Scott Complexity of Countable Structures

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Abstract

We define the Scott complexity of a countable structure to be the least complexity of a Scott sentence for that structure. This is a finer notion of complexity than Scott rank: it distinguishes between whether the simplest Scott sentence is Σ_α, Π_α, or d-Σ_α. We give a complete classification of the possible Scott complexities, including an example of a structure whose simplest Scott sentence is $\Sigma_{\lambda+1}$ for λ a limit ordinal. This answers a question left open by A. Miller.

We also construct examples of computable structures of high Scott rank with Scott complexities $\Sigma_{\omega_1^{CK}+1}$ and d-$\Sigma_{\omega_1^{CK}+1}$. There are three other possible Scott complexities for a computable structure of high Scott rank: $\Pi_{\omega_1^{CK}}$, $\Pi_{\omega_1^{CK}+1}$, $\Sigma_{\omega_1^{CK}+1}$. Examples of these were already known. Our examples are computable structures of Scott rank $\omega_1^{CK}+1$ which, after naming finitely many constants, have Scott rank ω_1^{CK}. The existence of such strutures was an open question.

1 Introduction

Scott [Sco65] showed that for a countable language L every countable structure can be described up to isomorphism among countable structures by a sentence of the infinitary logic $L_{\omega_1\omega}$. Such a sentence is called a *Scott sentence* of the structure. The standard proof uses back-and-forth relations and the key step is to show that for each countable structure $\mathcal{A}$ there is an ordinal α such that any two tuples which are α-back-and-forth-equivalent are actually in the same automorphism orbit. The least such α is a measure of the internal complexity of the structure and is one definition of the *Scott rank* of the structure.

Annoyingly, there are many similar but non-equivalent definitions of Scott rank in the literature, most of which agree up to a small factor. In an attempt to standardize the notion of Scott rank, Montalbán introduced a definition which connects the internal complexity of the automorphism orbits with the external complexity of describing the structure via a Scott sentence. Scott sentences, as with all formulas of $L_{\omega_1\omega}$, can be classified up to equivalence by the number of quantifier alternations, where infinite conjunctions are viewed as universal quantifiers and infinite disjunctions as existential quantifiers. The Σ_n sentences have n alternations of quantifiers, beginning with existential quantifiers; the Π_n sentences are similar but begin with a universal quantifier. The hierarchy continues in the natural way through the transfinite.

Definition 1.1 (Montalbán [Mon15])**.** The *Scott rank* of a countable structure $\mathcal{A}$ is the least α such that $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.

This definition is robust in the sense that there are many equivalent characterizations.

Theorem 1.2 (Montalbán [Mon15])**.** *Let $\mathcal{A}$ be a countable structure, and α a countable ordinal. The following are equivalent:*

1. *$\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.*
2. *Every automorphism orbit in $\mathcal{A}$ is Σ_α-definable without parameters.*
3. *$\mathcal{A}$ is uniformly (boldface) $\mathbf{\Delta}^0_\alpha$-categorical.*

The Scott ranks assigned by this definition are again not too different from the Scott ranks assigned by the definitions using back-and-forth relations.

Scott rank is a coarse measure of complexity in that it does not differentiate between, for example, whether the simplest Scott sentence for a structure is Σ_α or $\Pi_{\alpha+1}$; in either case, the Scott rank is α. Recently there have been a number of interesting results at this finer level of detail. We make the following informal definition:

Definition 1.3. The *Scott complexity* of a countable structure $\mathcal{A}$ is the least complexity of a Scott sentence for $\mathcal{A}$.

Now, Σ_α and Π_α are not the only possible complexities of a sentence of $L_{\omega_1\omega}$. For example [KS18, CHK+12, Ho17, HTH18], many finitely generated groups, including all abelian groups and free groups, have a Scott sentence which is the conjunction of a Σ_2 and a Π_2 sentence, but no simpler sentence. We call such a sentence a d$-\Sigma_2$ sentence (with "d" standing for "difference"); so d$-\Sigma_2$ should also be a possible Scott complexity.

There are also other types of $L_{\omega_1\omega}$ formulas. For example, a formula might be the disjunction of a Σ_α formula φ and Π_α formula ψ. However, such a formula cannot be the simplest Scott sentence of a structure $\mathcal{A}$, for if $\varphi \vee \psi$ is a Scott sentence for $\mathcal{A}$, then $\mathcal{A} \vDash \varphi \vee \psi$ and so either $\mathcal{A} \vDash \varphi$ in which case φ is a Σ_α Scott sentence for $\mathcal{A}$, or $\mathcal{A} \vDash \psi$ in which case ψ is a Π_α Scott sentence for $\mathcal{A}$. In practice, it seems that Σ_α, Π_α, and d$-\Sigma_\alpha$ are the only possible Scott complexities. These are arranged from the simplest on the left to most complex on the right as follows:

$$\begin{array}{ccccccccccccc} & & \Sigma_1 & & & & \Sigma_2 & & & & \Sigma_3 & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ \Sigma_0 & & & & \text{d-}\Sigma_1 & & & & \text{d-}\Sigma_2 & & & & \text{d-}\Sigma_3 \to \cdots \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & \Pi_1 & & & & \Pi_2 & & & & \Pi_3 & & \end{array}$$

Moreover, A. Miller [Mil83] showed that if a structure has both a Σ_α and a Π_α Scott sentence, then it has a d$-\Sigma_\beta$ Scott sentence for some $\beta < \alpha$.

Are these the only possible Scott complexities? The problem is that there is no formal notion of what it means to be "a complexity class" of $L_{\omega_1\omega}$ formulas. However, it is well-known that there are connections between the Scott sentences of a stucture $\mathcal{A}$ and the topological complexity of $\text{Iso}(\mathcal{A})$, the set of isomorphic copies of $\mathcal{A}$ with domain ω, viewed as a subset of Baire space. For any reasonably robust notion of the complexity of $L_{\omega_1\omega}$ formulas, the complexity of a Scott sentence for a structure $\mathcal{A}$ is in correspondence, via Vaught's version of the López-Escobar theorem, with the topological complexity of $\mathrm{Iso}(\mathcal{A})$. The fact that $\mathcal{A}$ has a Scott sentence implies that $\mathrm{Iso}(\mathcal{A})$ is always Borel. The topological complexity of $\mathrm{Iso}(\mathcal{A})$ can be measured by its Wadge degree. This motivates the following formal definition:

Definition 1.4. The *Scott complexity* of a structure $\mathcal{A}$ is the Wadge degree of $\mathrm{Iso}(\mathcal{A})$.

In the first part of this paper, we show that the Scott complexity of a countable structure must be one of $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, and $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\alpha$; so, for example, if $\mathcal{A}$ has a Π_α Scott sentence but no Σ_α Scott sentence, then $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{\Pi}^0_\alpha$-complete under Wadge reducibility.

By Vaught's theorem, we associate each of these Wadge degrees of $\mathrm{Iso}(\mathcal{A})$ with the complexity class of the corresponding Scott sentence for $\mathcal{A}$, e.g. identifying the Wadge degree $\mathbf{\Pi}^0_\alpha$ with the complexity Π_α. So we need not consider Scott sentences of any complexity other than Σ_α, Π_α, and $\mathrm{d}\text{-}\Sigma_\alpha$. Under the natural correspondence between Wadge degrees and complexities of Scott sentences, we can also define:

Definition 1.5. The *Scott complexity* of a structure $\mathcal{A}$ is the least complexity, from among Σ_α, Π_α, and $\mathrm{d}\text{-}\Sigma_\alpha$, of a Scott sentence for $\mathcal{A}$.

