mathcal{A}$ is the set of all Turing degrees that can compute an isomorphic copy of $\mathcal{A}$.[1] This is a measurement of the difficulty of computing a copy of $\mathcal{A}$ and captures the information that is encoded in the structure $\mathcal{A}$. Classifying the degree spectra of countable structures is one of the central problems of computable structure theory. Though we have a large number of results showing that certain classes of Turing degrees are degree spectra, and some small number of results showing that particular classes are not degree spectra, we still seem far from a complete classification. Classifying the degree spectra means determining which

[1] Often the degree spectrum is defined as the set of Turing degrees of atomic diagrams copies of $\mathcal{A}$, but by a result of Knight [Kni86] these definitions are equivalent except for the automorphically trivial structures.

computability-theoretic properties of sets of natural numbers can be encoded as the set of solutions to the problem of building a copy of a particular structure. A particular class of degrees being a degree spectra is a $\mathbf{\Sigma}^1_1$—and in fact, by Scott's isomorphism theorem [Sco65], Borel—definition of that class which takes a very particular form, and so a classification would determine which upwards-closed Borel sets of degrees admit a definition of this form. The above result of Slaman and Wehner shows that the non-computable degrees form a degree spectrum and hence are definable in this way.

The simplest degree spectra are the upwards cones, which consist of all degrees that compute a particular set $A$. Enumeration cones, the set of all degrees that can enumerate a set $A$, are also degree spectra. In addition to the Slaman-Wehner spectrum of non-computable degrees, other examples of degree spectra include the high degrees [Joc72], the hyperimmune degrees [CK10], the array non-recursive degrees and degrees without retracable jump [DGT13], the non-low$_\alpha$ degrees [GHK$^+$05, KF18], and the non-$K$-trivial degrees [KF18]. On the other hand one can probably count on a single hand the results which show that some class of degrees is not a degree spectrum. The most well-known is that a non-trivial countable union of upwards cones (or enumeration cones) is not a degree spectrum. (See Section V.3.2 of [Mon21] where Montalbán attributes this to Knight and her collaborators.) Furthermore, building on ideas of Andrews and Miller [AM15], Montalbán shows that the upwards closure of any $F_\sigma$ set of reals is never a degree spectrum unless it is trivial, i.e., an enumeration cone. As a corollary of this, we get the result of Andrews and Miller that the DNC degrees, ML-random degrees, and PA degrees are not degree spectra.

While there are uncountably many possible degree spectra, there are only countably many co-null degree spectra and so a good amount of work has concentrated on these. (To see this Greenberg, Montalbán, and Slaman [GMS11] show—and, in that paper, note that Kalimullin and Nies had independently announced—that any such degree spectrum must include the Turing degree of Kleene's $\mathcal{O}$, the complete $\Pi^1_1$ set.) The Slaman-Wehner degree spectrum of the non-computable degrees is an example of such a degree spectrum.

Greenberg, Montalbán, and Slaman [GMS13] asked whether the Slaman-Wehner theorem holds if we replace Turing reducibility by further higher reducibilities. They showed that it does hold for hyperarithmetic reducibility, where $A$ is hyperarithmetic in $B$ if $A$ is computable from some ordinal iterate $B^{(\beta)}$ of the Turing jump of $B$ for some $B$-computable $\beta$, or equivalently, if $A$ is $\Delta^1_1$ in $B$.

**Theorem 1.2** (Greenberg, Montalbán, Slaman [GMS13]). *There is a countable structure $\mathcal{M}$ such that for any set $X$, $X$ computes an isomorphic copy of $\mathcal{M}$ if and only if $X$ is not hyperarithmetic.*

Thus the non-hyperarithmetic degrees form a degree spectrum. On the other hand, in [GMS11] they show that this is not true for relative constructibility.

Arithmetic reducibility lies between Turing (computable) reducibility and hyperarithmetic reducibility. A set $A$ is arithmetic in $B$ if there is a formula $\varphi$ in that language of arithmetic defining $A$ in $\mathbb{N}$ with parameter $B$, i.e., $A = \{n : \mathbb{N} \vDash \varphi(n, B)\}$. Equivalently, $A$ is arithmetic in $B$ if $A$ is computable from some finite iterate $B^{(n)}$ of the Turing jump of $B$. Greenberg, Montalbán, and Slaman asked whether the Slaman-Wehner theorem holds for arithmetic reducibility: Are the non-arithmetic degrees a degree spectrum? For the past fifteen or so years, of all classes of Turing degrees the non-arithmetic degrees have been the most prominent class for which we have not been able to show that they either are or are not a degree spectrum. The main result of this paper is a resolution of this question.

**Theorem 1.3.** *There is countable structure $\mathcal{M}$ such that for any set $X$, $X$ computes an isomorphic copy of $\mathcal{M}$ if and only if $X$ is not arithmetic.*

Thus the non-arithmetic degrees are a degree spectrum. Our strategy is influenced by several recent developments on the computational properties of Scott sentences, and on the road to proving our main theorem we add to these developments. In the next section we take a detour to describe these. The main new technique is a new kind of Marker extensions and jump inversions, which we call *unfriendly Marker extensions* and *unfriendly jump inversions* (because they lead to structures which are not friendly in the sense of Ash and Knight [AK00]).

We end this section with a more technical discussion of why it was hard to resolve the question of whether non-arithmetic is a degree spectrum. One can think of the Slaman-Wehner result as saying that the non-$\Delta^0_1$ degrees are a degree spectrum. If we could show that for every $n$, the non-$\Delta^0_n$ degrees are a degree spectrum, and if there was a uniform list of examples, then non-arithmetic would be the degree spectrum of the multisorted structure combining these examples. Though this is true for the non-$\Delta^0_2$ degrees, which Kalimullin [Kal08] showed do form a degree spectrum by a modification of the Slaman-Wehner strategy, for $n \geq 3$ the non-$\Delta^0_n$ degrees are not a degree spectrum. This was shown by Andrews, Cai, Kalimullin, Lempp, Miller, and Montalbán [ACK+16]. For $n = 3$ for example, their strategy was as follows. If $\mathcal{M}$ was a structure whose degree spectrum contained all of the non-$\Delta^0_3$ degrees, then the existential theory of $\mathcal{M}$ would be c.e. They find a $\Delta^0_3$ degree $\mathbf{a}$ and a non-$\Delta^0_3$ degree $\mathbf{b}$ such that $\mathbf{a}' = \mathbf{b}'''$ and $\mathbf{b}''$ has $\mathbf{a}$-c.e. degree. Then $\mathbf{b}$ computes a copy of $\mathcal{M}$. They show, by an argument using Robinson low guessing, that $\mathbf{a}$ can guess at the third jump of $\mathbf{b}$ and thus build a copy of $\mathcal{M}$, showing that $\mathcal{M}$ has a $\Delta^0_3$ copy. However no strategy like this could be applied to show that the non-arithmetic degrees are not a degree spectrum: No finite number of jumps of an arithmetic degree can compute a non-arithmetic degree, let alone any of its jumps.

On the other hand, the strategy of Greenberg, Montalbán, and Slaman [GMS13] for showing that the non-hyperarithmetic degrees are a degree spectrum relies on properties of infinite jumps. Namely, together with an argument involving non-standard jump hierarchies, they use the fact that a degree is hyperarithmetic if and only if it is low$_\alpha$ for some computable ordinal $\alpha$, as $\mathbf{0}^{(\alpha)}$ is low$_{\alpha\cdot\omega}$ because

$$\left(\mathbf{0}^{(\alpha)}\right)^{(\alpha\cdot\omega)} = \mathbf{0}^{(\alpha+\alpha\cdot\omega)} = \mathbf{0}^{(\alpha\cdot\omega)}.$$

The analogous statement is far from being the case within the arithmetic degrees: $\mathbf{0}'$ is arithmetic but not low$_n$ for any $n$. Instead we need to develop new techniques.

# 2 Other applications of unfriendly jump inversions

In this section we will describe some other applications of our method of unfriendly jump inversions, which will also provide some initial motivation for the method.

One of the main themes in computable structure theory over the least thirty years, especially championed by Ash and Knight in their book [AK00] and more recently Montalbán in his books [Mon21, Mon], has been connections between computational properties and structural or definable properties of structures. The logic we use is the infinitary logic $\mathcal{L}_{\omega_1\omega}$ which allows countably infinite conjunctions and disjunctions. The complexity of the formulas defining a property are particularly important, where by complexity we mean the number of alternations between existential and universal quantifiers.

- A formula is $\Sigma_0$ and $\Pi_0$ if it is finitary quantifier-free.
- A formula is $\Sigma_{n+1}$ if it is of the form

$$\bigvee_i \exists \bar{x}_i \psi_i(\bar{x}_i)$$

  where each $\psi_i$ is $\Pi_n$.
- A formula is $\Pi_{n+1}$ if it is of the form

$$\bigwedge_i \forall \bar{x}_i \psi_i(\bar{x}_i)$$

  where each $\psi_i$ is $\Pi_n$.

(This complexity hierarchy continues through the infinite ordinals but in this paper we will mostly consider the case of finite $n$. This is both for simplicity and because all of the phenomena that we are interested in appear there.)

The intuition is that whenever a structure has some computational property there is a structural reason as expressed in infinitary logic at an appropriate level of complexity. While this is a useful guiding principal it is often only strictly true when we consider all copies of a structure and not just computable copies. For example, consider computable categoricity:

(1) A computable structure $\mathcal{A}$ is *computably categorical* if every computable copy $\mathcal{B}$ of $\mathcal{A}$ is isomorphic to $\mathcal{A}$ via an isomorphism that is computable.

(2) A computable structure $\mathcal{A}$ is *relatively computably categorical* if every copy $\mathcal{B}$ of $\mathcal{A}$ is isomorphic to $\mathcal{A}$ via an isomorphism that is computable relative to $\mathcal{B}$.

(3) A structure $\mathcal{A}$ is *computably categorical on a cone* if there is $X$ such that for all $Y \geq_T X$, every $Y$-computable copy $\mathcal{B}$ of $\mathcal{A}$ is isomorphic to $\mathcal{A}$ via an isomorphism that is $Y$-computable.

Relative computable categoricity and computable categoricity on a cone admit characterizations in terms of syntactic definability: $\mathcal{A}$ is computably categorical on a cone if and only if the automorphism orbits of $\mathcal{A}$ are definable by existential formulas, and $\mathcal{A}$ is relatively computably categorical if and only if the automorphism orbits of $\mathcal{A}$ are definable by existential formulas and there is a c.e. set of such formulas [AKMS89, Chi90]. On the other hand there are computable structures which are computably categorical but not relatively computably categorical [Gon77], and computable categoricity does not admit even a hyperarithmetic characterization [DKL+15]. The situation is similar more generally for $\Delta^0_\alpha$-categoricity where we ask for $\Delta^0_\alpha$ isomorphisms.

However under certain effectivity conditions on the structures we are considering we can often recover some reasonable characterization. Goncharov [Gon75] showed that if $\mathcal{A}$ is a computable structure whose finitary $\forall\exists$ diagram is computable, then $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ is relatively computably categorical if and only if there is a c.e. collection of existential formulas defining the automorphism orbits of $\mathcal{A}$. For $\Delta^0_\alpha$-categoricity we have similar theorems of Ash [Ash87], where in the effectiveness conditions the key is to understand the back-and-forth relations of the structures.

**Definition 2.1.** We define the standard asymmetric back-and-forth relations $(\mathcal{A},\bar{a}) \leq_n (\mathcal{B},\bar{b})$ inductively as follows.

(1) $(\mathcal{A},\bar{a}) \leq_0 (\mathcal{B},\bar{b})$ if and only if $\bar{a}$ and $\bar{b}$ satisfy the same atomic and negated atomic formulas (using the first $|\bar{a}|$-many symbols).

(2) $(\mathcal{A},\bar{a}) \leq_n (\mathcal{B},\bar{b})$ if for every $\bar{b}' \in \mathcal{B}$ there is $\bar{a}' \in \mathcal{A}$ such that $(\mathcal{B},\bar{b}\bar{b}') \leq_{n-1} (\mathcal{A},\bar{a}\bar{a}')$.

We write $\mathcal{A} \leq_n \mathcal{B}$ if $(\mathcal{A},\varnothing) \leq_n (\mathcal{B},\varnothing)$. We also write $\mathcal{A} \equiv_n \mathcal{B}$ if $\mathcal{A} \leq_n \mathcal{B}$ and $\mathcal{B} \leq_n \mathcal{A}$.

This particular definition of the back-and-forth relations is intimately connected with the infinitary logic: $\mathcal{A} \leq_n \mathcal{B}$ if and only if every $\Pi_n$ sentence true of $\mathcal{A}$ is true of $\mathcal{B}$, and also if and only if every $\Sigma_n$ sentence true of $\mathcal{B}$ is true of $\mathcal{A}$ [Kar65].

A structure or class of structures is said by Ash and Knight [AK00] to be $n$-friendly if the $n$-back-and-forth relations are c.e. Montalbán in his recent book [Mon21] requires instead that the back-and-forth relations be computable (but does not use the term "friendly"). Taking poetic license, we will adopt the terminology of Ash and Knight but take the stronger definition from Montalbán (a decision which is of no real consequence as we will use friendliness only in a motivational role and not in our theorems or proofs).

**Definition 2.2.** Let $\mathbb{K} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots\}$ be a uniformly computable list of finitely many or countably many structures. $\mathbb{K}$ is $n$-friendly if the $n$-back-and-forth relations between tuples in structures from $\mathbb{K}$ are computable, i.e., if

$$\{(i,\bar{a},j,\bar{b},m) : m \leq n \text{ and } (\mathcal{A}_i,\bar{a}) \leq_m (\mathcal{A}_j,\bar{b})\}$$

is computable.

