A REPRESENTATION THEOREM FOR POSSIBILITY MODELS

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Abstract. Possibility models are models for modal logic that generalize the standard Kripke models by allowing partial possibilities in addition to total worlds. A heuristic sometimes used to motivate possibility models takes each possibility to be a set of total worlds. Taking inspiration from this, we prove a representation theorem for possibility models: for every countable possibility model satisfying natural conditions, there is a Kripke model with respect to which we can identify the possibilities as certain sets of total worlds. More formally, we introduce a notion of a possibilization of a Kripke model, and we show that every separative and strong countable possibility model is isomorphic to a possibilization of a Kripke model.

1. Introduction

Humberstone [Hum81] introduced *possibility models* as an alternative semantics for propositional modal logic. In a possibility model, the worlds of a Kripke model are replaced by partial possibilities. A possibility determines some parts or aspects of a world; as Edgington [Edg85] explains (see also Chapter 10 of [Hal13] and Chapter 6 of [Rum15]):

> Possibilities differ from possible worlds in leaving many details unspecified... I am counting the possibility that the die lands six-up as one possibility. There are indefinitely many possible worlds compatible with this one possibility which vary not only in the precise location and orientation of the landed die, but also as to whether it is raining in China at the time, or at any other time, and so on ad infinitum . . . . (564)

While a world in a Kripke model determines the truth or falsity of every sentence in the language under consideration, a possibility might determine the truth of some sentences, the falsity of others, and leave the truth value of further sentences undetermined. One possibility might *refine* another. If a possibility Y refines a possibility X, then any propositional variable true at X remains true at Y and any propositional variable false at X remains false at Y; however, of the remaining propositional variables, some may become true at Y, some may become false, and others may remain undecided. There is also a modal accessibility relation between possibilities.

An extensive theory of possibility models was developed by Holliday in [Hol15]. There, adapting an example of Litak [Lit05], it was shown that at the level of *frames*, possibility semantics is more general than Kripke semantics: there are possibility frames as in Definition 2.1 below whose logic is a normal modal logic that is not the

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logic of any class of Kripke frames. For other recent work on possibility semantics, see [Gar13], [Hol14], [BBH17], [Yam17], and [HT16] (and the related [BH16]).

A heuristic sometimes used to motivate possibility models takes each possibility to be a set of total worlds. Sometimes, as in [Cre04], possibilities are even defined as sets of total worlds; the possibility that the die lands six-up can be identified or at least associated with the set of total worlds in which the die lands six-up. There are objections to this on philosophical grounds—Edgington [Edg85, p. 564] asserts that when one thinks of a possibility, one is not thinking of a single possible world or even a set of possible worlds, but rather some other type of object—but the idea of possibilities as sets-of-total-worlds can be thought of as a simplified motivating example of possibility models. Given a Kripke model, we can form a possibility model by taking, as the possibilities, every (non-empty) set of total worlds. We call this the *powerset possibilization* (see Definition 3.1). We can also view a Kripke model as a possibility model, taking the possibilities to be the singleton sets of total worlds (which we identify with their elements). In both of these cases, there is a key connection between the possibility model and the Kripke model: a sentence φ is true (in the possibility model) at a possibility if and only if it is true (in the Kripke model) at every world contained in that possibility. Of course, there are many other ways to generate a possibility model from a Kripke model, taking as possibilities some, but not all, sets of total worlds. We call such a possibility model a *possibilization* of the Kripke model, and we require that, as before, a sentence φ be true at a possibility if and only if it is true at every world in that possibility (see Definition 3.2).

In an abstract possibility model, the possibilities are simply primitive elements and need not be sets of total worlds, and indeed there need not be any connection with Kripke models at all. This leads to the main question of the paper:

Main Question. *Is every possibility model (isomorphic to) the possibilization of a Kripke model?*

It is not hard to see that there are possibility models which are not isomorphic to a possibilization. For example, the possibility model could include duplication of possibilities—two possibilities which have exactly the same refinements—which cannot happen in a possibilization. However, every possibility model can be transformed (see Propositions 3.5 and 3.7) in a natural way into one which is *separative* and *strong*; such possibility models avoid these sorts of issues.