In the second part of the paper, we will give examples of structures with particular Scott complexities in order to give a complete classification of the possible Scott complexities. A. Miller [Mil83] has given a number of examples, but the problem of constructing a structure of Scott complexity $\Sigma_{\lambda+1}$ for λ a limit ordinal was still open. We give such an example in Theorem 4.1. He also showed that for a language with no function symbols, if a structure has a Σ_2 Scott sentence, then it has a $\mathrm{d}\text{-}\Sigma_1$ Scott sentence. We give a proof of this fact for all languages in Theorem 5.1.

The complete classification is as follows:

Theorem 1.6. *The possible Scott complexities of countable structures $\mathcal{A}$ are:*

1. Π_α *for* $\alpha \geq 1$,
2. Σ_α *for* $\alpha \geq 3$ *a successor ordinal,*
3. $\mathrm{d}\text{-}\Sigma^0_\alpha$ *for* $\alpha \geq 1$ *a successor ordinal.*

There is a countable structure with each of these Wadge degrees.

Proof. In Theorem 2.6 we show that the Scott complexity of a structure must be one of Π_α, Σ_α (for α a non-limit), and $\mathrm{d}\text{-}\Sigma_\alpha$ (for α a non-limit). A. Miller [Mil83] showed that no structure has a Σ_1 Scott sentence. In Theorem 5.1 we show that Σ_2 cannot be the Scott complexity of a countable structure.

A. Miller [Mil83, §4] gave examples of structures which have Scott complexity Π_α for $\alpha \geq 1$ and Σ_α and $\mathrm{d}\text{-}\Sigma_\alpha$ for $\alpha \geq 3$ if α is not a limit ordinal or the successor of a limit ordinal. The group $\mathbb{Z}$ has Scott complexity $\mathrm{d}\text{-}\Sigma_2$. An infinite structure with a single unary operator holding of exactly one element has Scott complexity $\mathrm{d}\text{-}\Sigma_1$. In Theorem 4.1, we give examples of structures of Scott complexity $\Sigma_{\lambda+1}$ for λ a successor ordinal. Then the disjoint union of such a structure and a structure of Scott complexity $\Pi_{\lambda+1}$ has Scott complexity $\mathrm{d}\text{-}\Sigma_{\lambda+1}$. □

In the last part of this paper, we will investigate the Scott complexity of computable structures of high Scott rank and give a new example of such structures. We will see that some important properties of structures of high Scott rank can be rephrased in terms of Scott complexity.

If $\mathcal{A}$ is a countable structure, then the Scott rank of $\mathcal{A}$ is at most $\omega_1^{\mathcal{A}}+1$ [Nad74]; equivalently, the Scott complexity of $\mathcal{A}$ is at most $\Pi_{\omega_1^{\mathcal{A}}+2}$. We say that $\mathcal{A}$ has high Scott rank if it has Scott rank $\geq \omega_1^{\mathcal{A}}$; equivalently, $\mathcal{A}$ has Scott complexity at least $\Pi_{\omega_1^{\mathcal{A}}}$, so we say:

Definition 1.7. A structure $\mathcal{A}$ has *high Scott complexity/rank* if it has Scott complexity at least $\Pi_{\omega_1^{\mathcal{A}}}$, or equivalently, it has Scott rank $\geq \omega_1^{\mathcal{A}}$.

There are two possible Scott ranks for a structure of high Scott rank, namely $\omega_1^{\mathcal{A}}$ and $\omega_1^{\mathcal{A}}+1$. There are examples of each of these [Har68, KM10]. However, there are five possible high Scott complexities: $\Pi_{\omega_1^{\mathcal{A}}}$, $\Pi_{\omega_1^{\mathcal{A}}+1}$, $\Sigma_{\omega_1^{\mathcal{A}}+1}$, d-$\Pi_{\omega_1^{\mathcal{A}}+1}$, and $\Pi_{\omega_1^{\mathcal{A}}+2}$. Of these, $\Pi_{\omega_1^{\mathcal{A}}}$ and $\Pi_{\omega_1^{\mathcal{A}}+1}$ correspond to Scott rank $\omega_1^{\mathcal{A}}$, and $\Sigma_{\omega_1^{\mathcal{A}}+1}$, d-$\Pi_{\omega_1^{\mathcal{A}}+1}$, and $\Pi_{\omega_1^{\mathcal{A}}+2}$ corresponds to Scott rank $\omega_1^{\mathcal{A}}+1$. We will show that all of these possibilities can be achieved:

Theorem 1.8. *There are computable structures of all possible high Scott complexities:* $\Pi_{\omega_1^{CK}}$, $\Pi_{\omega_1^{CK}+1}$, $\Sigma_{\omega_1^{CK}+1}$, d-$\Pi_{\omega_1^{CK}+1}$, *and* $\Pi_{\omega_1^{CK}+2}$.

The standard example of a structure of Scott rank $\omega_1^{CK}+1$, the Harrison linear order $\omega_1^{CK}\cdot(1+\mathbb{Q})$, has Scott complexity $\Pi_{\omega_1^{CK}+2}$. There are also known examples of computable structures with Scott complexity $\Pi_{\omega_1^{CK}}$ and $\Pi_{\omega_1^{CK}+1}$. This is due to the following fact which we prove in Section 3.

Proposition 1.9. *Let $\mathcal{A}$ be a computable structure of high Scott complexity with a $\Pi_{\omega_1^{CK}+1}$ Scott sentence (i.e., with Scott rank ω_1^{CK}). Then:*

- *If the computable infinitary theory of $\mathcal{A}$ is $\aleph_0$-categorical, then $\mathcal{A}$ has Scott complexity $\Pi_{\omega_1^{CK}}$.*
- *Otherwise, $\mathcal{A}$ has Scott complexity $\Pi_{\omega_1^{CK}+1}$.*

The standard examples of computable structures of Scott rank ω_1^{CK}, constructed by Knight and Millar strengthening a construction of Makkai [Mak81, KM10], were known to have an $\aleph_0$-categorical computable infinitary theory and hence have Scott complexity $\Pi_{\omega_1^{CK}}$. For some time the most important open question about structures of high Scott rank was whether there is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical—which, by the corollary above, is exactly the same as asking for a computable structure of Scott complexity $\Pi_{\omega_1^{CK}+1}$. Eventually Harrison-Trainor, Igusa, and Knight [HTIK18] provided such an example.

Another important open question about structures of high Scott rank, which we answer in this paper, is whether there is a structure of Scott rank $\omega_1^{CK}+1$ which becomes a structure of Scott rank ω_1^{CK} after naming finitely many constants. This is equivalent to asking whether there is a computable structure of Scott complexity $\Sigma_{\omega_1^{CK}+1}$ or d-$\Sigma_{\omega_1^{CK}+1}$. The equivalence follows from the following fact, which is less obvious than it seems:

Proposition 1.10. *Let $\mathcal{A}$ be a countable structure. Then:*

1. *$\mathcal{A}$ has a $\Sigma_{\alpha+1}$ Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence.*

2. *$\mathcal{A}$ has a d–Σ_α Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence and the automorphism orbit of $\bar{c}$ is Σ_α-definable.*

This theorem was first stated without proof by Montalbán [Mon15], but the proof was more subtle than it appeared at first; it was proved for Σ_3 sentences in [Mon17]. We give a proof here.