In Chapter 15 of [AK00], Ash and Knight show that a number of structures, such as the standard copies of the computable ordinals, are $n$-friendly for all $n$.

We will give another example which is the pair-of-structures theorem of Ash and Knight [AK90]. This can be thought of as an invariant version of the Louveau and Saint-Raymond separation theorem [LR87]. Given two structures $\mathcal{A}$ and $\mathcal{B}$ we ask how hard it is to tell $\mathcal{A}$ and $\mathcal{B}$ apart. If there is a $\Sigma_\alpha$ sentence true of $\mathcal{A}$ and $\mathcal{B}$, then this gives a $\mathbf{\Sigma}^0_\alpha$ way to tell them apart. The pair of structures theorem says that otherwise it is $\mathbf{\Pi}^0_\alpha$-hard to tell them apart. We begin by stating the relative or boldface version.

**Theorem 2.3** (Ash and Knight)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be structures such that $\mathcal{B} \leq_n \mathcal{A}$. Then for any $\mathbf{\Pi}^0_n$ set $S \subseteq 2^{\mathbb{N}}$, there is a continuous map $x \mapsto \mathcal{C}_x$ such that*

$$\mathcal{C}_x \cong \begin{cases} \mathcal{A} & x \in S \\ \mathcal{B} & x \notin S \end{cases}.$$

The continuous construction is computable relative to some parameter. To get a computable or lightface version of the theorem—which was the original version proved by Ash and Knight—we need the structures to be $n$-friendly.

**Theorem 2.4** (Ash and Knight)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be computable structures such that $\mathcal{B} \leq_n \mathcal{A}$ and $\{\mathcal{A}, \mathcal{B}\}$ is $n$-friendly. Then for any $\Pi^0_n$ set $S \subseteq 2^{\mathbb{N}}$, there is a computable map $x \mapsto \mathcal{C}_x$ such that*

$$\mathcal{C}_x \cong \begin{cases} \mathcal{A} & x \in S \\ \mathcal{B} & x \notin S \end{cases}.$$

Even if a structure or class of structures is not $n$-friendly, the back-and-forth relations

$$\{(i, \bar{a}, j, \bar{b}, m) : m \leq n \text{ and } (\mathcal{A}_i, \bar{a}) \leq_m (\mathcal{A}_j, \bar{b})\}$$

are always $\mathbf{0}^{(2n)}$-computable. This comes from simply writing out the inductive characterization of the back-and-forth relations. (Chen, Gonzalez, Harrison-Trainor [CGHT] have shown that the relations $\leq_n$ cannot be defined with fewer than $2n$ alternations of quantifiers, in the sense that $\{(\mathcal{A}, \mathcal{B}) : \mathcal{A} \leq_n \mathcal{B}\}$ is $\mathbf{\Pi}^0_{2n}$-complete.) Thus, for example, in the computable pairs-of-structures theorem, even if $\{\mathcal{A}, \mathcal{B}\}$ is not $n$-friendly then we can get a $\mathbf{0}^{(2n)}$-computable map $x \mapsto \mathcal{C}_x$. Similarly, if a computable structure is $\Delta^0_n$-categorical on a cone, then the base of the cone can be taken to be $\mathbf{0}^{(2n)}$.

Though there are many known examples of structures which are not $n$-friendly, we generally do not have examples which are maximally unfriendly. In general, our examples are structures in which the $n$-back-and-forth relations compute $\mathbf{0}^{(n+1)}$ or $\mathbf{0}^{(n+2)}$, but they do not compute the upper bound of $\mathbf{0}^{(2n)}$. Our new technique of unfriendly jump inversions is exactly suited for proving such results, and so we show:

**Theorem 2.5.** *For each $n$ there is a computable structure $\mathcal{A}$ such that the $n$-back-and-forth relation on $\mathcal{A}$ is $\Pi^0_{2n}$-complete and hence computes $\mathbf{0}^{(2n)}$.*

Moreover, for all of the various theorems in which $n$-friendliness is assumed our unfriendly jump inversions give a way to construct the best possible counterexamples in the absence of $n$-friendliness. We give several applications, some of which answer questions which have been posed in the literature, but this is by no means a complete list of the possibilities. We list these applications now, with the proofs appears in Section 6.

## 2.1 Computability of Scott sentences

Let $\mathcal{A}$ be a countable structure, and suppose that we want to characterize $\mathcal{A}$ up to isomorphism by writing down a description of $\mathcal{A}$ in some formal language. If we work in elementary first-order logic then, as a consequence of compactness, we cannot do this for most countable structures $\mathcal{A}$. However Scott [Sco65] showed that if we use the infinitary logic $\mathcal{L}_{\omega_1\omega}$ then we can characterize any countable structure up to isomorphism. For any countable structure $\mathcal{A}$ there is a sentence $\varphi$ of $\mathcal{L}_{\omega_1\omega}$ such that for all countable $\mathcal{B}$,

$$\mathcal{B} \models \varphi \iff \mathcal{A} \cong \mathcal{B}.$$

We call such a sentence a *Scott sentence* for $\mathcal{A}$. While being isomorphic to a fixed structure $\mathcal{A}$ is naively analytic, because $\mathcal{A}$ has a Scott sentence it is actually Borel (in $\mathrm{Mod}(\mathcal{L})$, the Polish space of presentations of countable $\mathcal{L}$-structures).

One way to measure the complexity of a structure is the least complexity of a Scott sentence for that structure.

**Definition 2.6** (Montalbán [Mon15], see also [Mon21])**.** The *(unparametrized) Scott rank* of $\mathcal{A}$ is the least $n$ such that $\mathcal{A}$ has a $\Pi_{n+1}$ Scott sentence.

This is a robust measure of the complexity of $\mathcal{A}$, and has several equivalent characterizations both externally in terms of other structures and internally in terms of automorphism orbits.

**Theorem 2.7** (Montalbán [Mon15])**.** *Let $\mathcal{A}$ be a countable structure, and $\alpha$ a countable ordinal. The following are equivalent:*

*(1) $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.*

*(2) The set $\{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{B} \cong \mathcal{A}\}$ of isomorphic copies of $\mathcal{A}$ is $\mathbf{\Pi}^0_{\alpha+1}$.*

*(3) Every automorphism orbit in $\mathcal{A}$ is $\Sigma_\alpha$-definable.*

*(4) $\mathcal{A}$ is (boldface) $\mathbf{\Delta}^0_\alpha$-categorical.*

Note that if $\mathcal{A}$ has a $\Pi_n$ Scott sentence, then for all countable $\mathcal{B}$,

$$\mathcal{A} \leq_n \mathcal{B} \iff \mathcal{B} \cong \mathcal{A}.$$

Writing out the definition of the left-hand-side, we get a $\Pi_{2n}$ definition. Thus we get the well-known fact that if a computable structure has a $\Pi_n$ Scott sentence, then it has a computable $\Pi_{2n}$ Scott sentence.

Alvir, Knight, and McCoy [AKM20] showed that there is a computable structure with a $\Pi_2$ Scott sentence but no computable $\Pi_2$ Scott sentence, and this can be adjusted to show that for any $n$ there is a computable structure with a $\Pi_n$ Scott sentence but no computable Scott sentence. Alvir, Csima, and Harrison-Trainor [ACHT] showed that in the case $n = 2$, the above bound is optimal.

**Theorem 2.8** (Alvir, Csima, Harrison-Trainor [ACHT])**.** *There is a computable structure with a $\Pi_2$ Scott sentence but no computable $\Sigma_4$ Scott sentence.*

They were only able to show that for each $n \geq 2$ there is a computable structure with a $\Pi_n$ Scott sentence but no computable $\Sigma_{n+2}$ Scott sentence. Using our unfriendly jump inversions, we resolve this question.

**Theorem 2.9.** *For each $n$ there is a computable structure with a $\Pi_n$ Scott sentence but no computable $\Sigma_{2n}$ Scott sentence.*

Indeed our new techniques of unfriendly jump inversions allow us to obtain the general result almost directly from the case $n = 2$ (at least, if we prove a slightly more general statement in the case $n = 2$).

Along with being an application of the method of unfriendly jump inversions, these results give some motivation to the method (and indeed the thought process below is how the author developed the method). Consider a computable structure $\mathcal{A}$ with a $\Pi_2$ Scott sentence but no computable $\Sigma_4$ Scott sentence. It has a computable $\Pi_4$ Scott sentence and (as we prove in Theorem 3.5) a $\mathbf{0}''$-computable $\Pi_2$ Scott sentence. We can consider the complexity of certain index sets of structures isomorphic to $\mathcal{A}$. Let $(\mathcal{B}_i)_{i\in\mathbb{N}}$ be a computable list of the (possibly partial) computable $\mathcal{L}$-structures. Consider the index set

$$\Delta^0_1\text{-Copies}(\mathcal{A}) = \{i \in \mathbb{N} : \mathcal{B}_i \cong \mathcal{A}\}$$

of computable structures isomorphic to $\mathcal{A}$. Because $\mathcal{A}$ has a computable $\Pi_4$ Scott sentence, this set is $\Pi^0_4$. On the other hand, because $\mathcal{A}$ has no computable $\Sigma_4$ Scott sentence, one might imagine that this set is not $\Sigma^0_4$. Though this does not follow directly, by a slight modification of the same argument can build $\mathcal{A}$ so that this index set is $\Pi^0_4$-complete. Thus: deciding whether a computable structure is isomorphic to $\mathcal{A}$ is a $\Pi^0_4$-complete problem.

On the other hand, consider a list $(\mathcal{C}_i^{\mathbf{0}''})_{i\in\mathbb{N}}$ of the (possibly partial) $\mathbf{0}''$-computable structures, and the index set

$$\Delta^0_3\text{-Copies}(\mathcal{A}) = \{i \in \mathbb{N} : \mathcal{C}_i^{\mathbf{0}''} \cong \mathcal{A}\}$$

of $\mathbf{0}''$-computable copies of $\mathcal{A}$. Because $\mathcal{A}$ has a $\mathbf{0}''$-computable $\Pi_2$ Scott sentence, this is also a $\Pi^0_4$ set! That is, the difficulty of deciding whether a computable structure is isomorphic to $\mathcal{A}$ is the same as the difficult of deciding whether a $\mathbf{0}''$-computable structure is isomorphic to $\mathcal{A}$. This is highly unusual. Usually, if the set of computable structures isomorphic to $\mathcal{A}$ is $\Pi^0_4$ then we would expect that the set of $0''$-computable structures isomorphic to $\mathcal{A}$ would be $\Pi^0_4$ relative to $0''$, hence $\Pi^0_6$. This sort of phenomenon underlies the unfriendly jump inversions.

## 2.2 Definability of back-and-forth types

Given $\bar{a} \in \mathcal{A}$, the appropriate notion of types in countable model theory is often the $\Pi_n$-type of $\bar{a}$, which is the collection of $\Pi_n$ formulas true of $\bar{a}$. We have that $\bar{b} \in \mathcal{B}$ satisfies the $\Pi_n$-type of $\bar{a}$ if and only if $(\mathcal{A}, \bar{a}) \leq_n (\mathcal{B}, \bar{b})$.

Within the countable structure $\mathcal{A}$ the $\Pi_n$-type of $\bar{a}$ is definable by a single $\Pi_n$ formula. This is essentially because there are only countably many other tuples to consider; see, e.g., Lemma II.62 of [Mon].

On the other hand, there is no reason for a $\Pi_n$-type to be definable by a $\Pi_n$ formula in general among all tuples in all structures. In [CGHT], Chen, Gonzalez, and Harrison-Trainor showed that a $\Pi_n$-type is definable by a $\Pi_{n+2}$ formula, and that this is best possible. The simplest case (which is equivalent to the general case by naming constants) is to consider 0-tuples, i.e., just structures. For each fixed structure $\mathcal{A}$ the set $\{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{A} \leq_n \mathcal{B}\}$ is (boldface) $\mathbf{\Pi}^0_{n+2}$, and there is a choice of $\mathcal{A}$ for which it is $\mathbf{\Pi}^0_{n+2}$-complete.

If $\mathcal{A}$ is computable, what the lightface complexity of this set is? Equivalently, what is the least complexity of a computable sentence $\varphi$ such that $\mathcal{A} \leq_n \mathcal{B}$ if and only if $\mathcal{B} \models \varphi$? The upper bound is $\Pi_{2n}$ as one can write down the definition of the back-and-forth relations replacing each quantifier over $\mathcal{A}$ with a countably infinite conjunction or disjunction over the elements of $\mathcal{A}$. In this paper, we show that this is best possible.