For countable models, the answer to the main question is affirmative: we are able to show that every countable, separative, and strong possibility model is (up to isomorphism) a possibilization.

Theorem 1.1 (Representation theorem for countable possibility models)**.** *Every countable, separative, and strong possibility model in a countable language is isomorphic to a possibilization of a countable Kripke model.*

The representation theorem is proved in Section 4.

There are two important hypotheses in the statement of the theorem. First, we require that the language be countable, and second, that the number of possibilities be countable. We produce a counterexample when the assumption that the language is countable is lifted (see Section 5). Our method of proof does not work when there are uncountably many possibilities; but it might be that some other method would allow us to extend our representation theorem to uncountable models.

Question. *Is every separative and strong possibility model, countable or uncountable but in a countable language, isomorphic to the possibilization of a Kripke model?*

1.1. **Worldifications.** The first step in proving our representation theorem is to find, in a possibility model, some total worlds. To this end, we introduce the notion of *worldification*, which is a dual notion to possibilization (see Definition 4.1). A worldification of a possibility model $\mathcal{M}$ is a Kripke model $\mathcal{K}$ such that every possibility in $\mathcal{M}$ is refined by some total world of $\mathcal{K}$ and so that every total world of $\mathcal{K}$ is the limit of more and more refined possibilities in $\mathcal{M}$; moreover, there must be a tight relation between truth and the accessibility relation in the two models. We show that every countable possibility model has a worldification.

Theorem 1.2. *Every countable possibility model in a countable language has a worldification.*

This theorem is proved in Section 4. The difficulty in proving this theorem is that the accessibility relation of the worldification $\mathcal{K}$ should come from the accessibility relation on $\mathcal{M}$; it is obvious how to extend each individual possibility to a total world, but it is not obvious how to do this simultaneously for each possibility of $\mathcal{M}$ while respecting the accessibility relation.

Our representation theorem is proved by embedding each possibility model $\mathcal{M}$ into a worldification $\mathcal{K}$ with a few additional properties. Then we interpret the possibilities in $\mathcal{M}$ as sets of total worlds from $\mathcal{K}$.

Theorem 1.2 is not true for uncountable models. We produce a counterexample using Aronszajn trees. (Aronszajn trees are trees with odd behaviour coming from set theory.) However, this example relies in an essential way on not being strong; we do not know whether every strong possibility model has a worldification.

Question. *Does every strong possibility model have a worldification?*

1.2. **Frames.** At the level of frames, we can define a notion of frame-worldification. A Kripke frame $\mathcal{F}$ is a frame-worldification of a possibility frame $\mathcal{G}$ if two conditions are met. First, the possibilities and the worlds of $\mathcal{F}$ and $\mathcal{G}$ are related as in a worldification of models (so that every possibility in $\mathcal{G}$ is refined by some total world of $\mathcal{F}$ and so that every total world of $\mathcal{F}$ is the limit of more and more refined possibilities in $\mathcal{G}$). Second, any Kripke model $\mathcal{K}$ based on $\mathcal{F}$ should induce a possibility model $\mathcal{M}$ based on $\mathcal{G}$ (and every possibility model $\mathcal{M}$ based on $\mathcal{G}$ should admit a Kripke model $\mathcal{K}$ based on $\mathcal{F}$) so that $\mathcal{K}$ is a worldification of $\mathcal{M}$.

There are countable possibility frames which have no frame-worldifications. This ties in to Holliday's [Hol15] result that there are possibility frames whose logic is not the logic of any class of Kripke frames. Such a possibility frame could not have a worldification. On the other hand, if we consider general possibility frames and general Kripke frames, we find that if a general possibility frame has countably many admissible sets, then it has a frame-worldification.

Theorem 1.3. *Every countable general possibility frame with countably many admissible sets has a frame-worldification.*

This theorem is proved in Section 6.