The Harrison linear order has Scott complexity $\Pi_{\omega_1^{CK}+2}$ because after naming finitely many constants it still has Scott rank $\omega_1^{CK}+1$. We construct new examples of computable structures of Scott complexity $\Sigma_{\omega_1^{CK}+1}$ and d–$\Sigma_{\omega_1^{CK}+1}$ in Section 4.

Theorem 1.11. *There is a computable structure of Scott rank $\omega_1^{CK}+1$ which, after naming finitely many constants, has Scott rank ω_1^{CK}.*

2 The Possible Complexities of Structures

2.1 Wadge Degrees

The Wadge degrees were introduced by Wadge in his PhD thesis [Wad83] to measure the topological complexity of subsets of Baire space ω^ω under continuous reductions.

Definition 2.1 (Wadge)**.** Let A and B be subsets of Baire space ω^ω. We say that A is *Wadge reducible* to B, and write $A \leq_W B$, if there is a continuous function f on ω^ω with $A = f^{-1}[B]$, i.e.

$$x \in A \iff f(x) \in B.$$

The equivalence classes under this pre-order are called the Wadge degrees; we write $[A]_W$ for the Wadge degree of A. The Wadge hierarchy is the set of Wadge degrees under continuous reductions.

With enough determinacy, the Wadge hierarchy is very well-behaved; it is well-founded and almost totally ordered (in the sense that any anti-chain has size at most two).

Theorem 2.2 (Martin and Monk, AD, see [VW78])**.** *The Wadge order is well-founded.*

Theorem 2.3 (Wadge's Lemma, AD, [Wad83])**.** *Given $A, B \subseteq \omega^\omega$, either $A \leq_W B$ or $B \leq_W \omega^\omega - A$.*

Since determinacy for Borel sets is provable in ZFC, this theorem holds in ZFC for such sets.

In general, for each of the pointclasses Γ from among $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Delta}^0_\alpha$, d–$\boldsymbol{\Sigma}^0_\alpha$, and other pointclasses arising from the Borel or difference hierarchies, if A is Wadge-reducible to a set in Γ, then A itself is in Γ; and moreover, there is a Γ-complete set. We denote by Γ the Wadge degree of a Γ-complete set. So, for example, $\boldsymbol{\Sigma}^0_1$ is the Wadge degree of open, but not clopen, sets.

2.2 The Lopez-Escobar Theorem

Fixing a language $\mathcal{L}$, we work in the Polish space $\mathrm{Mod}(\mathcal{L})$ of structures in the language $\mathcal{L}$. Given a structure $\mathcal{A}$, we can view the set $\mathrm{Iso}(\mathcal{A})$ of isomorphic copies of $\mathcal{A}$ as a subset of Baire space ω^ω. The syntactic form of a Scott sentence for $\mathcal{A}$ puts a topological restriction on $\mathrm{Iso}(\mathcal{A})$. For example, if $\mathcal{A}$ has a Σ_1 Scott sentence, then $\mathrm{Iso}(\mathcal{A})$ is an open ($\mathbf{\Sigma}^0_1$) subset of ω^ω. More generally, if $\mathcal{A}$ has a Σ_α (respectively Π_α) Scott sentence, then $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$) in the hierarchy of Borel sets. Since every structure has a Scott sentence, $\mathrm{Iso}(\mathcal{A})$ is always Borel. By Vaught's strengthening of the Lopez-Escobar [LE65] theorem, the correspondence of complexities also reverses.

Theorem 2.4 (Vaught [Vau75])**.** *Let $\mathbb{K}$ be a subclass of* $\mathrm{Mod}(\mathcal{L})$ *which is closed under isomorphism. Then $\mathbb{K}$ is $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$, $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\alpha$, $\neg\mathbf{d}\text{-}\mathbf{\Sigma}^0_\alpha$) in the Borel hierarchy if and only if $\mathbb{K}$ is axiomatized by an infinitary Σ_α (respectively, Π_α, $\mathrm{d}\text{-}\Sigma_\alpha$, $\neg\mathrm{d}\text{-}\Sigma_\alpha$) sentence.*

This theorem was later effectivized by Vanden Boom [VB07]. The inclusion of $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\alpha$ and $\neg\mathbf{d}\text{-}\mathbf{\Sigma}^0_\alpha$ in this theorem was not originally proved by Vaught, but it is well-known and not hard to show, e.g., by the forcing argument used by Vanden Boom. Moreover, one imagines that the theorem would extend to any reasonably defined class.

2.3 Possible Scott Complexities

We will also use the following fact which has a short proof due to Alvir.

Theorem 2.5 (A. Miller [Mil83])**.** *Let $\mathcal{A}$ be a countable structure. Then if $\mathcal{A}$ has both a Σ_α and a Π_α Scott sentence, it has a $\mathrm{d}\text{-}\Sigma_\beta$ Scott sentence for some $\beta < \alpha$.*

Proof. If α is a limit ordinal, then the theorem is trivial: one of the disjuncts of the Σ_α Scott sentence is true in $\mathcal{A}$, and this is a Σ_β Scott sentence for $\mathcal{A}$, $\beta < \alpha$. Otherwise, let $\exists\bar{x}\Phi(\bar{x})$ be a Σ_α Scott sentence for $\mathcal{A}$, with Φ being $\Pi_{\alpha-1}$. Let $\bar{a}$ be such that $\mathcal{A} \models \Phi(\bar{a})$. Since $\mathcal{A}$ has a Π_α Scott sentence, by Theorem 1.2 the automorphism orbit of $\bar{a}$ is definable by a $\Sigma_{\alpha-1}$ formula; call this formula $\psi_{\bar{a}}(\bar{x})$. Then $\exists\bar{x}\psi_{\bar{a}}(\bar{x}) \wedge \forall\bar{x}\,(\psi_{\bar{a}}(\bar{x}) \to \Phi(\bar{x}))$ is a $\mathrm{d}\text{-}\Sigma_{\alpha-1}$ Scott sentence for $\mathcal{A}$. □

We can now show:

Theorem 2.6. *The only possible Wadge degrees of* $\mathrm{Iso}(\mathcal{A})$ *for countable structures $\mathcal{A}$ are $\mathbf{\Pi}^0_\alpha$ (for any α), $\mathbf{\Sigma}^0_\alpha$ (for α a non-limit), and $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\alpha$ (for α a non-limit).*

Proof. We will not have to use the Axiom of Determinacy because for any structure $\mathcal{A}$, $\mathrm{Iso}(\mathcal{A})$ is Borel.

Let α be least such that $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{\Sigma}^0_\alpha$. First note that α cannot be a limit ordinal; if it were, then $\mathcal{A}$ would have a Σ_α Scott sentence, and one of those disjuncts would be a Σ_β Scott sentence for $\mathcal{A}$ for some $\beta < \alpha$. If the Wadge degree of $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{\Sigma}^0_\alpha$, then we are done. Otherwise, by Wadge's Lemma, $\mathbf{\Sigma}^0_\alpha \nleq_W [\mathrm{Iso}(\mathcal{A})]_W$ and so $[\mathrm{Iso}(\mathcal{A})]_W \leq_W \mathbf{\Pi}^0_\alpha$. Thus $\mathrm{Iso}(\mathcal{A})$ is both $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Sigma}^0_\alpha$; by Vaught's theorem, $\mathcal{A}$ has both a Σ_α and a Π_α Scott sentence. Then, by Theorem 2.5, $\mathcal{A}$ has a $\mathrm{d}\text{-}\Sigma_\beta$ Scott sentence for some $\beta < \alpha$.