**Theorem 2.10.** *There is a computable structure $\mathcal{A}$ such that the set*

$$\{\mathcal{B} \in Mod(\mathcal{L}) : \mathcal{A} \leq_n \mathcal{B}\}$$

*is not (lightface) $\Sigma^0_{2n}$.*

## 2.3 Separating structures

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are computable structures and that $\mathcal{A} \nleq_n \mathcal{B}$. This means that there is a witnessing $\Pi_n$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ but $\mathcal{B} \not\models \varphi$. What kind of non-computable separation can we find? Ash and Knight show that if $\mathcal{A}$ and $\mathcal{B}$ are $n$-friendly and have computable existential diagrams, then there is a computable $\Pi_n$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ but $\mathcal{B} \not\models \varphi$. In general, the best we can do is a computable $\Pi_{2n-1}$ sentence $\varphi$ (obtained by choosing $\bar{b} \in \mathcal{B}$ such that there is no $\bar{a} \in \mathcal{A}$ with $(\mathcal{A}, \bar{a}) \geq_{n-1} (\mathcal{B}, \bar{b})$, and then by writing out the definition $\geq_{n-1}$) such that $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \leq_n \mathcal{B}$; then $\mathcal{A} \models \varphi$ but $\mathcal{B} \not\models \varphi$. We show that this is the best we can do:

**Theorem 2.11.** *Fix $n \geq 2$. There are computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B} \nleq_n \mathcal{A}$ but for any computable $\Sigma_{2n-1}$ sentence $\varphi$, if $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.*

# 3 Computing optimal Scott sentences and separating sentences

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are computable structures, and $\mathcal{A} \nleq_n \mathcal{B}$ which means that there is a $\Pi_n$ sentence true of $\mathcal{A}$ but not of $\mathcal{B}$. Ash and Knight [AK00] prove the theorem below which, as a corollary, gives conditions—that $\{\mathcal{A}, \mathcal{B}\}$ be $n$-friendly with computable existential diagrams—under which we can compute such a sentence. The theorem is set in a more general context. A tuple $\bar{a} \in \mathcal{A}$, the $\Pi_n$-type of $\bar{a}$ is known to be definable within $\mathcal{A}$ by a $\Pi_n$ sentence, and moreover, this remains true as long as we consider a only countable set of structures. (As shown in [CGHT], $\Pi_n$-types are not definable by $\Pi_n$ formulas in general.) In the theorem, we get conditions under which the $\Pi_n$-types becomes definable by computable $\Pi_n$ formulas.

**Theorem 3.1** (Theorem 15.2 of [AK00])**.** *Let $(\mathcal{A}_i)_{i \in \mathbb{N}}$ be a computable list of $n$-friendly structures with uniformly computable existential diagrams. Then for any $m$ such that $1 \leq m \leq n$, and any tuple $\bar{a} \in \mathcal{A}_i$, we can find a computable $\Pi_m$ formula $\varphi^m_{i,\bar{a}}$ such that*

$$(\mathcal{A}_i, \bar{a}) \leq_m (\mathcal{A}_j, \bar{b}) \iff \mathcal{A}_j \models \varphi^m_{i,\bar{a}}(\bar{b}).$$

Suppose that we do not know that the pair of structures is $n$-friendly or that they have computable existential diagrams. How hard is it to compute such formulas? Our main goal is the following theorem, along the way to which we will prove an analogue of Theorem 3.1.

**Theorem 3.2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be computable structures, and suppose that $\mathcal{A} \nleq_n \mathcal{B}$. Then there is a $\mathbf{0}^{(n-1)}$-computable $\Pi_n$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$.*

We will also obtain in Theorem 3.5 that if a computable structure has a $\Pi_n$ Scott sentence then it has a $\mathbf{0}^{(n)}$-computable $\Pi_n$ Scott sentence.

In the proofs we will make multiple uses of the following fact which can be found as Proposition 7.12 of the book by Ash and Knight [AK00].

**Proposition 3.3.** *Suppose that a formula $\varphi(\bar{x})$ is a disjunction of a $\Sigma^0_n$ set of (indices for) computable $\Sigma_n$ formulas. Then $\varphi(\bar{x})$ is equivalent to a computable $\Sigma_n$ formula, and we can find this computable $\Sigma_n$ formula uniformly.*

Essentially the idea is that the complexity of which formulas are in the disjunction can be absorbed into the complexity of the formulas themselves. This proposition relativizes, and we will use it in relativized form.

We now prove our theorem about defining $\Pi_n$ types, the analogue of Theorem 3.1.

**Theorem 3.4.** *Let $(\mathcal{A}_i)_{i\in\omega}$ be a computable list of computable structures. For each $n$ and $\bar{a} \in \mathcal{A}_i$ there is a $\mathbf{0}^{(n)}$-computable $\Pi_n$ sentence $\varphi^n_{i,\bar{a}}$ such that for all $\bar{b} \in \mathcal{A}_j$*

$$(\mathcal{A}_i, \bar{a}) \leq_n (\mathcal{A}_j, \bar{b}) \iff \mathcal{A}_j \models \varphi^n_{i,\bar{a}}(\bar{b}).$$

*Moreover, this is uniform in $i$, $\bar{a}$, and $n$.*

*Proof.* We argue inductive, noting that for each $n$ the statement of the theorem also implies that the set of pairs $\{(i,\bar{a},j,\bar{b}) : (\mathcal{A}_i,\bar{a}) \leq_n (\mathcal{A}_j,\bar{b})\}$ is $\mathbf{0}^{(2n)}$-computable.

For $n = 1$, we have $(\mathcal{A}_i,\bar{a}) \leq_1 (\mathcal{A}_j,\bar{b})$ if and only if every $\forall$-formula true of $\bar{a}$ is true of $\bar{b}$. For each $(\mathcal{A}_i,\bar{a})$, using $\mathbf{0}'$ we can list out the (finitary) $\forall$-formulas true of $\bar{a}$, and take the conjunction of these. This gives a $\Pi_1$ formula $\varphi^1_{i,\bar{a}}$ such that

$$(\mathcal{A}_i, \bar{a}) \leq_1 (\mathcal{A}_j, \bar{b}) \iff \mathcal{A}_j \models \varphi^1_{i,\bar{a}}(\bar{b}).$$

Now we will verify the result for $n+1$ assuming we know it for $n$. For each $i$ and $\bar{c} \in \mathcal{A}_i$ let $\varphi^n_{i,\bar{c}}(\bar{x})$ be a $\mathbf{0}^{(n)}$-computable $\Pi_n$ formula such that

$$(\mathcal{A}_i, \bar{c}) \leq_n (\mathcal{A}_j, \bar{b}) \iff \mathcal{A}_j \models \varphi^n_{i,\bar{c}}(\bar{b}).$$

Then for each $i$ and each tuple $\bar{a} \in \mathcal{A}_i$ consider the $\Pi_{n+1}$ formula

$$\varphi^{n+1}_{i,\bar{a}}(\bar{x}) = \bigwedge_{j} \bigwedge_{\bar{b},\bar{b}' \in \mathcal{A}_j} \forall \bar{y} \bigvee_{\bar{a}' \in \mathcal{A}_i} \begin{cases} \neg\varphi^n_{j,\bar{b}\bar{b}'}(\bar{x},\bar{y}) & (\mathcal{A}_j,\bar{b}\bar{b}') \nleq_n (\mathcal{A}_i,\bar{a}\bar{a}') \\ \top & (\mathcal{A}_j,\bar{b}\bar{b}') \leq_n (\mathcal{A}_i,\bar{a}\bar{a}') \end{cases}.$$

By the inductive hypothesis, for fixed $\bar{b}, \bar{b}' \in \mathcal{A}_j$ and $\bar{a} \in \mathcal{A}_i$, deciding for some $\bar{a}' \in \mathcal{A}_i$ whether $(\mathcal{A}_j, \bar{b}\bar{b}') \leq_n (\mathcal{A}_i, \bar{a}\bar{a}')$ is $\mathbf{0}^{(2n)}$-computable. We use the proposition from [AK00] highlighted above as Proposition 3.3 relativized to $\mathbf{0}^{(n+1)}$. The subformula

$$\bigvee_{\bar{a}' \in \mathcal{A}_i} \begin{cases} \neg \varphi^n_{j,\bar{b}\bar{b}'}(\bar{x}, \bar{y}) & (\mathcal{A}_j, \bar{b}\bar{b}') \nleq_n (\mathcal{A}_i, \bar{a}\bar{a}') \\ \top & (\mathcal{A}_j, \bar{b}\bar{b}') \leq_n (\mathcal{A}_i, \bar{a}\bar{a}') \end{cases}$$

is a disjunction over a $\mathbf{0}^{(2n)}$-computable (and hence $\Sigma_{2n+1}$ or equivalently $\Sigma_n$ relative to $\mathbf{0}^{(n+1)}$) set of indices for $\mathbf{0}^{(n)}$-computable $\Sigma_n$ formulas, and is thus equivalent to a $\mathbf{0}^{(n+1)}$-computable $\Sigma_n$ formula. Thus $\varphi^{n+1}_{i,\bar{a}}(\bar{x})$ is equivalent to a $\mathbf{0}^{(n+1)}$-computable $\Pi_{n+1}$ formula. We must show that $\varphi^{n+1}_{i,\bar{a}}(\bar{x})$ is such that

$$(\mathcal{A}_i, \bar{a}) \leq_{n+1} (\mathcal{A}_j, \bar{b}) \iff \mathcal{A}_j \models \varphi^{n+1}_{i,\bar{a}}(\bar{b}).$$

- First, we note that $\varphi^{n+1}_{i,\bar{a}}$ is true of $\bar{a}$, as for any $j$ and $\bar{b}, \bar{b}' \in \mathcal{A}_j$ and $\bar{a}' \in \mathcal{A}_i$ (the value of $\bar{y}$), we have that the disjunct corresponding to $\bar{a}'$ is true: if $(\mathcal{A}_j, \bar{b}\bar{b}') \leq_n (\mathcal{A}_i, \bar{a}\bar{a}')$, then the disjunct is $\top$, and if $(\mathcal{A}_j, \bar{b}\bar{b}') \nleq_n (\mathcal{A}_i, \bar{a}\bar{a}')$ then we have $\neg\varphi^n_{j,\bar{b}\bar{b}'}(\bar{a}, \bar{a}')$. Since $\varphi^{n+1}_{i,\bar{a}}$ is $\Pi_{n+1}$ and true of $\bar{a}$, it is true of any $\bar{b} \in \mathcal{A}_j$ with $(\mathcal{A}_i, \bar{a}) \leq_{n+1} (\mathcal{A}_j, \bar{b})$.
- Suppose that $\varphi^{n+1}_{i,\bar{a}}$ is true of $\bar{b}$. Given $\bar{b}' \in \mathcal{A}_j$, taking the conjunct corresponding to $j$ and $\bar{b}, \bar{b}' \in \mathcal{A}_j$ and taking $\bar{y} = \bar{b}'$, we have that

$$\bigvee_{\bar{a}'} \begin{cases} \neg \varphi^n_{j,\bar{b}\bar{b}'}(\bar{b}, \bar{b}') & (\mathcal{A}_j, \bar{b}\bar{b}') \nleq_n (\mathcal{A}_i, \bar{a}\bar{a}') \\ \top & (\mathcal{A}_j, \bar{b}\bar{b}') \leq_n (\mathcal{A}_i, \bar{a}\bar{a}') \end{cases}$$

  Since $(\mathcal{A}_j, \bar{b}\bar{b}') \leq_n (\mathcal{A}_j, \bar{b}\bar{b}')$ we have $\varphi^n_{j,\bar{b}\bar{b}'}(\bar{b}, \bar{b}')$ and so it must be that there is some disjunct that is $\top$, i.e., some $\bar{a}' \in \mathcal{A}_i$ with $(\mathcal{A}_j, \bar{b}\bar{b}') \leq_n (\mathcal{A}_i, \bar{a}\bar{a}')$. Thus $(\mathcal{A}_i, \bar{a}) \leq_{n+1} (\mathcal{A}_j, \bar{b})$.

This completes the inductive argument. □

We are now ready to prove our intended application.

*Proof of Theorem 3.2.* Suppose that $\mathcal{A}$ and $\mathcal{B}$ are computable structures and that $\mathcal{A} \nleq_n \mathcal{B}$. Then there is $\bar{b} \in \mathcal{B}$ such that for all $\bar{a} \in \mathcal{A}$ we have $(\mathcal{B}, \bar{b}) \nleq_{n-1} (\mathcal{A}, \bar{a})$. By the previous theorem (using just $\mathcal{A}$ and $\mathcal{B}$ as the sequence of structures), there is a $\mathbf{0}^{(n-1)}$-computable $\Pi_{n-1}$ formula $\psi(\bar{x})$ such that

$$(\mathcal{B}, \bar{b}) \leq_{n-1} (\mathcal{A}, \bar{a}) \iff \mathcal{A} \models \psi(\bar{a}).$$

Let $\hat{\varphi} = \exists \bar{x}\ \psi(\bar{x})$. This is a $\mathbf{0}^{(n-1)}$-computable $\Sigma_n$ sentence such that $\mathcal{B} \models \varphi$ and $\mathcal{A} \not\models \varphi$. The negation of $\varphi$ is a $\mathbf{0}^{(n-1)}$-computable $\Pi_n$ sentence true in $\mathcal{A}$ but not $\mathcal{B}$, proving the theorem. □

Since we have proved the lemma, we will also prove the following fact which is of independent interest (and which was also the author's initial motivation for the analysis above).