1.3. **Brief philosophical remarks.** Arguments have been given, from a philosophical perspective, for and against the idea of constructing worlds out of limits of possibilities. Rumfitt [Rum15], for example, describes how one might try to construct a total world:

> [T]he possibility that I have red hair leaves it undetermined whether Ed Miliband will win a General Election. But there is also the possibility that I have red hair while Miliband wins an election, and the distinct possibility that I have red hair while he does not. By iterating this process, it may be suggested, we shall eventually reach fully determinate possibilities which do settle the truth or falsity of all statements. These possibilities will be the points of modal space . . . . (159)

Our construction follows essentially the strategy described above, which has similarities to the construction of a generic in set theory (see [Coh66], [Jec03], or [Kun80]). However, our proof of Theorem 1.2 requires much more than the strategy just described; most of the work that we will do goes into picking appropriate sequences of refinements so that one can define the modal accessibility relation between the constructed points. It must also be noted that Rumfitt [Rum15] expresses doubts about the construction of the points themselves:

> [T]he business of making a possibility more determinate seems open-ended. There are possibilities that the child at home should be a boy, a six-year-old boy, a six-year-old boy with blue eyes, a six-year-old boy with blue eyes who weighs 3 stone, and so forth. So far from terminating in a fully determinate possibility, we seem to have an indefinitely long sequence of increasingly determinate possibilities, any one of which is open to further determination. But then, so far from conceiving of our rational activities as discriminating between regions of determinate points, we appear to have no clear conception of such a point at all. (159)

Here it is important that we restrict our attention to a countable set of propositional variables, so that we can define a countable sequence of possibilities such that each propositional variable is decided at some point in the sequence. Although every possibility in the sequence may be open to further determination, we can take the countable sequence itself as a "world" which decides each propositional variable in the given countable set. This is compatible with Rumfitt's assertion that we may never reach a possibility that "settles the truth or falsity of all statements" without restriction.

2. Possibility Models

2.1. **Possibility Semantics.** Let P be a set of propositional variables, and let $\mathcal{L}(P)$ be the standard language of propositional modal logic with modal operators $\Box$ and $\Diamond$ and propositional variables coming from P.

The following frames may be viewed as a special case of the "full possibility frames" of [Hol15] and as a generalization of the frames of [Hum81].[1]

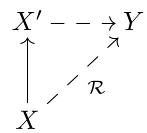
[1] Holliday [Hol15] writes '$X \sqsubseteq Y$' to mean that X is a refinement of Y, going "down" rather than "up" for refinements, while [Hum81] writes '$X \geqslant Y$' to mean that X is a refinement of Y. We will write '$X \geq Y$' to mean that X is a refinement of Y.

Definition 2.1. A (*basic*) *possibility frame* is a tuple $\mathcal{F} = (\mathcal{P}, \mathcal{R}, \leq)$ where:

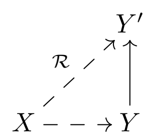
(1) $\mathcal{P}$ is a non-empty set of *possibilities*,
(2) $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a binary *accessibility relation*, and
(3) $\leq$ is a partial order on $\mathcal{P}$, the *refinement relation*,

satisfying the following three properties:

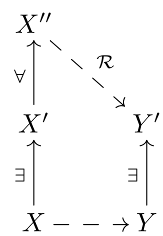
P1: For all X, X', and Y with $X' \geq X$, if $X'\mathcal{R}Y$ then $X\mathcal{R}Y$.



P2: For all X, Y, and Y' with $Y' \geq Y$, if $X\mathcal{R}Y$ then $X\mathcal{R}Y'$.



R: For all X and Y, if $X\mathcal{R}Y$ then there is $X' \geq X$ such that for all $X'' \geq X'$, there is $Y' \geq Y$ such that $X''\mathcal{R}Y'$.