So now, assume that $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\beta$ but not $\mathbf{\Sigma}^0_\beta$. First, if $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{\Pi}^0_\beta$, then we claim that $[\mathrm{Iso}(\mathcal{A})]_W =_W \mathbf{\Pi}^0_\beta$. Indeed, if $\mathbf{\Pi}^0_\beta \nleq_W [\mathrm{Iso}(\mathcal{A})]_W$, then by Wadge's Lemma, $[\mathrm{Iso}(\mathcal{A})]_W \leq_W \mathbf{\Sigma}^0_\beta$, and we know that this is not the case.

Finally, we are left with the case that $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\beta$ but neither $\mathbf{\Sigma}^0_\beta$ nor $\mathbf{\Pi}^0_\beta$. We can argue as before that β cannot be a limit ordinal (or $\mathrm{Iso}(\mathcal{A})$ would be $\mathbf{\Pi}^0_\beta$). We argue by contradiction that the Wadge degree of $\mathrm{Iso}(\mathcal{A})$ is $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\beta$. If $\mathbf{d}\text{-}\mathbf{\Sigma}^0_\beta \nleq_W [\mathrm{Iso}(\mathcal{A})]_W$ then by Wadge's Lemma $[\mathrm{Iso}(\mathcal{A})]_W \leq_W \neg\mathbf{d}\text{-}\mathbf{\Sigma}^0_\beta$ where $\neg\mathbf{d}\text{-}\mathbf{\Sigma}^0_\beta$ is the subsets of Baire space which are a union of a Σ^0_β set and a Π^0_β set. So $\mathrm{Iso}(\mathcal{A})$ is of this form, and by Vaught's Theorem, it has a Scott sentence which is a disjunction of a Σ_β and a Π_β sentence. But $\mathcal{A}$, being a single structure, must satisfy one of these two disjuncts, and that disjunct is by itself a Scott sentence for $\mathcal{A}$. So $\mathrm{Iso}(\mathcal{A})$ is either $\mathbf{\Sigma}^0_\beta$ or $\mathbf{\Pi}^0_\beta$, a contradiction. □

So—by the correspondence between the complexity of Scott sentences and Wadge degrees—the only possible Scott complexities are Σ_α, Π_α, and d-Σ_α. Note that this was all non-effective, and we do not know what happens in the lightface case.

3 Characterizations of structures of High Scott Complexity

In this section we give the proofs of Propositions 1.9 and 1.10. We will need the back-and-forth relations. We will use the symmetric back-and-forth relations: given $\bar{a} \in \mathcal{A}$ and $\bar{b} \in \mathcal{B}$, we define $(\mathcal{A}, \bar{a}) \equiv_0 (\mathcal{B}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ have the same atomic type, and $(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b})$ if for every $\bar{a}'$ and $\beta < \alpha$ there is $\bar{b}'$ such that $(\mathcal{A}, \bar{a}\bar{a}') \equiv_\alpha (\mathcal{B}, \bar{b}\bar{b}')$, and for every $\bar{b}'$ and $\beta < \alpha$ there is $\bar{a}'$ such that $(\mathcal{A}, \bar{a}\bar{a}') \equiv_\alpha (\mathcal{B}, \bar{b}\bar{b}')$.

We also define $\mathcal{A} \leq_\alpha \mathcal{B}$ if every Π_α sentence true of $\mathcal{A}$ is true of $\mathcal{B}$. We have that $\mathcal{A} \equiv_\alpha \mathcal{B}$ implies $\mathcal{A} \leq_\alpha \mathcal{B}$ and $\mathcal{B} \leq_\alpha \mathcal{A}$, but not vice versa. Note that this is different from being an α-elementary substructure, which is sometimes denoted using the same notation; to have $\mathcal{A} \leq_\alpha \mathcal{B}$ does *not* require that $\mathcal{A}$ be a substructure of $\mathcal{B}$.

Proposition 3.1. *Let $\mathcal{A}$ be a countable structure and α a countable limit ordinal. The following are equivalent:*

1. *$\mathcal{A}$ has a Π_α Scott sentence.*
2. *Whenever $\mathcal{A} \leq_\alpha \mathcal{B}$ for a countable structure $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$.*

Proof. For $1 \Rightarrow 2$, if $\mathcal{A} \leq_\alpha \mathcal{B}$ then every Π_α sentence true of $\mathcal{A}$ (including the Scott sentence for $\mathcal{A}$) is true of $\mathcal{B}$; thus $\mathcal{B} \cong \mathcal{A}$.

For $2 \Rightarrow 1$ we use the standard technique for showing the back and forth relations are definable, proceeding by induction on β. For each $\bar{a} \in A$, let $\varphi^1_{\bar{a}}(\bar{x})$ be the conjunction of all Π_1 and Σ_1 formulas which are true of $\bar{a}$ in A. For $\beta > 1$ define

$$\varphi^\beta_{\bar{a}}(\bar{x}) = \left(\bigwedge_{0<\gamma<\beta} \forall \bar{y} \bigvee_{\bar{a}' \in \mathcal{A}} \varphi^\gamma_{\bar{a}\bar{a}'}(\bar{x}\bar{y}) \right) \wedge \left(\bigwedge_{0<\gamma<\beta} \bigwedge_{\bar{a}' \in \mathcal{A}} \exists \bar{y}\ \varphi^\gamma_{\bar{a}\bar{a}'}(\bar{x}\bar{y}) \right).$$

Note that we may assume $|\bar{a}| = |\bar{x}|$. By induction, we can show that

$$(\mathcal{B}, \bar{b}) \models \varphi_{\bar{a}}^{\beta} \iff (\mathcal{A}, \bar{a}) \equiv_{\beta} (\mathcal{B}, \bar{b}).$$

It also follows by induction on $\beta \geq 1$ that $\varphi_{\bar{a}}^{\beta}$ is a $\Pi_{2\cdot\beta}$ formula.

We claim that $\Phi = \bigwedge\!\!\!\bigwedge_{0<\beta<\alpha} \varphi_{\varnothing}^{\beta}$ is a Π_α Scott sentence for $\mathcal{A}$. It is Π_α because $2\cdot\beta < \alpha$ when $0 < \beta < \alpha$ and α is a limit ordinal. If $\mathcal{B} \models \Phi$, then $\mathcal{A} \equiv_\beta \mathcal{B}$ for all $0 < \beta < \alpha$, and so $\mathcal{A} \leq_\alpha \mathcal{B}$ since α is a limit ordinal; hence $\mathcal{B} \cong \mathcal{A}$. □

Proposition 1.9. *Let $\mathcal{A}$ be a computable structure of high Scott complexity with a $\Pi_{\omega_1^{CK}+1}$ Scott sentence (i.e., with Scott rank ω_1^{CK}). Then:*

- *If the computable infinitary theory of $\mathcal{A}$ is $\aleph_0$-categorical, then $\mathcal{A}$ has Scott complexity $\Pi_{\omega_1^{CK}}$.*
- *Otherwise, $\mathcal{A}$ has Scott complexity $\Pi_{\omega_1^{CK}+1}$.*

Proof. Since $\mathcal{A}$ has high Scott complexity and has a $\Pi_{\omega_1^{CK}+1}$ Scott sentence, the only possible Scott complexities are $\Pi_{\omega_1^{CK}}$ and $\Pi_{\omega_1^{CK}+1}$. So it suffices to show that $\mathcal{A}$ has a $\Pi_{\omega_1^{CK}}$ Scott sentence if and only if the computable infinitary theory of $\mathcal{A}$ is $\aleph_0$-categorical.