**Theorem 3.5.** *If a computable structure $\mathcal{A}$ has a $\Pi_n$ Scott sentence then it has a $\mathbf{0}^{(n)}$-computable $\Pi_n$ Scott sentence.*

*Proof.* It will be more convenient to prove the theorem for $n+1$ rather than $n$. Suppose that $\mathcal{A}$ has a $\Pi_{n+1}$ Scott sentence; we must show that it has a $\mathbf{0}^{(n+1)}$-computable Scott sentence. Since $\mathcal{A}$ has a $\Pi_{n+1}$ Scott sentence we know that for each tuple $\bar{a}$ there is a $\Sigma_n$ formula $\theta_{\bar{a}}(\bar{x})$ such that

$$\bar{a} \cong \bar{b} \iff \bar{a} \leq_n \bar{b} \iff \mathcal{A} \models \theta_{\bar{a}}(\bar{b}).$$

We may assume that $\theta_{\bar{a}}(\bar{x})$ is of the form

$$\theta_{\bar{a}}(\bar{x}) = \exists \bar{y}\ \varphi^{n-1}_{\bar{a}\bar{a}'}(\bar{x}, \bar{y})$$

for some $\bar{a}'$, where $\varphi^{n-1}_{\bar{a},\bar{a}'}$ is the $\mathbf{0}^{(n-1)}$-computable $\Pi_{n-1}$ formula obtained from Theorem 3.4 with $\bar{a}, \bar{a}' \leq_{n-1} \bar{b}\bar{b}'$ if and only if $\mathcal{A} \models \varphi^{n-1}_{\bar{a},\bar{a}'}(\bar{b}, \bar{b}')$. The reason for this is as follows. If $\theta_{\bar{a}}(\bar{x})$ was of the form $\bigvee\!\!\!\bigvee_k\ \exists \bar{y}_k\ \psi_k(\bar{x}, \bar{y}_k)$ where $\psi_k$ is $\Pi_{n-1}$ then take any $k$ and $\bar{a}'$ such that $\mathcal{A} \models \psi_k(\bar{a}, \bar{a}')$. Since $\psi_k(\bar{x}, \bar{y})$ is $\Pi_{n-1}$ and holds of $\bar{a}, \bar{a}'$, $\varphi^{n-1}_{\bar{a},\bar{a}'}(\bar{x}, \bar{y})$ implies $\psi_k(\bar{x}, \bar{y})$ within $\mathcal{A}$. Thus $\exists \bar{y}\ \varphi^{n-1}_{\bar{a}\bar{a}'}(\bar{x}, \bar{y})$ implies $\theta_{\bar{a}}(\bar{x})$ within $\mathcal{A}$. On the other hand, if $\mathcal{A} \models \theta_{\bar{a}}(\bar{b})$ then there is an automorphism taking $\bar{a}$ to $\bar{b}$, and so $\mathcal{A} \models \exists \bar{y}\ \varphi^{n-1}_{\bar{a},\bar{a}'}(\bar{b}, \bar{y})$. Thus $\exists \bar{y}\ \varphi^{n-1}_{\bar{a},\bar{a}'}(\bar{b}, \bar{y})$ and $\theta_{\bar{a}}(\bar{x})$ are equivalent within $\mathcal{A}$.

This formula $\theta_{\bar{a}}(\bar{x})$ is a $\mathbf{0}^{(n-1)}$-computable $\Sigma_n$ formula as $\varphi^{n-1}_{\bar{a}\bar{a}'}(\bar{x}, \bar{y})$ is a $\mathbf{0}^{(n-1)}$-computable $\Pi_{n-1}$ formula. Given $\bar{a}$, we may find the appropriate $\bar{a}'$ and thus the formula $\theta_{\bar{a}}(\bar{x})$ using $\mathbf{0}^{(2n)}$ by searching for a $\bar{a}'$ such that for all $\bar{b}\bar{b}'$, if $\mathcal{A} \models \varphi^{n-1}_{\bar{a}\bar{a}'}(\bar{b}\bar{b}')$ then $\bar{a} \leq_n \bar{b}$. For each $\bar{a}'$, this can be done using $\mathbf{0}^{(2n)}$ since we can decide whether $\mathcal{A} \models \varphi^{n-1}_{\bar{a}\bar{a}'}(\bar{b}\bar{b}')$ using $\mathbf{0}^{(2n-2)}$ and deciding whether $\bar{a} \leq_n \bar{b}$ is $\mathbf{0}^{(2n)}$-computable.

Now we can build the canonical Scott sentence for $\mathcal{A}$ as follows:

$$\chi = \bigwedge\!\!\!\bigwedge_{\bar{a} \in \mathcal{A}} \forall \bar{x} \left[ \left( \theta_{\bar{a}}(\bar{x}) \longrightarrow D(\bar{x}) = D(\bar{a}) \right) \wedge \left( \bigwedge\!\!\!\bigwedge_{a' \in \mathcal{A}} \exists y\ \theta_{\bar{a}a'}(\bar{x}, y) \right) \wedge \left( \forall y \bigvee\!\!\!\bigvee_{a' \in \mathcal{A}} \theta_{\bar{a}a'}(\bar{x}, y) \right) \right].$$

We write $D(\bar{x})$ for the atomic diagram of $\bar{x}$, which (if the language is infinite) we assume to consist of only the first $|\bar{x}|$-many symbols in the language. This is a $\mathbf{0}^{(2n)}$-computable $\Pi_{n+1}$ formula. We now use the proposition of Ash and Knight [AK00] highlighted above as Proposition 3.3 to improve this. For fixed $\bar{a}$, the disjunction

$$\bigvee\!\!\!\bigvee_{a' \in \mathcal{A}} \theta_{\bar{a}a'}(\bar{x}, y)$$

is a disjunction of a $\mathbf{0}^{(2n)}$-computable, and hence $\Sigma_n$ relative to $\mathbf{0}^{(n+1)}$, set of (indices for) $\mathbf{0}^{(n-1)}$-computable $\Sigma_n$ formulas. Thus the disjunction is equivalent to a $\mathbf{0}^{(n+1)}$-computable $\Sigma_n$ formula. Also for fixed $\bar{a}$, the conjunction

$$\bigwedge\!\!\!\bigwedge_{a' \in \mathcal{A}} \exists y\ \theta_{\bar{a}a'}(\bar{x}, y)$$

is a conjunction of a $\mathbf{0}^{(2n)}$-computable, and hence $\Pi_{n+1}$ relative to $\mathbf{0}^{(n)}$, set of (indices for) $\mathbf{0}^{(n-1)}$-computable $\Sigma_n$ (hence $\mathbf{0}^{(n-1)}$-computable $\Pi_{n+1}$) formulas. Thus the conjunction is equivalence to a $\mathbf{0}^{(n)}$-computable $\Pi_{n+1}$ formula. It is $\mathbf{0}^{(2n)}$-computable to find these formulas given $\bar{a}$.

Finally, $\chi$ is a conjunction of a $\mathbf{0}^{(2n)}$-computable, hence $\Pi_{n+1}$ relative to $\mathbf{0}^{(n+1)}$, set of (indices for) $\mathbf{0}^{(n+1)}$-computable $\Pi_{n+1}$ formulas. Thus $\chi$ is equivalent to a $\mathbf{0}^{(n+1)}$-computable $\Pi_{n+1}$ formula. ☐

# 4 Unfriendly jump inversions

We begin by describing the related methods of jump inversion and Marker extensions as they are generally used. We follow, for example, Chapter X.3 of [Mon21]. Given $n$, let $\mathcal{A}$ and $\mathcal{B}$ be computable structures such that $\mathcal{A} \equiv_n \mathcal{B}$ but $\mathcal{A} \nleq_{n+1} \mathcal{B}$ and $\mathcal{B} \nleq_{n+1} \mathcal{A}$. Suppose further that $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_{n+2}$ Scott sentences and that $\mathcal{A}$ and $\mathcal{B}$ are $(n+1)$-friendly. By Theorem 3.1 (which is Theorem 15.2 of [AK00]) there are computable $\Pi_{n+1}$ sentences satisfied by $\mathcal{A}$ but not by $\mathcal{B}$ and by $\mathcal{B}$ but not by $\mathcal{A}$. $\mathcal{A}$ and $\mathcal{B}$ also have computable $\Pi_{n+2}$ Scott sentences and are $\Delta^0_n$-categorical. Montalbán asks in addition that $\mathcal{A}$ and $\mathcal{B}$ be rigid.

If we have a structure $\mathcal{M}$ which we know is isomorphic to either $\mathcal{A}$ or $\mathcal{B}$, we can figure out which in a $\Delta^0_{n+1}$ way relative to $\mathcal{M}$ by asking whether $\mathcal{M}$ satisfies the computable $\Pi_{n+1}$ sentence true of $\mathcal{A}$ but not $\mathcal{B}$, or the computable $\Pi_{n+1}$ sentence true of $\mathcal{B}$ but not $\mathcal{A}$. On the other hand, distinguishing between $\mathcal{A}$ and $\mathcal{B}$ is $\Delta^0_{n+1}$-hard: By the Ash and Knight pairs-of-structures theorem, in the form of Theorem 18.7 of [AK00], for any $\Delta^0_{n+1}$ set $S$ there is a uniformly computable sequence of structures $\mathcal{C}_i$ such that

$$i \in S \Longleftrightarrow \mathcal{C}_i \cong \mathcal{A}$$

and

$$i \notin S \Longleftrightarrow \mathcal{C}_i \cong \mathcal{B}.$$

This also relativizes.

Given a structure, e.g., for the sake of simplicity a graph $\mathcal{G}$, we can define the $n$th jump inversion $\mathcal{G}^{(-n)}$ of $\mathcal{G}$ by replacing each edge in $\mathcal{G}$ by a copy of $\mathcal{A}$, and each non-edge in $\mathcal{G}$ by a copy of $\mathcal{B}$. For a more general $k$-ary relation, for each $k$-tuple we attach a copy of either $\mathcal{A}$ or $\mathcal{B}$ depending on whether the relation holds of the $k$-tuple or not. The idea is that the edge relation of $\mathcal{G}$ becomes a definable relation in $\mathcal{G}^{(-n)}$, but the complexity has increased to being $\Delta^0_{n+1}$. There are various different forms of jump inversions, with different properties, but typically one proves something like the following facts.

**Theorem 4.1.** *If $\mathcal{A}$ and $\mathcal{B}$ are rigid, then the $n$th jump of $\mathcal{G}^{(-n)}$ is effectively bi-interpretable with $\mathcal{G}$.*[2] *Even if $\mathcal{A}$ and $\mathcal{B}$ are not rigid, we have the following:*

*(1) For every $\mathbf{d}$-computable $\Pi_{n+k}$ formula $\varphi$ there is a $\mathbf{d}$-computable $\Pi_k$ formula $\varphi_*$ such that $\mathcal{G} \models \varphi_*$ if and only if $\mathcal{G}^{(-n)} \models \varphi$.*

*(2) For every $\mathbf{d}$-computable $\Pi_k$ formula $\varphi$ there is a $\mathbf{d}$-computable $\Pi_{n+k}$ formula $\varphi^*$ such that $\mathcal{G} \models \varphi$ if and only if $\mathcal{G}^{(-n)} \models \varphi^*$.*

*(3) If $\mathcal{G}$ has a $\Pi_k$ Scott sentence then $\mathcal{G}^{(-n)}$ has a $\Pi_{n+k}$ Scott sentence.*

*(4) Given a $\mathbf{d}$-computable copy of $\mathcal{G}^{(-n)}$ there is a $\mathbf{d}^{(n)}$-computable copy of $\mathcal{G}$.*

*(5) Given a $\mathbf{d}^{(n)}$-computable copy of $\mathcal{G}$ there is a $\mathbf{d}$-computable copy of $\mathcal{G}^{(-n)}$.*

[2] We will not define these terms, but the reader can find them in [Mon21] and [Mon]. In any case, properties (1)-(5) give a sufficient idea of what is going on.

For a Marker extension we have a similar process, except that for different relations in the language we might use different structures $\mathcal{A}$ and $\mathcal{B}$, or we might leave certain relations alone. So, for example, in a language including relations $R_1, R_2, \ldots$ by taking a Marker extension we might make the relation $R_n$ into a $\Delta_{n+1}$-definable relation.

Instead of using friendly structures, for our unfriendly jump inversions we will use structures $\mathcal{A}$ and $\mathcal{B}$ that are maximally unfriendly. These structures are given by the following theorem which we will prove later.

**Theorem 4.2.** *Fix $n$. Let $S \subseteq \mathbb{N}$ be a complete $\Sigma^0_{2n+1}$ set. There are computable structures $\mathcal{A} \ncong \mathcal{B}$ and a sequence of computable structures $\mathcal{C}_i$ such that*

$$i \in S \Longrightarrow \mathcal{C}_i \cong \mathcal{A}$$

*and*

$$i \notin S \Longrightarrow \mathcal{C}_i \cong \mathcal{B}.$$

*Moreover:*

*(1) $\mathcal{B} \nleq_{n+1} \mathcal{A}$, which by Theorem 3.2 implies that there is a $\mathbf{0}^{(n)}$-computable $\Sigma_{n+1}$ sentence $\varphi$ such that $\mathcal{A} \vDash \varphi$ and $\mathcal{B} \nvDash \varphi$;*

*(2) $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_{n+2}$ Scott sentences.*

*This is uniform in $n$.*

Given the theorem for some particular value of $n$, we can use these structures $\mathcal{A}$ and $\mathcal{B}$ to define a particular jump inversion operation which we will use for the construction of a structure with non-arithmetic degree spectrum. This jump inversion operation will be described in full detail in Section 4.1, where we also prove the desired properties (as the analogue of Theorem 4.1). Montalbán notes that even if two computable structures are not $n$-friendly, they become $n$-friendly relative to $\mathbf{0}^{(2n)}$. Thus we get different properties depending on what degree we are working relative to. The key difference from the standard friendly jump inversions is:

(1) Given a $\mathbf{0}^{(2n)}$-computable copy of $\mathcal{G}$, there is a computable copy of $\mathcal{G}^{(-n)}$.

(2) Given a $\mathbf{d}$-computable copy of $\mathcal{G}^{(-n)}$ for $\mathbf{d} \geq \mathbf{0}^{(n)}$, there is a $\mathbf{d}^{(n)}$-computable copy of $\mathcal{G}$.

These might seem contradictory, but (1) does not relativize and (2) is only true for $\mathbf{d} \geq \mathbf{0}^{(n)}$. These imply, for example, that if $\mathcal{G}^{(-n)}$ has a $\mathbf{0}^{(n)}$-computable copy then it has a computable copy. This is usually not true for jump inversions.