If the computable infinitary theory of $\mathcal{A}$ is $\aleph_0$-categorical, then the conjunction of these sentences is a $\Pi_{\omega_1^{CK}}$ Scott sentence for $\mathcal{A}$. On the other hand, suppose that $\mathcal{A}$ has a $\Pi_{\omega_1^{CK}}$ Scott sentence. Then by Proposition 3.1, whenever $\mathcal{A} \leq_{\omega_1^{CK}} \mathcal{B}$ for a countable structure $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$. In the proof of Proposition 3.1 we constructed for each $\alpha < \omega_1^{CK}$ a computable $\Pi_{2\cdot\alpha}$ sentence φ_α such that

$$\mathcal{B} \models \varphi_\alpha \iff \mathcal{A} \equiv_\alpha \mathcal{B}.$$

Thus if $\mathcal{B}$ satisfies the computable infinitary theory of $\mathcal{A}$, then for all $\alpha < \omega_1^{CK}$ we have $\mathcal{A} \equiv_\alpha \mathcal{B}$, hence $\mathcal{A} \leq_{\omega_1^{CK}} \mathcal{B}$, and so $\mathcal{A} \cong \mathcal{B}$. □

Part (1) of the next theorem was first stated by Montálban for successor ordinals in [Mon15]. The original proof proceeds by observing that the following are equivalent:

1. A is Δ_α^0-categorical on a cone.
2. A has a $\Sigma_{\alpha+2}$ Scott sentence.
3. There is a tuple $\bar{c}$ such that $(A, \bar{c})$ has a $\Pi_{\alpha+1}$ Scott sentence.

That (1) and (3) are equivalent is obtained by considering Theorem 10.14 of [AK00] on a cone. Given this equivalence, the proof of (1) ⇒ (2) is immediate. After noticing that the proof of (2) ⇒ (1) was not trivial, Montalbán gave a proof of the fact (for $\alpha = 1$) in [Mon17] via Henkin construction. There is not yet a published proof in the literature for $\alpha > 1$, so we give a proof here; our proof uses Theorem 1.2.

Lemma 3.2. *Let $\mathcal{A}$ be a structure. If the automorphism orbit of every tuple in $\mathcal{A}$ is definable by a Σ_α formula, then each such orbit is definable by a Π_α formula.*

Proof. Fix, for each $\bar{a} \in \mathcal{A}$, a Σ_α definition $\varphi_{\bar{a}}$ for $\bar{a}$. Given a tuple $\bar{c}$, the Π_α formula

$$\bigwedge_{|\bar{a}|=|\bar{c}|,\ \bar{a}\not\equiv\bar{c}} \neg\varphi_{\bar{a}}(\bar{x})$$

defines the orbit of $\bar{c}$. ☐

Proposition 1.10. *Let $\mathcal{A}$ be a countable structure. Then:*

1. *$\mathcal{A}$ has a $\Sigma_{\alpha+1}$ Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence.*
2. *$\mathcal{A}$ has a d$-\Sigma_\alpha$ Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence and the automorphism orbit of $\bar{c}$ in $\mathcal{A}$ is Σ_α-definable.*

Proof. (1) The right-to-left direction is easy. For the left-to-right direction, suppose that $\mathcal{A}$ has a $\Sigma_{\alpha+1}$ Scott sentence $\exists\bar{x}\varphi(\bar{x})$, with φ being Π_α. Let $\bar{c}$ be such that $\mathcal{A} \models \varphi(\bar{c})$. Note that the Scott sentence for $\mathcal{A}$ is $\Pi_{\alpha+2}$ and so by Theorem 1.2 each automorphism orbit is $\Sigma_{\alpha+1}$-definable. Let $\exists\bar{y}\psi(\bar{x},\bar{y})$ be a $\Sigma_{\alpha+1}$ formula defining the orbit of $\bar{c}$, with ψ being Π_α. Let $\bar{d}$ be such that $\mathcal{A} \models \psi(\bar{c},\bar{d})$. By Lemma 3.2 the automorphism orbit of $\bar{c}$ is also $\Pi_{\alpha+1}$-definable, say by γ; so $(\mathcal{A},\bar{c})$ has a $\Pi_{\alpha+1}$ Scott sentence $\varphi(\bar{c}) \wedge \gamma(\bar{c})$. Thus by Theorem 1.2 and Lemma 3.2, the orbit of $\bar{d}$ over $\bar{c}$ is Π_α- definable, say by $\theta(\bar{c},\bar{y})$. Then $\varphi(\bar{c}) \wedge \psi(\bar{c},\bar{d}) \wedge \theta(\bar{c},\bar{d})$ is a Π_α Scott sentence for $(\mathcal{A}, \bar{c}\bar{d})$.

(2) We may assume that α is not a limit ordinal. For the left-to-right direction, suppose that $\mathcal{A}$ has a d$-\Sigma_\alpha$ Scott sentence $\exists\bar{x}\varphi(\bar{x}) \wedge \gamma$ with φ being Π_β ($\beta < \alpha$) and γ being Π_α. Let $\bar{c}$ be such that $\mathcal{A} \models \varphi(\bar{c})$. Since $\mathcal{A}$ also has a $\Pi_{\alpha+1}$ Scott sentence, every automorphism orbit is Σ_α-definable; by Lemma 3.2, every automorphism orbit is also Π_α-definable. Let $\psi(\bar{x})$ be a Π_α formula defining the orbit $\bar{c}$ in $\mathcal{A}$. Then $\gamma \wedge \varphi(\bar{c}) \wedge \psi(\bar{c})$ is a Π_α Scott sentence for $(\mathcal{A}, \bar{c})$.

For the right-to-left direction, suppose that $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence $\varphi(\bar{c})$ and the automorphism orbit of $\bar{c}$ is Σ_α-definable by a formula $\psi(\bar{x})$. Then $\mathcal{A}$ has a d$-\Sigma_\alpha$ Scott sentence $\exists\bar{x}\psi(\bar{x}) \wedge \forall\bar{x}(\psi(\bar{x}) \to \varphi(\bar{x}))$. ☐

4 Structures of Scott Complexity $\Sigma_{\lambda+1}$

In this section, we will show that for λ a limit ordinal, there are structures of Scott complexity $\Sigma_{\lambda+1}$; this resolves an open question from [Mil83].

Theorem 4.1. *For every limit ordinal λ, there is a structure of Scott complexity $\Sigma_{\lambda+1}$.*

We will also show that there are computable structures of Scott complexity $\Sigma_{\omega_1^{CK}+1}$; these are structures of Scott rank $\omega_1^{CK}+1$ which, after naming finitely many constants, have Scott rank ω_1^{CK}. This answers another open question.

Theorem 4.2. *For every limit ordinal $\lambda \leq \omega_1^{CK}$, there is a computable structure of Scott complexity $\Sigma_{\lambda+1}$.*

From these, taking disjoint unions with known structures we can easily get structures with Scott complexity d$-\Sigma_{\lambda+1}$.

Corollary 4.3. *For every limit ordinal $\lambda \leq \omega_1^{CK}$, there is a computable structure of Scott complexity* d-$\Sigma_{\lambda+1}$.