The proof of Theorem 4.2 will be an inductive argument which we give in Section 4.2. We begin by proving the case $n = 1$ as Lemma 4.4. To prove the case $n+1$, we first use Lemma 4.4 relativized to $\mathbf{0}^{(2n)}$, and then us the jump inversion operation obtained from the inductive hypothesis, which is the case $n$ of Theorem 4.2, to obtain Theorem 4.2 for $n+1$.

## 4.1 Construction and properties of unfriendly jump inversions

Suppose that we are given Theorem 4.2 for some value of $n$. Let $S$ be a complete $\Sigma^0_{2n+1}$ set, and fix computable structures $\mathcal{A} \ncong \mathcal{B}$ with $\Pi_{n+2}$ Scott sentences, a computable sequence of structures $\mathcal{C}_i$, and a $\mathbf{0}^{(n)}$-computable $\Sigma_{n+1}$ sentence $\varphi$ as in Theorem 4.2.

Consider a graph $\mathcal{G}$ (or, by simple modifications, a structure in any relational language). Define $\mathcal{G}^{(-n)}$ to be the structure consisting of the vertices of $\mathcal{G}$, and associated to each pair $(u,v)$ of vertices we have an ordered pair of structures isomorphic to $(\mathcal{A},\mathcal{B})$ if there is an edge $u - v$, and isomorphic to $(\mathcal{B},\mathcal{A})$ if there is no edge $u - v$. More formally, let $\mathcal{L}$ be the language of $\mathcal{A}$ and $\mathcal{B}$. We work in the language $\mathcal{L}^* = \mathcal{L} \cup \{U, E, F\}$ where $U$ is a unary relation and $E$ and $F$ are ternary relations. $\mathcal{G}^{(-n)}$ will be the structure defined as follows. The domain of $\mathcal{G}^{(-n)}$ will consist of the elements of $\mathcal{G}$ together with, for each $u, v \in \mathcal{G}$, disjoint infinite sets $C_{u,v}$ and $D_{u,v}$. $U$ will hold only of the elements of $\mathcal{G}$. We put $E(u,v,x)$ if and only if $x \in C_{u,v}$ and $F(u,v,x)$ if and only if $x \in D_{u,v}$. The relations of $\mathcal{L}$ will hold only of tuples of elements all in the same set $C_{u,v}$ or $D_{u,v}$, and we will have $C_{u,v} \cong \mathcal{A}$ and $D_{u,v} \cong \mathcal{B}$ if there is an edge $u - v$, and $C_{u,v} \cong \mathcal{B}$ and $D_{u,v} \cong \mathcal{A}$ if there is no edge $u - v$.

Note that $\mathcal{G}$ is interpretable in $\mathcal{G}^{(-n)}$ using computable $\Delta_{2n+1}$ formulas (in the sense of $\mathcal{L}_{\omega_1\omega}$ interpretations as introduced in [HTMM18a] and [HTMM18b]; see Section VII.8 of [Mon]). Namely, the domain of $\mathcal{G}$ in $\mathcal{G}^{(-n)}$ is defined by $U$, and we define the edge relation on pairs $u, v \in U$ using computable $\Delta_{2n+1}$ formulas, as

$$uEv \iff E(u,v,\cdot) \models \varphi \iff F(u,v,\cdot) \models \neg\varphi.$$

Here, by $E(u,v,\cdot) \models \varphi$ we mean that the $\mathcal{L}$-structure on the set $\{x : E(u,v,x)\}$ is a model of $\varphi$. Relative to $\mathbf{0}^{(n)}$, these definitions become $\Delta_{n+1}$. Note that this interpretation of $\mathcal{G}$ in $\mathcal{G}^{(-n)}$ is uniform in $\mathcal{G}$. Hence $\mathcal{G} \cong \mathcal{H}$ if and only if $\mathcal{G}^{(-n)} \cong \mathcal{H}^{(-n)}$.

We prove several properties of this jump inversion. Some of them, like (3), (4), and (5), are properties that any form of jump inversion should satisfy. (1) and (2) are special properties of unfriendly jump inversions.

**Theorem 4.3.**

*(1) Given a $\mathbf{0}^{(2n)}$-computable copy of $\mathcal{G}$, there is a computable copy of $\mathcal{G}^{(-n)}$.*

*(2) Given a $\mathbf{0}^{(n)}$-computable copy of $\mathcal{G}^{(-n)}$, there is a $\mathbf{0}^{(2n)}$-computable copy of $\mathcal{G}$.*

*(3) Given a $\Sigma_\ell$ sentence $\varphi$, there is a $\Sigma_{n+\ell}$ sentence $\varphi^*$ such that for all $\mathcal{G}$, $\mathcal{G} \models \varphi$ if and only if $\mathcal{G}^{(-n)} \models \varphi^*$.*

*(4) If $\mathcal{G}$ has a $\Pi_\ell$ Scott sentence for $\ell \geq 2$, then $\mathcal{G}^{(-n)}$ has a $\Pi_{n+\ell}$ Scott sentence.*

*(5) If $\mathcal{G}$ has a $\Sigma_\ell$ Scott sentence for $\ell \geq 3$, then $\mathcal{G}^{(-n)}$ has a $\Sigma_{n+\ell}$ Scott sentence.*

*Moreover, (1) and (2) are uniform.*

*Proof.* For (1): Given a $\mathbf{0}^{(2n)}$-computable copy of $\mathcal{G}$, the edge relation of $\mathcal{G}$ is both $\Sigma^0_{2n+1}$ and $\Pi^0_{2n+1}$. Thus there computable functions $f$ and $g$ such that

$$uEv \iff f(u,v) \in S$$

and

$$uEv \iff g(u,v) \notin S.$$

Then we can build a computable copy of $\mathcal{G}^{(-n)}$: given $u$ and $v$, associate to them the pair of structures $(\mathcal{C}_{f(u,v)}, \mathcal{C}_{g(u,v)})$. Then

$$uEv \iff \mathcal{C}_{f(u,v)} \cong \mathcal{A} \text{ and } \mathcal{C}_{g(u,v)} \cong \mathcal{B}$$

and

$$u \not{E}\, v \iff \mathcal{C}_{f(u,v)} \cong \mathcal{B} \text{ and } \mathcal{C}_{g(u,v)} \cong \mathcal{A}.$$

For (2): Given a $\mathbf{0}^{(n)}$-computable copy of $\mathcal{G}^{(-n)}$ we have the $\mathbf{0}^{(n)}$-computable $\Sigma_{n+1}$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$. Given vertices $u$ and $v$, it is $\Delta^0_{n+1}$ relative to $\mathbf{0}^{(n)}$ to decide whether they have associated to them a copy of $(\mathcal{A}, \mathcal{B})$ or $(\mathcal{B}, \mathcal{A})$. $\Delta^0_{n+1}$ relative to $\mathbf{0}^{(n)}$ is $\mathbf{0}^{(2n)}$-computable, and so we can use $\mathbf{0}^{(2n)}$ to compute a copy of $\mathcal{G}$.

For (3): Given a $\Sigma_\ell$ sentence $\varphi$, we can create a new sentence $\varphi^*$ by replacing every instance of $uEv$ with the $\Sigma_{n+1}$ formula $E(u,v,\cdot) \models \varphi$ or the $\Pi_{n+1}$ formula $F(u,v,\cdot) \models \neg\varphi$, and every instance of $u \not{E}\, v$ replaced by the $\Sigma_{n+1}$ formula $F(u,v,\cdot) \models \varphi$ or the $\Pi_{n+1}$ formula $E(u,v,\cdot) \models \neg\varphi$. Thus $\varphi^*$ is a $\Sigma_{n+\ell}$ sentence, and $\mathcal{G} \models \varphi$ if and only if $\mathcal{G}^{(-n)} \models \varphi^*$.

For (4): We can write a $\Pi_{n+\ell}$ Scott sentence for $\mathcal{G}^{(-n)}$ as follows:

(a) The following $\Pi_2$ sentences:

   (i) If $E(u,v,x)$ or $F(u,v,x)$ then $u \neq v$, $u,v \in U$, and $x \notin U$.

   (ii) $E(u,v,x)$ if and only if $E(v,u,x)$ and $F(u,v,x)$ if and only if $F(v,u,x)$.

   (iii) For all $x \notin U$, there are $u$ and $v$ such that $E(u,v,x)$ or $F(u,v,x)$.

   (iv) If $E(u,v,x)$ (or $F(u,v,x)$) and $E(u',v',x)$ (or $F(u',v',x)$) then $\{u,v\} = \{u',v'\}$.

(b) For all $u,v \in U$, either $E(u,v,\cdot) \cong \mathcal{A}$ and $F(u,v,\cdot) \cong \mathcal{B}$, or alternatively $E(u,v,\cdot) \cong \mathcal{B}$ and $E(u,v,\cdot) \cong \mathcal{A}$. Since $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_{n+2}$ Scott sentences, this is $\Pi_{n+2}$.

(c) The $\Pi_\ell$ Scott sentence of $\mathcal{G}$, but with quantifiers restricted to $U$ and every instance of $uEv$ replaced by the $\Sigma_{n+1}$ formula $E(u,v,\cdot) \models \varphi$ or the $\Pi_{n+1}$ formula $F(u,v,\cdot) \models \neg\varphi$, and every instance of $u \not{E}\, v$ replaced by $\Sigma_{n+1}$ formula $F(u,v,\cdot) \models \varphi$ or the $\Pi_{n+1}$ formula $E(u,v,\cdot) \models \neg\varphi$. This is $\Pi_{\ell+n}$.

Here (a) and (b) imply that the structure is $\mathcal{H}^{(-n)}$ for some $\mathcal{H}$, and (c) expresses that $\mathcal{H} \cong \mathcal{G}$.

For (5): Similar to (4), but (c) is now $\Sigma_{\ell+n}$. □

## 4.2 Constructing the required structures

We begin by proving as a lemma the case $n = 1$ of Theorem 4.2.

**Lemma 4.4.** *Let $S \subseteq \mathbb{N}$ be a complete $\Sigma_3^0$ set. There are computable structures $\mathcal{A} \ncong \mathcal{B}$ and a sequence of computable structures $\mathcal{C}_i$ such that*

$$i \in S \Longrightarrow \mathcal{C}_i \cong \mathcal{A}$$

*and*

$$i \notin S \Longrightarrow \mathcal{C}_i \cong \mathcal{B}.$$

*Moreover:*

*(1) $\mathcal{B} \nleq_2 \mathcal{A}$ and so there is a $\mathbf{0}'$-computable $\Sigma_2$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$;*

*(2) $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_3$ Scott sentences.*

*Proof.* These structures $\mathcal{A}$ and $\mathcal{B}$ will be "bouquet graphs". A bouquet graph consists of a number of different connected components ("flowers") with each connected component having a central vertex and then some number of loops attached to the central vertex. If there is a loop of length, say, $n+3$, then we think of the component as having been labeled with the number $n$. Thus each flower corresponds to some subset of $\mathbb{N}$ consisting of the labels it has received, and the graph corresponds to a collection of subsets of $\mathbb{N}$. We can think of the labels as being c.e. since as the graph is being constructed we can always add new loops/labels, but once a label is added it cannot be removed. It will be more convenient to us to think of the structures as consisting of infinitely many points which are labeled with c.e. labels coming from the set of labels $\ell_n$. Putting a label $\ell_n$ on a element means to add a loop of length $n+3$ to the flower whose center is that element.

To make $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_3$ Scott sentences we must ensure that each automorphism orbit is definable by a $\Sigma_2$ formula, which we will accomplish by making sure that for each point $x$ there is a finite set $P_x$ of labels and a possibly infinite set $N_x$ of labels such that whenever any $y$ satisfies all of the labels in $P_x$ and none of the labels in $N_x$, then $y$ satisfies the same set of labels as $x$ (and hence they are in the same automorphism orbit). We also need $\mathcal{B} \nleq_2 \mathcal{A}$, and so we need a $\Sigma_2$ sentence $\varphi$ true of $\mathcal{A}$ but not of $\mathcal{B}$. To this end, we will make sure that $\mathcal{A}$ has an element $a$ such that every element $b \in \mathcal{B}$ has a label that $a$ does not have.

We reserve a special label $\ell_{\mathsf{kill}}$, which we call a killing label. Whenever any element has the label $\ell_{\mathsf{kill}}$, it will also have all other labels. During the construction we may say that we kill an element; by this, we mean that we put the label $\ell_{\mathsf{kill}}$ on that element, and also every other label.

We build, stage-by-stage, computable structures $\mathcal{B}$, $\mathcal{A}$, and $\mathcal{C}_i$ for $i \in \omega$ with each $\mathcal{C}_i$ satisfying

$$i \in S \Longrightarrow \mathcal{C}_i \cong \mathcal{A}$$

and

$$i \notin S \Longrightarrow \mathcal{C}_i \cong \mathcal{B}.$$

Let $(W^{i,k})_{i,k \in \mathbb{N}}$ be a uniform list of c.e. sets such that

$$i \in S \Longleftrightarrow \exists j \;\; W^{i,j} \text{ is infinite}$$

and such that, when $i \in S$, there is a unique such $j$. As we describe the elements of these structures $\mathcal{B}$, $\mathcal{A}$, and $\mathcal{C}_i$, each element will have infinitely many copies with the same labels as it; we will generally not mention this explicitly in the construction, but when we add some new element, we add infinitely many copies of it, and when we add a new label to an element, we add that label to all copies of that element.

$\mathcal{A}$ will consist of $\mathcal{B}$ together with an additional element $a$ (and hence infinitely many copies of $a$). Each $\mathcal{C}_i$ will consist of $\mathcal{B}$ together with infinitely many new elements $c_j$ (and of course infinitely many copies of these). Whenever we add a new element to $\mathcal{B}$, we implicitly also add it to $\mathcal{A}$ and the $\mathcal{C}_i$. We will never add any further new elements to $\mathcal{A}$ or to some $\mathcal{C}_i$ without adding it to $\mathcal{B}$, though we may add new labels.