Proof. It is known that there exists a computable structure $\mathcal{C}$ of Scott complexity $\Pi_{\lambda+1}$ ([Mil83] for the case $\lambda < \omega_1^{CK}$, and [HTIK18] for the case $\lambda = \omega_1^{CK}$). Let $\mathcal{A}$ be a computable structure of Scott complexity $\Sigma_{\lambda+1}$. Then $\mathcal{A} \sqcup \mathcal{C}$, where we add a unary relation to the language to distinguish $\mathcal{A}$ and $\mathcal{C}$, has Scott complexity d-$\Sigma_{\lambda+1}$. □

Corollary 4.4. *For every limit ordinal λ, there is a structure of Scott complexity* d-$\Sigma_{\lambda+1}$.

We prove Theorems 4.1 and 4.2 using a construction which takes as input a sequence of trees and produces a structure whose Scott complexity depends on the input trees. By feeding different sequences of trees into this construction, we get the two theorems.

In general in computable structure theory, ω_1^{CK} behaves quite differently from computable limit ordinals, and so for Theorem 4.2 one would expect the construction of such a structure to proceed differently based on whether $\lambda < \omega_1^{CK}$ or $\lambda = \omega_1^{CK}$. On the other hand, the difference between Theorem 4.1 and the case $\lambda < \omega_1^{CK}$ of Theorem 4.2 is just the difference between effectiveness and non-effectiveness. We are able to isolate these differences to the proof of the following lemma.

Lemma 4.5. *Given a limit ordinal λ, there are a pair of sequences of trees $(S_i)_{i\in\omega}$ and $(T_i)_{i\in\omega}$ such that the following all hold:*

1. *Each S_i and T_i has Scott complexity at most Π_λ;*

2. *If $S_i \not\cong T_i$, then there is a $\beta < \lambda$ such that $S_i \not\equiv_\beta T_i$; and*

3. *For each $\beta < \lambda$, there is an i with $S_i \not\cong T_i$ but $S_i \equiv_\beta T_i$.*

If $\lambda \leq \omega_1^{CK}$, then we can take these sequences to be uniformly computable.

Proof. For $\lambda < \omega_1^{CK}$, there are many examples appearing in the literature. For instance, we can fix a computable increasing sequence of successor ordinals $(\alpha_i)_{i\in\omega}$ which converge to λ and let $S_i = \mathcal{A}_{\alpha_i}$ and $T_i = \mathcal{E}_{\alpha_i}$, where these are the trees defined by Hirschfeldt and White [HW02]. If we do not care about effectiveness (e.g., if $\lambda > \omega_1^{CK}$), we can take these same trees, relatived to some oracle which makes λ computable. We explain how to verify the properties (1), (2), and (3):

1. Each $\mathcal{A}_\alpha$ and $\mathcal{E}_\alpha$, $\alpha < \lambda$, is a tree of rank $< \lambda$. One can argue inductively by rank that a tree of rank γ has a Scott sentence of complexity $\Pi_{2\cdot\gamma}$.

2. This is Lemma 3.5 of [HW02].

3. It suffices to see that $\mathcal{A}_\alpha \not\cong \mathcal{E}_\alpha$, but $\mathcal{A}_\alpha \equiv_{\alpha-1} \mathcal{E}_\alpha$. The (relativization of) Proposition 3.2 of [HW02] implies that $\mathcal{A}_\alpha \equiv_{\alpha-1} \mathcal{E}_\alpha$, as if $\mathcal{A}_\alpha$ and $\mathcal{E}_\alpha$ disagreed on an X-computable $\Sigma_{\alpha-1}$ formula, then for $\mathcal{P}(n)$ a $\Sigma_\alpha(X)$-complete predicate, we could not produce the sequence $\mathcal{T}_n$ as in the proposition.

For $\lambda = \omega_1^{CK}$, Harrison-Trainer, Igusa and Knight [HTIK18] constructed appropriate trees. Fixing a presentation $\mathcal{H}$ of the Harrison order, they constructed trees $(T_a)_{a\in\mathcal{H}}$ such that if a is from the ill-founded part of $\mathcal{H}$, then $T_a \cong T^*$, where T^* is a fixed tree of Scott complexity $\Pi_{\omega_1^{CK}}$, and if a is from the well-founded part of $\mathcal{H}$ and the part of $\mathcal{H}$ to the left of a has order-type α, then T_a is well-founded (so $T_a \not\equiv_\beta T^*$ for some $\beta < \omega_1^{CK}$) but $T_a \equiv_\alpha T^*$. Then we let $(a_i)_{i\in\omega}$ be an enumeration of $\mathcal{H}$ and set $T_i = T_{a_i}$ and $S_i = T^*$. To verify the properties:

1. T^* has Scott complexity $\Pi_{\omega_1^{CK}}$ because its computable infinitary theory is $\aleph_0$-categorical. If a is from the well-founded part of $\mathcal{H}$ and the part of $\mathcal{H}$ to the left of a has order-type α, then T_a has tree rank at most $\omega\cdot(\alpha+1)$, and hence has a $\Pi_{2\cdot(\omega\cdot(\alpha+1))}$ Scott sentence. See the first paragraph after Theorem 2.1 of [HTIK18], as well as [CKM06].

2. If $S_i \not\cong T_i$, then this means that $T_i = T_{a_i} \not\cong T^*$ and a_i is in the well-founded part of $\mathcal{H}$. So T_{a_i} is well-founded, and hence has tree rank $< \omega_1^{CK}$. Thus there is $\gamma < \omega_1^{CK}$ such that $T_{a_i} \not\equiv_\gamma T^*$.

3. Given $\beta < \lambda$, let a_i be an element of the well-founded part of $\mathcal{H}$ such that the predecessors of a_i in $\mathcal{H}$ have order type greater than β. Then $S_i = T_{a_i} \not\cong T^* = T_i$, but $S_i = T_{a_i} \equiv_\beta T^* = T_i$. □

Then the common construction is contained in the following theorem:

Theorem 4.6. *Given a limit ordinal λ and a pair of sequences of trees $(S_i)_{i\in\omega}$ and $(T_i)_{i\in\omega}$ as in the previous lemma, there is a structure of Scott complexity $\Sigma_{\lambda+1}$. If the sequences are uniformly computable, then the structure is computable.*

Proof. The language for our structure $\mathcal{A}$ will have binary relations P, R and E_i for $i \in \omega$. The universe of our structure will be

$$[\omega]^{<\omega} \sqcup \bigsqcup_{i\in\omega} S_i \sqcup \bigsqcup_{i\in\omega} T_i.$$

Thus we have disjoint copies of each of the S_i and T_i, and also we have every finite subset of ω.

On each of the T_i and S_i, P is the tree relation. P has no other structure; that is, it does not hold for any pair (x,y) not drawn from the same tree.

Each of the E_i is defined on $[\omega]^{<\omega}$ by $E_i(F,G) \leftrightharpoons F \triangle G = \{i\}$. The E_i have no other structure; that is, they do not hold for any pair (x,y) not both drawn from $[\omega]^{<\omega}$.

We can understand the structure on $[\omega]^{<\omega}$ as an affine space acted on by $\bigoplus_{i\in\omega}\mathbb{Z}/2$, where the action is $F + e_i = F \triangle \{i\}$. By an affine space, we mean a vector space except that we forget the origin. Then $E_i(F,G) \iff F + e_i = G$. Alternatively, we can think of the structure as the vertices of an infinite dimensional cube, where E_i is the edge relation in the "i direction".

Finally, we define $R(x,y)$ to hold if and only if one of the following is true:

- $x \in [\omega]^{<\omega}$, $y \in S_i$, and $i \notin x$; or
- $x \in [\omega]^{<\omega}$, $y \in T_i$, and $i \in x$.

The affine space $[\omega]^{<\omega}$ is partitioned into two hyperplanes perpendicular to e_i: the first is $\{F \in [\omega]^{<\omega} : i \notin F\}$, and the second is $\{F \in [\omega]^{<\omega} : i \in F\}$. Instead considering the infinite dimensional cube, these are the two connected components that result if we delete all the E_i edges. One of these sets is associated, via R, with S_i, and the other with T_i.

Claim 6.1. Let $\mathcal{B}$ be the substructure $\mathcal{A} \restriction_{[\omega]^{<\omega}}$. The automorphisms of B are precisely the maps of the form $g(F) = F \triangle H$ for some fixed $H \in [\omega]^{<\omega}$.

Proof. To see that such a map is an automorphism, observe that

$$\begin{aligned} E_i(F,G) &\iff F \triangle G = \{i\} \\ &\iff F \triangle G \triangle \varnothing = \{i\} \\ &\iff F \triangle G \triangle (H \triangle H) = \{i\} \\ &\iff (F \triangle H) \triangle (G \triangle H) = \{i\} \\ &\iff E_i(g(F), g(G)). \end{aligned}$$

The relations R and P are empty on $\mathcal{B}$, and so this suffices.

Conversely, suppose g is an automorphism of $\mathcal{B}$. Let $H = g(\varnothing)$. We prove by induction on $|F|$ that $g(F) = F \triangle H$. The case $|F| = 0$ is immediate. For $|F| > 0$, fix $i \in F$, and let $G = F - \{i\}$. Then $E_i(F,G)$, so $E_i(g(F), g(G))$, and thus

$$\begin{aligned} g(F) &= g(G) \triangle \{i\} \\ &= (G \triangle H) \triangle \{i\} \\ &= (G \triangle \{i\}) \triangle H \\ &= F \triangle H. \end{aligned}$$

□

Claim 6.2. For every $\beta < \lambda$ there is an $H \in [\omega]^{<\omega}$ such that $(\mathcal{A}, \varnothing) \equiv_\beta (\mathcal{A}, H)$, but $(\mathcal{A}, \varnothing) \not\cong (\mathcal{A}, H)$.

Proof. Fix an i such that $S_i \not\cong T_i$ but $S_i \equiv_{2\beta} T_i$, and let $H = \{i\}$. (Recall that since λ is a limit, if $\beta < \lambda$, then $2\beta < \lambda$.)

The elements of $[\omega]^{<\omega}$ are definable in $\mathcal{A}$ by $\exists y\, E_0(x,y)$, and so any isomorphism $g : (\mathcal{A}, \varnothing) \cong (\mathcal{A}, H)$ must restrict to an isomorphism $g : (\mathcal{B}, \varnothing) \cong (\mathcal{B}, H)$. Thus $g(F) = F \triangle \{i\}$ on $\mathcal{B}$, by the previous claim. The tree S_i is associated via R with all the elements of $\{F \in [\omega]^{<\omega} : i \notin F\}$, and it is the only tree associated with all of these elements. Similarly, the tree T_i is the only tree associated with all the elements of $\{F \in [\omega]^{<\omega} : i \in F\}$. As g interchanges these two sets, g must map S_i to T_i. But $S_i \not\cong T_i$, a contradiction. Thus $(\mathcal{A}, \varnothing) \not\cong (\mathcal{A}, H)$.

Claim 6.2.1. Fix $\alpha \le \beta$. Suppose $\bar{F}, \bar{G} \in [\omega]^{<\omega}$, $\bar{x}, \bar{z} \in S_i$, $\bar{y}, \bar{w} \in T_i$, $\bar{q} \in \mathcal{A} - [\omega]^{<\omega} - S_i - T_i$ are such that:

- $|\bar{F}| = |\bar{G}|$ and for all $j < |\bar{F}|$, $F_j \triangle G_j = \{i\}$; and
- $|\bar{x}| = |\bar{y}|$, $|\bar{z}| = |\bar{w}|$ and $(S_i, \bar{x}, \bar{z}) \equiv_{2\alpha} (T_i, \bar{y}, \bar{w})$.

Then $(\mathcal{A}, \bar{F}, \bar{x}, \bar{w}, \bar{q}) \equiv_\alpha (\mathcal{A}, \bar{G}, \bar{y}, \bar{z}, \bar{q})$.

Proof. We argue by induction on α. The case $\alpha = 0$ is simply a matter of checking that R is preserved.

For $\alpha > 0$, without loss of generality we must argue that for any $\gamma < \alpha$ and any finite extension of $(\bar{F}, \bar{x}, \bar{w}, \bar{q})$, there is a corresponding extension of $(\bar{G}, \bar{y}, \bar{z}, \bar{q})$ which is γ-equivalent in $\mathcal{A}$. Fixing a finite extension of $(\bar{F}, \bar{x}, \bar{w}, \bar{q})$, we partition this extension into $(\bar{F}', \bar{x}', \bar{w}', \bar{q}')$, where $\bar{F}' \in [\omega]^{<\omega}$, and similarly for the other entries.

Define $\bar{G}'$ by $G_j = F_j' \triangle \{i\}$ for all $j < |\bar{F}'|$. As $2\alpha > 2\gamma+1$, there is $\bar{y}' \supseteq \bar{y}$ with $(S_i, \bar{x}', \bar{z}) \equiv_{2\gamma+1} (T_i, \bar{y}', \bar{w})$. So there is $\bar{z}' \supseteq \bar{z}$ with $(S_i, \bar{x}', \bar{z}') \equiv_{2\gamma} (T_i, \bar{y}', \bar{w}')$.

By the inductive hypothesis, $(\mathcal{A}, \bar{F}', \bar{x}', \bar{w}', \bar{q}') \equiv_{\gamma} (\mathcal{A}, \bar{G}', \bar{y}', \bar{z}', \bar{q}')$. □

It follows that $(\mathcal{A}, \varnothing) \equiv_{\beta} (\mathcal{A}, H)$. □

So the automorphism orbit of $\varnothing \in \mathcal{A}$ is not definable by a Σ_α formula for any $\alpha < \lambda$. Then by Theorem 1.2, $\mathcal{A}$ does not have a $\Pi_{\lambda+1}$ Scott sentence, and so the Scott complexity of $\mathcal{A}$ is at least $\Sigma_{\lambda+1}$. To show that it is precisely $\Sigma_{\lambda+1}$, it suffices to show that $(\mathcal{A}, \varnothing)$ has Scott complexity Π_λ.

Observe that $(\mathcal{B}, \varnothing)$ is rigid, because any automorphism must be an automorphism of $\mathcal{B}$ that fixes $\varnothing$, which by our characterization of the automorphisms of $\mathcal{B}$ must be the identity. In fact, every element of $(\mathcal{B}, \varnothing)$ is Σ_1 definable: $F = \{i_0, i_1, \ldots, i_{k-1}\}$ is the unique element z of $(\mathcal{A}, \varnothing)$ satisfying

$$\exists x_0, \ldots, x_k \left[x_0 = \varnothing \wedge x_k = z \wedge \bigwedge_{j<k} E_{i_j}(x_j, x_{j+1})\right].$$

This is the key fact that drops the Scott complexity of $(\mathcal{A}, \varnothing)$. It follows that each S_i and T_i is Π_2 definable in $(\mathcal{A}, \varnothing)$:

$$S_i = \left\{ y : \bigwedge_{i \notin F} \exists x\,(x = F) \wedge R(x, y) \right\},$$

where "$x = F$" represents the appropriate Σ_1 formula given above. T_i is similar.