*Construction.*

*Stage 0.* $\mathcal{B}$ consists of infinitely many killed elements. For each $j \in \omega$, $\mathcal{B}$ will also have an element $\hat{c}_{i,j,0}$. $\mathcal{A}$ will contain all of the elements of $\mathcal{B}$ together with a distinguished element $a$ which begins with no labels. For each $i$, $\mathcal{C}_i$ will contain all of the elements of $\mathcal{B}$ together with an additional element $c_{i,j}$ for each $j$, with labels $\ell_{i,j,k}$ for $k \in \omega$.

*Stage $s+1$.* For each $i$ and $j$ with $|W^{i,j}_{s+1}| > |W^{i,j}_s|$, letting $k = |W^{i,j}_s|$, do the following:

(1) Put the label $\ell_{i,j,k}$ on all elements of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}_j$.

(2) Kill $\hat{c}_{i,j,k}$ (which is in $\mathcal{B}$ and hence also in $\mathcal{A}$ and the $\mathcal{C}_j$).

(3) Create a new element $\hat{c}_{i,j,k+1}$ (in $\mathcal{B}$, and hence also in $\mathcal{A}$ and the $\mathcal{C}_j$). Give $\hat{c}_{i,j,k+1}$ the same labels as $c_{i,j}$ has at this stage.

*End construction.*

It is not hard to see that $\mathcal{B}$ will consist of the following elements:

- Infinitely many killed elements with the label $\ell_{\mathsf{kill}}$ and also all other labels.
- For each $i, j$ with $k = |W^{i,j}| < \infty$, infinitely many elements $\hat{c}_{i,j,k}$ with the labels $\ell_{i,j,k'}$ for $k' \in \mathbb{N}$ and also the labels $\ell_{i',j',k'}$ for $k' < |W^{i',j'}|$.

$\mathcal{A}$ will consist of these elements, and in addition:

- Infinitely many elements $a$ with the labels $\ell_{i,j,k}$ for $k < |W^{i,j}|$.

$\mathcal{C}_i$ will consist of the elements of $\mathcal{B}$, and in addition:

- For each $j$, infinitely many elements $c_{i,j}$ with the labels $\ell_{i,j,k}$ for $k \in \mathbb{N}$ and also the labels $\ell_{i',j',k'}$ for $k' < |W^{i',j'}|$.

The following four claims complete the proof.

*Claim* 1. If $i \notin S$ then $\mathcal{C}_i \cong \mathcal{B}$.

*Proof.* For each $j$, $W^{i,j}$ is finite, say of size $k$, and so the elements $c_{i,j}$ in $\mathcal{C}_i$ and $\hat{c}_{i,j,k}$ in $\mathcal{B} \subseteq \mathcal{A}$ have exactly the same labels. Since every element of $\mathcal{C}_i$ has the same labels as some element of $\mathcal{B}$, and since $\mathcal{C}_i$ contains $\mathcal{B}$, and there are infinitely many copies of each element, $\mathcal{C}_i \cong \mathcal{B}$. □

*Claim* 2. If $i \in S$ then $\mathcal{C}_i \cong \mathcal{A}$.

*Proof.* Since $i \in S$, there is a unique $j$ such that $W_{i,j}$ is infinite. Then the elements $c_{i,j}$ in $\mathcal{C}_i$ and $a$ in $\mathcal{A}$ have exactly the same labels. For each $j' \neq j$, $W^{i,j'}$ is finite, say of size $k$, and so the elements $c_{i,j'}$ in $\mathcal{C}_i$ and $\hat{c}_{i,j',k}$ in $\mathcal{B} \subseteq \mathcal{A}$ have exactly the same labels. Thus $\mathcal{C}_i \cong \mathcal{A}$. □

*Claim* 3. There is a $\Sigma_2$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$.

*Proof.* Let $N$ be the set of labels which do not hold of $a$ in $\mathcal{A}$. Let $\varphi = \exists x \bigwedge\!\!\!\bigwedge_{\ell \in N} \neg \ell(x)$, that is, $\varphi$ says that there is an $x$ such that any label holding of $x$ also holds of $a$. We must argue that every element of $\mathcal{B}$ has a label that $a$ does not. Since $a$ does not have the label $\ell_{\mathsf{kill}}$, we can restrict our attention to the elements $\hat{c}_{i,j,k}$ of $\mathcal{C}_i$ with $k = |W^{i,j}|$. Since $k+1 > |W^{i,j}|$, $a$ does not have the label $\ell_{i,j,k+1}$. However this label holds of $\hat{c}_{i,j,k}$. □

*Claim* 4. $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_3$ Scott sentences.

*Proof.* Two elements are in the same automorphism orbit if and only if they have the same labels. We will show that for each element $x$ of $\mathcal{A}$ or $\mathcal{B}$ there is a $\Sigma_2$ formula such that whenever that formula holds of some $y$ in $\mathcal{A}$ or $\mathcal{B}$, $x$ and $y$ have the same labels; this will be enough to see that $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_3$ Scott sentences. For the killed elements, this formula simply says that $\ell_{\mathsf{kill}}$ holds of that element. For $\hat{c}_{i,j,k}$ with $k = |W^{i,j}|$, it is that $\ell_{i,j,k+1}$ holds. For $a$, it is the formula $\bigwedge\!\!\!\bigwedge_{\ell \in N} \neg \ell(x)$ where $N$ is the set of labels which do not hold of $a$ in $\mathcal{A}$. The label $\ell_{\mathsf{kill}}$ is in $N$, separating $a$ from each killed element, and for each $i,j$ with $k = |W^{i,j}| < \infty$, there is a label $\ell_{i,j,k+1}$ in $N$ holding of the elements $\hat{c}_{i,j,k}$ but not of $a$. This distinguishes $a$ from all of the elements of $\mathcal{B}$, which make up the other elements of $\mathcal{A}$. □

The claims complete the proof of the theorem. □

Now we give the inductive step of the proof of Theorem 4.2, completing its proof.

*Proof of Theorem 4.2 for $n+1$ given Theorem 4.2 for $n$.* By Lemma 4.4 relativized to $\mathbf{0}^{(2n)}$, we get the following. Let $S \subseteq \mathbb{N}$ be a complete $\Sigma^0_{2n+3}$ set, which is a complete $\Sigma^0_3$ relative to $\mathbf{0}^{(2n)}$ set. There are $\mathbf{0}^{(2n)}$-computable structures $\mathcal{A}$ and $\mathcal{B}$ with $\Pi_3$ Scott sentences and a sequence of $\mathbf{0}^{(2n)}$-computable structures $\mathcal{C}_i$ such that

$$i \in S \Longrightarrow \mathcal{C}_i \cong \mathcal{A}$$

and

$$i \notin S \Longrightarrow \mathcal{C}_i \cong \mathcal{B}.$$

Moreover, there is a $\Sigma_2$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$.

Now let $\mathcal{A}^* = \mathcal{A}^{(-n)}$, $\mathcal{B}^* = \mathcal{B}^{(-n)}$, and $\mathcal{C}_i^* = {\mathcal{C}_i}^{(-n)}$. We use the properties of Theorem 4.3. Then $\mathcal{A}^*$ and $\mathcal{B}^*$ are a computable structures with $\Pi_{n+3}$ Scott sentences. $\mathcal{C}_i^*$ is a computable sequence of structures such that

$$i \in S \Longrightarrow \mathcal{C}_i^* \cong \mathcal{A}^*$$

and

$$i \notin S \Longrightarrow \mathcal{C}_i^* \cong \mathcal{B}^*.$$

Finally, we obtain from Theorem 4.3 a $\Sigma_{n+2}$ sentence $\varphi^*$ be such that $\mathcal{A}^* \models \varphi$ and $\mathcal{B}^* \not\models \varphi$. □

# 5 Non-arithmetic is a degree spectrum

In this section we prove the main theorem of the paper.

**Theorem 1.3.** *There is countable structure $\mathcal{M}$ such that for any set $X$, $X$ computes an isomorphic copy of $\mathcal{M}$ if and only if $X$ is not arithmetic.*

*Proof.* Our structure will consist, for each $n$, of a sort which diagonalizes against the $\mathbf{0}^{(n)}$-computable structures. For each $n$, we verify the following lemma which is of interest on its own. It says that for each $n$ there is a structure whose degree spectrum is contained in the non-$\Delta^0_{n+1}$ degrees, and contains all of the non-$\Delta^0_{2n+1}$ degrees.

**Lemma 5.1.** *There is a structure $\mathcal{M}_n$ which has no $\mathbf{0}^{(n)}$-computable copy, but for each set $X$ with $X \nleq_T \mathbf{0}^{(2n)}$, there is an $X$-computable copy of $\mathcal{M}_n$.*

With a little extra work we could make this uniform in $X$, but in the construction we give there will be two uniform procedures from building $\mathcal{M}_n$ from $X$, one which works if $X$ is not $\Sigma^0_{2n+1}$, and the other working if $\overline{X}$ is not $\Sigma^0_{2n+1}$. If $X$ is non-arithmetic, then it is not $\Sigma^0_{2n+1}$ for any $n$, and so there will be a uniform construction of $\mathcal{M}_n$ from $X$.

Letting $\mathcal{M}$ be the structure whose $n$th sort consists of a copy of $\mathcal{M}_n$, the degree spectrum of $\mathcal{M}$ is the non-arithmetic degrees. For each arithmetic set $X$, $X \leq_T \mathbf{0}^{(n)}$ for some $n$, and so $X$ cannot even compute a copy of the $n$th sort of $\mathcal{M}$. On the other hand, for each non-arithmetic set $X$, for each $n$ we have $X \nleq_T \mathbf{0}^{(2n)}$ and so $X^{(2n)} \nleq_T \mathbf{0}^{(2n)}$; thus, $X$ can compute a copy of $\mathcal{M}_n$. Moreover, since $X$ is not $\Sigma^0_{2n+1}$, $X$ can uniform construct a copy of $\mathcal{M}_n$, and hence $X$ can compute a copy of $\mathcal{M}$. Thus it remains to prove the lemma.

*Proof of lemma.* Fix $n$. We will construct $\mathcal{M} = \mathcal{M}_n$ as in the claim. $\mathcal{M}$ will essentially have the form of an unfriendly jump inversion, though we will have to build $\mathcal{M}$ directly rather than defining it as the jump inversion of some other structure and applying Theorem 4.3. (Technically, it will be constructed using a Marker extension rather than a jump inversion.)

Let $\mathcal{A} \ncong \mathcal{B}$ be computable $\mathcal{L}$-structures as in Theorem 4.2, i.e., for $S$ a complete $\Sigma^0_{2n+1}$ set there is a computable sequence of structures $\mathcal{C}_i$ such that

$$i \in S \iff \mathcal{C}_i \cong \mathcal{A}$$

and

$$i \notin S \iff \mathcal{C}_i \cong \mathcal{B}.$$

Moreover, we have a $\mathbf{0}^{(n)}$-computable $\Sigma_{n+1}$ sentence $\varphi$ such that $\mathcal{A} \vDash \varphi$ and $\mathcal{B} \nvDash \varphi$.

Consider the language $\{U_k : k \in \omega\} \cup \{R_\ell : \ell \in \omega\}$ where all of these are unary relations, and we think of the $R_\ell$ as being c.e. relations. This is (one choice for) the usual language in which the Slaman-Wehner structures are defined; the $U_k$ are sorts, and we think of an element $a$ as c.e.-coding the set $\{\ell : R_\ell(a)\}$; the standard construction is to have the elements of the $k$th sort code exactly the finite sets $\{W_e : W_e \text{ finite and } W_e \neq W_k\}$. The $R_\ell$ are made into c.e. relations by some form of Marker extension such as by attaching a loop of length $\ell + 3$ to $a$ to signify that $R_\ell(a)$.

In constructing $\mathcal{M}$ we will also apply a variant of the unfriendly Marker extensions to the relations $R_\ell$ while leaving the $U_k$ alone. The language will consist of the unary relations $U_k$

as well as additional binary relations $D_\ell$ and $\overline{D}_\ell$, an equivalence relation $E$, and the symbols from $\mathcal{L}$ the language of $\mathcal{A}$ and $\mathcal{B}$. The $U_k$ will still be sorts, but we now have elements not satisfying any of the $U_k$. For each $x \in U_k$, the relations $D_{\ell,s}$ and $\overline{D}_{\ell,s}$ will associate with $x$ two infinite sets

$$D_\ell(x) = \{y : D_\ell(x,y)\} \text{ and } \overline{D}_\ell(x) = \{y : \overline{D}_\ell(x,y)\}.$$

These sets will partition the elements which are not in any of the sorts $U_k$. $E$ will be an equivalence relation on all of the non-$U$ elements and will divide each of these sets $D_\ell(x)$ and $\overline{D}_\ell(x)$ into infinitely many infinite equivalence classes. Each equivalence class will be split across both $D_\ell(x)$ and $\overline{D}_\ell(x)$ for some $x$. Each equivalence class, when restricted to $D_\ell(x)$ or $\overline{D}_\ell(x)$, will be the domain of an $\mathcal{L}$-structure. Thus each of these $\mathcal{L}$-structures in $D_\ell(x)$ will have an associated $\mathcal{L}$-structure in $\overline{D}_\ell(x)$ in the same equivalence class, which we call its dual. The elements of these $\mathcal{L}$-structures will be elements outside of the sorts $U_n$, and these structures will all be disjoint from each other. Among each $\mathcal{L}$-structure and its dual, one will be isomorphic to $\mathcal{A}$ and the other to $\mathcal{B}$. See Figure 1 for a depiction of the elements associated to some $x$.

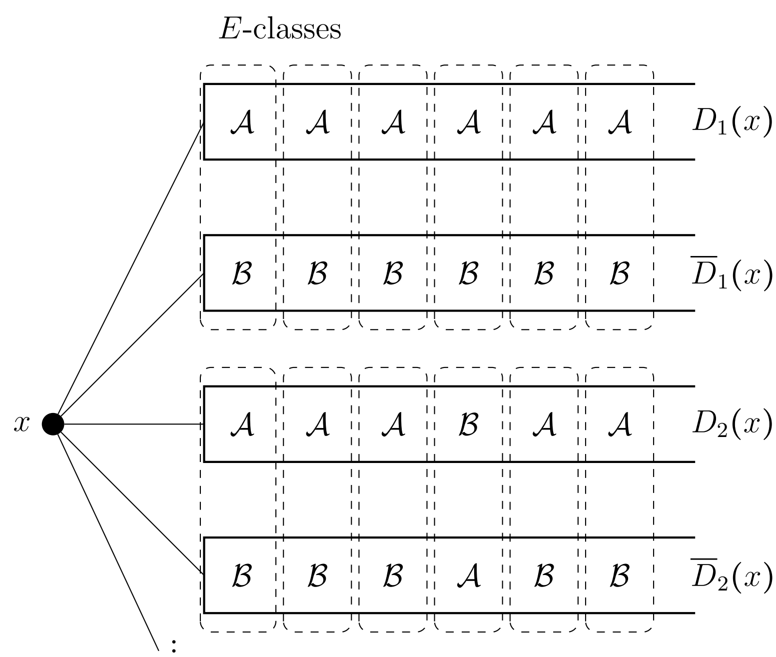


Figure 1: An element $x \in U_k$ and all of its associated elements. The element $x$ shown here satisfies $\neg R_1(x)$ because each $D_1(x)$-structure is isomorphic to $\mathcal{A}$ and each $\overline{D}_1(x)$-structure is isomorphic to $\mathcal{B}$. It satisfied $R_2(x)$ as witnessed by the fourth equivalence class in $D_2(x)$ and $\overline{D}_2(x)$.

For each $x$, either all of the $\mathcal{L}$-structures in $D_\ell(x)$ will be isomorphic to $\mathcal{A}$ (and hence all of the $\mathcal{L}$-structures in $\overline{D}_\ell(x)$ will be isomorphic to $\mathcal{B}$) or one will be isomorphic to $\mathcal{B}$ and the others all isomorphic to $\mathcal{A}$ (and hence one $\mathcal{L}$-structure in $\overline{D}_\ell(x)$ will be isomorphic to $\mathcal{A}$ and the others to $\mathcal{B}$). Thus we have the definable relation $R_\ell(x)$ which holds of $x$ if and only if one of the $\mathcal{L}$-structures of $D_\ell(x)$ is isomorphic to $\mathcal{B}$ if and only if one of the $\mathcal{L}$-structures of $\overline{D}_\ell(x)$ is isomorphic to $\mathcal{A}$. Note that $R_\ell$ is definable by a $\mathbf{0}^{(n)}$-computable $\Sigma_{n+1}$ formula.[3]

To a structure $\mathcal{N}$ of the above type, we can assign certain families of sets coded by the structure. Each point $x$ in one of the sorts $U_n$ codes a set

$$\begin{aligned} F(x) &= \{\ell : R_\ell(x)\} \\ &= \{\ell : \text{one of the structures attached to } x \text{ via } D_\ell \text{ is} \cong \mathcal{B}\} \\ &= \{\ell : \text{one of the structures attached to } x \text{ via } \overline{D}_\ell \text{ is} \cong \mathcal{A}\} \end{aligned}$$

Then the $k$th sort of $\mathcal{N}$ codes the family of sets

$$\mathcal{F}_k = \{F(x) : x \text{ is in the } n\text{t sort } U_n \text{ of } \mathcal{N}\}.$$

Thus $\mathcal{N}$ codes the sequence of families of sets $(\mathcal{F}_k)_{k\in\omega}$. Moreover, to each sequence of families $\mathcal{F} = (\mathcal{F}_k)_{k\in\omega}$ we can construct a structure coding $\mathcal{F}$, with $\mathcal{F}_k$ coded in the $k$th sort

Let $\mathcal{M}$ be the structure whose $k$th sort corresponds to the sequence of families

$$\{F : F \text{ finite and } F \neq W_k^{\mathbf{0}^{(2n)}}\}$$

where $W_k^{\mathbf{0}^{(2n)}}$ is the $k$th $\Sigma^0_{2n+1}$ set.

*Claim* 1. There is no $\mathbf{0}^{(n)}$-computable copy of $\mathcal{M}$.

*Proof.* In a $\mathbf{0}^{(n)}$-computable copy of $\mathcal{M}$, let $a_k$ be the first element of the $k$th sort, and let $V_k = F(a_k)$. Then $V_k$ is $\Sigma^0_{2n+1}$ as $\ell \in V_k$ if and only if one of the structures attached to $a_k$ via $D_\ell$ is isomorphic to $\mathcal{A}$ (and we have a $\mathbf{0}^{(n)}$-computable $\Sigma_{n+1}$ formula distinguishing $\mathcal{A}$ from $\mathcal{B}$). Moreover, we can find a $\Sigma_{2n+1}$ index for $V_k$ uniformly in $V_k$ in $k$, and hence a c.e.-relative-to-$\mathbf{0}^{(2n)}$ index for $V_k$. Then, by the recursion theorem, for some $k$, we have $V_k = W_k^{\mathbf{0}^{(2n)}}$. But then this was not a copy of $\mathcal{M}$ after all. ☐

In the second claim we make use of the idea of what we call $S$-*indices* for $\Sigma^0_{2n+1}$ facts. By an $S$-index we mean some $I \in \mathbb{N}$. $I$ represents $\mathsf{true}$ or 1 if $I \in S$, in which case we write $I \equiv 1$, and represents $\mathsf{false}$ or 0 if $I \notin S$, in which case we write $I \equiv 0$. Because $S$ is $\Sigma^0_{2n+1}$-complete, any $\Sigma^0_{2n+1}$ fact has an $S$-index, and we can generally compute such an index in reasonable situations. For example, given some positive Boolean combination of facts represented by $S$-indices, we can compute an $S$-index for the result, e.g., given $S$-indices $I_1$, $I_2$, and $I_3$ we can compute an $S$-index $J$ representing "($I_1$ and $I_2$) or $I_3$", that is,

$$J \in S \iff I_1 \text{ and } I_2 \text{ are both in } S, \text{ or } I_3 \text{ is in } S.$$

We can also represent $\Delta^0_{2n+1}$ facts by giving a pair of $S$-indices $I$ and $\overline{I}$ with $I \equiv 1 \iff \overline{I} \equiv 0$.

The importance of $S$-indices is that given an $S$-index $I$,

$$\mathcal{C}_I \cong \mathcal{A} \iff I \equiv 1 \text{ and } \mathcal{C}_I \cong \mathcal{B} \iff I \equiv 0.$$

The $\mathcal{C}_I$ are computable, but in a construction of some larger structure we might non-computably build some sequence of $S$-indices $I_1, I_2, \ldots$ and then incorporate the structures $\mathcal{C}_{I_1}, \mathcal{C}_{I_2}, \ldots$ into the larger structure.

[3]The reader might wonder why we cannot just attach to each $x$ for each $m$ an $\mathcal{L}$-structure isomorphic to either $\mathcal{A}$ or $\mathcal{B}$, with $R_m(x)$ if and only if the structure is isomorphic to $\mathcal{B}$. We would still have that $R_m$ is $\Sigma^0_{2n+1}$ in any computable copy. However in non-computable copies the complexities would not be correct.

*Claim* 2. Suppose that $X$ is not $\Sigma^0_{2n+1}$. Then there is (uniformly in $X$) an $X$-computable copy $\mathcal{N}$ of $\mathcal{M}$.

*Proof.* Fix $k$. We want to build (uniformly in $X$ and $k$) a copy $\mathcal{N}_k^X$ of the $k$th sort coding

$$\{F : F \text{ finite and } F \neq W_k^{\mathbf{0}^{(2n)}}\}.$$

If we can do this uniformly, then we can compute uniformly in $X$ a copy of $\mathcal{M}$. Our elements will be of the form $a_{F,s}$ for each finite set $F$ and $s \in \omega$. We will build our structure to be $X$-computable in the following way. For each $a_{F,s}$, $\ell$, and $t$, we will have, $X$-computably, two $S$-indices $I(F,s,\ell,t)$ and $\overline{I}(F,s,\ell,t)$ which are complementary, i.e., $I(F,s,\ell,t) \not\equiv \overline{I}(F,s,\ell,t)$. $I(F,s,\ell,t)$ and $\overline{I}(F,s,\ell,t)$ describe which structures should be attached to $a_{F,s}$ in $\mathcal{N}_k^X$.

The $t$th structure attached to $a_{F,s}$ in $D_\ell(a_{F,s})$ will be:

- $\mathcal{B}$, if $I(F,s,\ell,t) \equiv 0$ (and $\overline{I}(F,s,\ell,t) \equiv 1$), and
- $\mathcal{A}$, if $I(F,s,\ell,t) \equiv 1$ (and $\overline{I}(F,s,\ell,t) \equiv 1$)

The $t$th structure attached to $a_{F,s}$ in $\overline{D}_\ell(a_{F,s})$ will be:

- $\mathcal{A}$, if $I(F,s,\ell,t) \equiv 0$ (and $\overline{I}(F,s,\ell,t) \equiv 1$), and
- $\mathcal{B}$, if $I(F,s,\ell,t) \equiv 1$ (and $\overline{I}(F,s,\ell,t) \equiv 0$).

The $t$th structures in $D_\ell(a_{F,s})$ and $\overline{D}_\ell(a_{F,s})$ will be in the same $E$-equivalence class. For each $F,s,\ell$ there will be at most one $t$ with $I(F,s,\ell,t) \equiv 1$ and $\overline{I}(F,s,\ell,t) \equiv 0$.

To compute a copy of the structure $\mathcal{N}_k^X$ using $X$, we put as the $t$th equivalence class in $D_\ell(a_{F,s})$ a copy of $\mathcal{C}_{I(F,s,\ell,t)}$ and in $\overline{D}_\ell(a_{F,s})$ a copy of $\mathcal{C}_{\overline{I}(F,s,\ell,t)}$. Then we have $\mathcal{C}_{I(F,s,\ell,t)} \cong \mathcal{A}$ and $\mathcal{C}_{\overline{I}(F,s,\ell,t)} \cong \mathcal{B}$ if $I(F,s,\ell,t) \equiv 1$ and $\overline{I}(F,s,\ell,t) \equiv 0$, and $\mathcal{C}_{I(F,s,\ell,t)} \cong \mathcal{B}$ and $\mathcal{C}_{\overline{I}(F,s,\ell,t)} \cong \mathcal{A}$ if $I(F,s,\ell,t) \equiv 0$ and $\overline{I}(F,s,\ell,t) \equiv 1$, as desired. Since the sequence $\mathcal{C}_i$ is computable and the indices $I(F,s,\ell,t)$ and $\overline{I}(F,s,\ell,t)$ are $X$-computable, this procedure builds an $X$-computable structure. It remains to define $I(F,s,\ell,t)$ and $\overline{I}(F,s,\ell,t)$.

We define, for each $t$,

$$R_{F,s}[t] = \{\ell : \exists t' \leq t \; I(F,s,\ell,t') \equiv 1\} = \{\ell : \exists t' \leq t \; \overline{I}(F,s,\ell,t') \equiv 0\}.$$

This is a $\Delta^0_{2n+1}$ set and for each $\ell$ we can compute $S$-indices for the facts "$\ell \in R_{F,s}[t]$" and "$\ell \notin R_{F,s}[t]$" from $S$-indices $I(F,s,\ell,t')$ and $\overline{I}(F,s,\ell,t')$. That is, given $F,s,\ell,t$ we can compute some $j, \overline{j}$ such that

$$\ell \in R_{F,s}[t] \iff j \in S$$

and

$$\ell \in R_{F,s}[t] \iff \overline{j} \in S.$$

Also, let $W_{k,t}^{\mathbf{0}^{(2n)}}$ be the stage $t$ approximation to $W_k^{\mathbf{0}^{(2n)}}$, i.e., $\ell \in W_{k,t}^{\mathbf{0}^{(2n)}}$ if and only if $\Phi_{k,t}^{\mathbf{0}^{(2n)}}(\ell)\downarrow$. $W_{k,t}^{\mathbf{0}^{(2n)}}$ is $\Delta^0_{2n+1}$ uniformly in $t$, and we can compute $S$-indices for "$\ell \in W_{k,t}^{\mathbf{0}^{(2n)}}$" and "$\ell \notin W_{k,t}^{\mathbf{0}^{(2n)}}$".

Define $I(F,s,\ell,t)$ and $\overline{I}(F,s,\ell,t)$ as follows, inductively on $t$. For each $t$ we define them for all $F,s,\ell$ and so we can use $R_{F,s}[t]$ in the definitions at stage $t+1$. We begin by defining $I(F,s,\ell,t)$ and $\overline{I}(F,s,\ell,t)$ by represented value, i.e., whether $I(F,s,\ell,t) \equiv 0$ or $I(F,s,\ell,t) \equiv 1$, before explaining how they can be defined as $S$-indices. The idea is that we carry out something like the standard Slaman-Wehner argument, relative to $\mathbf{0}^{(2n)}$, using $X$ as an oracle.