By assumption, each S_i and T_i has a Π_λ Scott sentence ϕ_i and ψ_i, respectively. We will construct a Π_λ Scott sentence for $(\mathcal{A}, \varnothing)$. The idea is that our ability to distinguish the elements of $\mathcal{B}$ and the various S_i and T_i lets us give a complete description of the structure on $\mathcal{B}$ and the relation R. We then use the ϕ_i and ψ_i to give descriptions of the S_i and T_i.

More precisely, let ϕ_i' be the sentence made from ϕ_i by restricting the quantifiers to S_i. That is, each instance of $\forall x\, \theta(x)$ becomes $\forall x\, [x \in S_i \implies \theta(x)]$, and each instance of $\exists x\, \theta(x)$ becomes $\exists x\, [x \in S_i \wedge \theta(x)]$, where "$x \in S_i$" represents the Π_2 formula given above. Similarly, ψ_i' is made from ψ_i by restricting the quantifiers to T_i. By an inductive argument on subformulas, ϕ_i' and ψ_i' are both Π_λ.

We are now ready to give a Π_λ Scott sentence for $(\mathcal{A},\varnothing)$:

$$\begin{aligned}
&\forall x \left(\bigvee\!\!\!\bigvee_{F\in[\omega]^{<\omega}} x = F \vee \bigvee\!\!\!\bigvee_{i\in\omega} x \in S_i \vee \bigvee\!\!\!\bigvee_{i\in\omega} x \in T_i \right)\\
&\wedge\neg\exists x \bigvee\!\!\!\bigvee_{i<\omega} \bigvee\!\!\!\bigvee_{F\in[\omega]^{<\omega}} x = F \wedge (x \in S_i \vee x \in T_i)\\
&\wedge\neg\exists x \bigvee\!\!\!\bigvee_{i<\omega} x \in S_i \wedge x \in T_i\\
&\wedge\neg\exists x \bigvee\!\!\!\bigvee_{i\neq j} (x \in S_i \vee x \in T_i) \wedge (x \in S_j \vee x \in T_j)\\
&\wedge\neg\exists x \bigvee\!\!\!\bigvee_{F\neq G} x = F \wedge x = G\\
&\wedge\forall x,y \bigwedge\!\!\!\bigwedge_{i<\omega} \left(E_i(x,y) \iff \bigvee\!\!\!\bigvee_{F\triangle G=\{i\}} x = F \wedge y = G \right)\\
&\wedge\forall x,y\, R(x,y) \implies \left(\bigvee\!\!\!\bigvee_{i<\omega} \bigvee\!\!\!\bigvee_{i\in F} x = F \wedge y \in S_i \right) \vee \left(\bigvee\!\!\!\bigvee_{i<\omega} \bigvee\!\!\!\bigvee_{i\notin F} x = F \wedge y \in T_i \right)\\
&\wedge \bigwedge\!\!\!\bigwedge_{i<\omega} \phi_i' \wedge \bigwedge\!\!\!\bigwedge_{i<\omega} \psi_i'
\end{aligned}$$

The first four lines partition the structure into $\mathcal{B}$, the S_i and the T_i. The fifth and sixth lines state that the various E_i are defined correctly on $\mathcal{B}$, and further that none of the E_i hold with any elements outside of $\mathcal{B}$. The seventh line states that R is defined correctly. The final line determines the isomorphism types of the S_i and T_i (it is here that P is defined). Note that apart from the final line, this sentence is Π_4. Since the ϕ_i' and ψ_i' are Π_λ, the entire sentence is a Π_λ Scott sentence for $(\mathcal{A},\varnothing)$, as desired. □

5 Scott complexity Σ_2 is impossible

Theorem 5.1. *There is no structure with Scott complexity Σ_2.*

Proof. Suppose that $\mathcal{A}$ is a structure with a Σ_2 Scott sentence. We may assume that the Scott sentence is of the form

$$\exists x_1,\ldots,x_n \varphi(\bar{x})$$

where $\varphi(\bar{x})$ is Π_1. Let $\bar{a} \in \mathcal{A}$ be such that $\mathcal{A} \models \varphi(\bar{a})$. Let $\mathcal{A}^*$ be the substructure of $\mathcal{A}$ generated by $\bar{a}$. Then since φ is Π_1, $\mathcal{A}^* \models \varphi(\bar{a})$. Thus $\mathcal{A}^* \cong \mathcal{A}$. So we may assume that $\bar{a}$ generates $\mathcal{A}$.

We repeat here Lemma 1.1 of [Mil83]:

Claim. $\mathcal{A}$ is saturated.

Proof. If $\mathcal{B}$ is any elementary extension of $\mathcal{A}$, then $\mathcal{B} \models \varphi(\bar{a})$ and so $\mathcal{B}$ is isomorphic to $\mathcal{A}$. So every type (with no parameters) in $Th(\mathcal{A})$ is realized in $\mathcal{A}$, i.e., $\mathcal{A}$ is weakly saturated. In particular, there are countably many types in $Th(\mathcal{A})$, and so $Th(\mathcal{A})$ has a countable saturated model $\mathcal{B}$. But then $\mathcal{A}$ elementarily embeds into $\mathcal{B}$, and so $\mathcal{B}$ is isomorphic to $\mathcal{A}$. Thus $\mathcal{A}$ is saturated. □

If $\mathcal{A}$ were infinite, then the type of an element which is not generated by $\bar{a}$ is consistent. But $\mathcal{A}$ is saturated, and so this type must be realized in $\mathcal{A}$. This cannot happen as $\mathcal{A}$ is generated by $\bar{a}$. So $\mathcal{A}$ is finite, say with n elements. List out these elements as $a_1, \ldots, a_n$. Let $\langle \psi_m(a_1, \ldots, a_n) \rangle$ be a list of the quantifier-free formulas true of $a_1, \ldots, a_n$. (Note that among these formulas are those saying that $a_1, \ldots, a_n$ are distinct.)

Then $\mathcal{A}$ has a d-Σ_1 Scott sentence: $\mathcal{A}$ is axiomatized by saying that there exists n elements, there are not more than n elements, and also for each m,

$$\forall x_1, \ldots, x_n \left[\bigwedge_{i \neq j} x_i \neq x_j \longrightarrow \bigvee_{\sigma \in S(n)} \bigwedge_{i \leq m} \psi_i(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \right]$$

where $S(n)$ is the set of permutations of $\{1, \ldots, n\}$. (Since $S(n)$ is finite, these conjunctions and disjunctions are all finite and hence do not increase the complexity.) Suppose that $\mathcal{B}$ is a model of these sentences; then $\mathcal{B}$ has exactly n elements $b_1, \ldots, b_n$. Since there are only finitely many permutations in $S(n)$, there must be some permutation σ such that for arbitrarily large m,

$$\mathcal{B} \models \bigwedge_{i \leq m} \psi_i(b_{\sigma(1)}, \ldots, b_{\sigma(n)}).$$

Then $a_i \mapsto b_{\sigma(i)}$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. □

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