(1) $I(F,s,\ell,t) \equiv 1$ and $\overline{I}(F,s,\ell,t) \equiv 0$ if $\ell \in F$.

(2) $I(F,s,\ell,t) \equiv 0$ and $\overline{I}(F,s,\ell,t) \equiv 1$ if $t \leq s$ and $\ell \notin F$.

(3) At stage $t+1$, find the first $t+1$ elements of $X$, and let

  (a) $I(F,s,\ell,t+1) \equiv 1$ and $\overline{I}(F,s,\ell,t+1) \equiv 0$ if $R_{F,s}[t] = W_{k,t}^{\mathbf{0}^{(2n)}}$ and $\ell$ is the least element of $X - W_{k,t}^{\mathbf{0}^{(2n)}}$;

  (b) $I(F,s,t+1) \equiv 0$ and $\overline{I}(F,s,\ell,t) \equiv 1$ otherwise.

Note that by define, we mean that we compute using oracle $X$ the $S$-indices for $I$ and $\overline{I}$. For $t=1$, in (1) and (2) we can simply fix $s_1 \in S$ and $s_0 \notin S$, setting, e.g., $I(F,s,\ell,0) = s_1$ if $I(F,s,\ell,0) \equiv 1$. For (3), note that the values of $I(F,s,\ell,t+1)$ and $\overline{I}(F,s,\ell,t+1)$ depend on finitely many $\Delta^0_{2n+1}$ questions that we need to ask, and we can compute which questions we must ask using $X$. It is important for this that $W_{k,t}^{\mathbf{0}^{(2n)}}$ has only elements $< t$. We have $S$-indices for each of these $\Delta^0_{2n+1}$ facts, and so we can combine these to get $S$-indices for $I(F,s,\ell,t+1)$ and $\overline{I}(F,s,\ell,t+1)$. The process of computing indices is $X$-computable.

Now we must verify that this construction works. This is essentially the usual Slaman-Wehner argument. First we show that for every $F,s$ we have that

$$R_{F,s} = \bigcup_t R_{F,s}[t]$$

is finite and not equal to $W_k^{\mathbf{0}^{(2n)}}$. Indeed, if we had $R_{F,s} = W_k^{\mathbf{0}^{(2n)}}$, then this would be because $R_{F,s} = W_k^{\mathbf{0}^{(2n)}} = F \cup X$ and $F \cup X$ is not $\Sigma^0_{2n+1}$ while $W_k^{\mathbf{0}^{(2n)}}$ is. On the other hand, for every finite set $F \neq W_k^{\mathbf{0}^{(2n)}}$, we have $R_{F,s} = F$ for large enough $s$: simply take $s$ so that for every $t \geq s$ we have $F \neq W_{k,t}^{\mathbf{0}^{(2n)}}$. Thus the collection

$$\{R_{F,s} : F \text{ finite, } s \in \mathbb{N}\} = \{F : F \text{ finite and } F \neq W_k^{\mathbf{0}^{(2n)}}\}.$$

Thus $\mathcal{N}_k^X$ is isomorphic to the $k$th sort of $\mathcal{M}$, and hence $\mathcal{N}^X \cong \mathcal{M}$ is an $X$-computably copy of $\mathcal{M}$. ☐

These claims complete the proof of the lemma. By Claim 1 there is no $\mathbf{0}^{(n)}$-computable copy of $\mathcal{M}$. If $X$ is not arithmetic then it is not $\Sigma^0_{2n+1}$ and so by Claim 2 there is an $X$-computable copy of $\mathcal{M}$ computable uniformly from $X$. This is uniform in $n$. ☐

We have now proved the lemma for all $n$. For each $n$ there is a structure $\mathcal{M}_n$ which has no $\mathbf{0}^{(n)}$-computable copy, but for each set $X$ which is not $\Sigma^0_{2n+1}$, we can build (uniformly in $X$) an $X$-computable copy of $\mathcal{M}_n$. Putting all of these structures together as described just after the statement of the lemma, we obtain a structure whose degree spectrum is exactly the non-arithmetic degrees. ☐

# 6 Other applications

## 6.1 Computable Scott sentences

In this section, we will prove that when a computable structure has a $\Pi_n$ Scott sentence, the best we can say is that it has a $\mathbf{0}^{(n)}$-computable $\Pi_n$ Scott sentence, and a computable $\Pi_{2n}$ Scott sentence. Recall that Alvir, Csima, and Harrison-Trainor [ACHT] proved that there is a computable structure with a $\Pi_n$ Scott sentence but no computable $\Sigma_4$ Scott sentence. We want to show that there is a computable structure with a $\Pi_n$ Scott sentence but no computable $\Sigma_{2n}$ Scott sentence. Morally speaking, we prove the theorem above by relativizing that fact and applying jump inversion. However, this argument does not quite work. Relativizing to $\mathbf{0}^{(2n-4)}$, there is a $\mathbf{0}^{(2n-4)}$-computable structure $\mathcal{A}$ with a $\Pi_2$ Scott sentence but no $\mathbf{0}^{(2n-4)}$-computable $\Sigma_4$ Scott sentence. Then $\mathcal{A}^{(-(n-2))}$ has a computable copy and a $\Pi_n$ Scott sentence; however, it is not clear that is has no computable $\Sigma_{2n}$ Scott sentence. Instead, we prove the following stronger fact.

**Theorem 6.1.** *Fix $n \geq 2$. Let $S \subseteq \mathbb{N}$ be a complete $\Pi^0_{2n}$ set. There is a computable structure $\mathcal{A}$ with a $\Pi_n$ Scott sentence and a uniformly computable sequence of structures $\mathcal{C}_i$ such that*

$$i \in S \Longrightarrow \mathcal{C}_i \cong \mathcal{A}$$

*and*

$$i \notin S \Longrightarrow \mathcal{C}_i \ncong \mathcal{A}.$$

*In particular, $\mathcal{A}$ has no computable $\Sigma_{2n}$ Scott sentence.*

The case $n = 1$ of this theorem is proved very similarly to the corresponding fact from [ACHT]. The proof is quite technical and contains only minor new ideas and so we omit it. Given this fact, we can apply the jump inversion as follows.

*Proof of the general case given the case $n = 2$.* Fix $n \geq 3$. Given a complete $\Pi^0_{2n}$ set $S$, it is a complete $\Pi_4$ set relative to $\mathbf{0}^{(2n-4)}$. Relativizing the case $n = 2$ to $\mathbf{0}^{(2n-4)}$, there is a $\mathbf{0}^{(2n-4)}$-computable structure $\mathcal{A}$ with a $\Pi_2$ Scott sentence, and a $\mathbf{0}^{(2n-4)}$-computable sequence of structures $\mathcal{C}_i$, such that

$$i \in S \Longrightarrow \mathcal{C}_i \cong \mathcal{A}$$

and

$$i \notin S \Longrightarrow \mathcal{C}_i \ncong \mathcal{A}.$$

Let $\mathcal{A}^* = \mathcal{A}^{(-(n-2))}$ and $\mathcal{C}_i^* = {\mathcal{C}_i}^{(-(n-2))}$ be unfriendly $(n-2)$-jump inversions. We use Theorem 4.3: $\mathcal{A}^*$ has a computable copy and there is a uniformly computable sequence of computable copies of $\mathcal{C}_i^*$. $\mathcal{A}$ has a $\Pi_n$ Scott sentence and

$$i \in S \Longrightarrow \mathcal{C}_i^* \cong \mathcal{A}^*$$

and

$$i \notin S \Longrightarrow \mathcal{C}_i^* \ncong \mathcal{A}^*.$$

This completes the argument. □

## 6.2 Back-and-forth types

Recall that in [CGHT], Chen, Gonzalez, and Harrison-Trainor showed that for each fixed $\mathcal{A}$ the set

$$\{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{A} \leq_n \mathcal{B}\}$$

is always (boldface) $\mathbf{\Pi}^0_{n+2}$. They asked, if $\mathcal{A}$ is computable, what the lightface complexity of this set is. The upper bound is $\Pi^0_{2n}$. For the structures from Theorem 6.1 this is best possible.

**Theorem 6.2.** *There is a computable structure $\mathcal{A}$ such that the set*

$$\{\mathcal{B} \in Mod(\mathcal{L}) : \mathcal{A} \leq_n \mathcal{B}\}$$

*is not (lightface) $\Sigma^0_{2n}$.*

## 6.3 Separating structures

Recall that Ash and Knight [AK00] showed that if $\mathcal{A}$ and $\mathcal{B}$ are computable structures, $n$-friendly with computable existential diagrams, and $\mathcal{B} \nleq_n \mathcal{A}$, then there is a computable $\Pi_n$ sentence $\varphi$ such that $\mathcal{B} \models \varphi$ and $\mathcal{A} \not\models \varphi$. If we drop the assumption that $\mathcal{A}$ and $\mathcal{B}$ are $n$-friendly with computable existential diagrams, then there is a computable $\Pi_{2n-1}$ sentence $\varphi$.[4] We get the following example showing that this is best possible.

**Theorem 2.11.** *Fix $n \geq 2$. There are computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B} \nleq_n \mathcal{A}$ but for any computable $\Sigma_{2n-1}$ sentence $\varphi$, if $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.*

*Proof.* Let $S$ be a $\Sigma^0_{2n-1}$-complete set. Fix, as in Theorem 4.2, computable structures $\mathcal{A} \ncong \mathcal{B}$ and a sequence of computable structures $\mathcal{C}_i$ such that

$$i \in S \Longrightarrow \mathcal{C}_i \cong \mathcal{A}$$

and

$$i \notin S \Longrightarrow \mathcal{C}_i \cong \mathcal{B}.$$

Moreover, $\mathcal{B} \nleq_n \mathcal{A}$. If there was a computable $\Pi_{2n-1}$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$, then

$$S = \{i : \mathcal{C}_i \models \varphi\}$$

would be $\Pi^0_{2n-1}$. Since it is $\Sigma^0_{2n-1}$-complete, there is no such $\varphi$. □

## 6.4 Maximally unfriendly structures

We have noted previously that if $\mathcal{A}$ is a computable structure the the back-and-forth relations $\leq_n$ on $\mathcal{A}$ are $\Pi_{2n}$. We show that this is best possible, that is, there is a maximally unfriendly structure.

[4]The proof is similar to that of Theorem 3.2 where we showed that if $\mathcal{A}$ and $\mathcal{B}$ are computable structures, and $\mathcal{A} \nleq_n \mathcal{B}$, then there is a $\mathbf{0}^{(n-1)}$-computable $\Pi_n$ sentence $\varphi$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$.

**Theorem 2.5.** *For each $n$ there is a computable structure $\mathcal{A}$ such that the $n$-back-and-forth relation on $\mathcal{A}$ is $\Pi^0_{2n}$-complete and hence computes $\mathbf{0}^{(2n)}$.*

*Proof.* Let $U$ be a $\Pi^0_{2n}$-complete set. Write

$$k \in U \iff \forall \ell \, (k, \ell) \in S$$

where $S$ is $\Sigma^0_{2n-1}$. We may assume that for each $n$ there is at most one $m$ with $(k, \ell) \notin S$. Fix, as in Theorem 4.2, computable structures $\mathcal{A} \ncong \mathcal{B}$ and a sequence of computable structures $\mathcal{C}_{k,\ell}$ such that

$$(k, \ell) \in S \Longrightarrow \mathcal{C}_{k,\ell} \cong \mathcal{A}$$

and

$$(k, \ell) \notin S \Longrightarrow \mathcal{C}_{k,\ell} \cong \mathcal{B}.$$

Moreover, $\mathcal{B} \nleq_n \mathcal{A}$.

$\mathcal{M}$ will have a unary relation $U$, binary relations $D_\ell$, and the symbols of the language of $\mathcal{A}$ and $\mathcal{B}$. For each element $a$ satisfying $U$, we will attach to $a$ a set of elements $D_\ell(a) = \{b : D_\ell(a, b)\}$. None of these elements will satisfy $U$, and the sets $D_\ell(a)$ will partition the non-$U$ elements. Each set $D_\ell(a)$ will be the domain of a structure isomorphic to $\mathcal{A}$ or $\mathcal{B}$. Essentially, one should think of $\mathcal{M}$ as consisting of infinitely many elements $a$, each of which has attached to it an ordered sequence of structures $D_\ell(a)$ each of which is isomorphic to $\mathcal{A}$ or $\mathcal{B}$.

$\mathcal{M}$ will have elements $a_k$, each of which has as the $\ell$th structure $D_\ell(a_k)$ attached to it a copy of $\mathcal{C}_{k,\ell}$. If $k \in U$, then each of these structures is isomorphic to $\mathcal{A}$, while if $k \notin U$, then at least one of these structures is isomorphic to $\mathcal{B}$. $\mathcal{M}$ will also have one additional element $a^*$ with each structure attached to it being a copy of $\mathcal{A}$.

Standard arguments with back-and-forth relations show that $(\mathcal{M}, a^*) \geq_n (\mathcal{M}, a_k)$ if and only if $k \in U$: If $k \in U$, then $a_k$ and $a^*$ have the same structures attached, all copies of $\mathcal{A}$, while if $k \notin U$, then for some $\ell$ the $\ell$th structure attached to $a_k$ is isomorphic to $\mathcal{B}$, but the $\ell$ structure attached to $a^*$ is isomorphic to $\mathcal{A}$, and $\mathcal{A} \ngeq_n \mathcal{B}$. This yields a 1-reduction from the $\Pi^0_{2n}$-complete set $U$ to the $n$-back-and-forth relation on $\mathcal{M}$ which is thus $\Pi^0_{2n}$-complete and computes $\mathbf{0}^{(2n)}$. ☐

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