We interpret $X\mathcal{R}Y$ as meaning that what is necessary at X is true at Y. $X \geq Y$ means that X determines each issue which Y does, in the same way, and possibly more. Our possibility frames are more general than those considered by Humberstone. Humberstone asked that a stronger version of the condition **R** be satisfied, namely:

R^{++}: For all X and Y, if $X\mathcal{R}Y$ then there is $X' \geq X$ such that for all $X'' \geq X'$, $X''\mathcal{R}Y$.[2]

See [Hol15, Section 2.3] for a discussion of why it is desirable to use the weaker condition on the refinability relation.

A partial function $f\colon D \to C$ is a function which is defined on some, possibly proper, subset of D. If $x \in D$ and f is defined at $x \in D$ and maps x to $y \in C$, we write $f(x) = y$; otherwise, if f is not defined at x, we write $f(x) = ?$.

Definition 2.2. A *possibility model* is a tuple $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ where $\mathcal{F} = (\mathcal{P}, \mathcal{R}, \leq)$ is a possibility frame and $V\colon \mathcal{P} \times P \to \{\mathtt{T}, \mathtt{F}\}$ is a partial function, the *valuation*, satisfying:

Persistence: For any $Y \geq X$ in $\mathcal{P}$ and any $p \in P$, if $V(X, p) = \mathtt{T}$ then $V(Y, p) = \mathtt{T}$, and similarly for $\mathtt{F}$.

Refinability: For any $X \in \mathcal{P}$, if $V(x, p) = ?$, then there exist $Y \geq X$ and $Z \geq X$ such that $V(Y, p) = \mathtt{F}$ and $V(Z, p) = \mathtt{T}$.

[2] There is also an intermediate condition $\mathbf{R}^{+}$ discussed in [HT16] and [Hol15].

$\mathcal{M}$ is said to be *based on* $\mathcal{F}$.

If $X \in \mathcal{P}$, then we read $V(X,p) = \mathtt{T}$ as "p is true at X (under V)", $V(X,p) = \mathtt{F}$ as "p is false at X (under V)", and $V(X,p) = ?$ as "p is undetermined at X (under V)".

Definition 2.3. Given a possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$, the *satisfaction relation* is defined inductively as follows:

(1) $\mathcal{M}, X \models p$ if $V(X,p) = \mathtt{T}$.
(2) $\mathcal{M}, X \models \varphi \wedge \psi$ if $\mathcal{M}, X \models \varphi$ and $\mathcal{M}, X \models \psi$.
(3) $\mathcal{M}, X \models \neg\varphi$ if for all $Y \geq X$, $\mathcal{M}, Y \not\models \varphi$.
(4) $\mathcal{M}, X \models \Box\varphi$ if for all $Y \in \mathcal{P}$ such that $X\mathcal{R}Y$, $\mathcal{M}, Y \models \varphi$.

Humberstone [Hum81] proves all of the following lemmas and Proposition 2.7 below (see [Hol15] for the proofs using the weaker refinability condition).

Lemma 2.4 (Persistence)**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. If $Y \geq X$ and $\mathcal{M}, X \models \varphi$, then $\mathcal{M}, Y \models \varphi$.*

Lemma 2.5 (Refinability)**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. If $\mathcal{M}, X \not\models \varphi$, then for some $Y \geq X$, $\mathcal{M}, Y \models \neg\varphi$.*

Lemma 2.6 (Double Negation Elimination)**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. $\mathcal{M}, X \models \varphi$ if and only if $\mathcal{M}, X \models \neg\neg\varphi$*

As usual, we say that a sentence φ is *globally true* in a possibility model $\mathcal{M}$ if $\mathcal{M}, X \models \varphi$ for all X, and φ is *valid* if it is globally true in all possibility models. A sentence φ is *satisfiable* if there is a model $\mathcal{M}$ and possibility X with $\mathcal{M}, X \models \varphi$.

Proposition 2.7 (Soundness and Completeness)**.** *For any sentence φ, the following are equivalent:*

(1) φ is valid over all possibility models,
(2) φ is valid over all Kripke models,
(3) φ is provable in the minimal normal modal logic K.

3. Possibilizations

3.1. **Possibilizations.** The simplest example of a possibilization is the powerset possibilization, where the set of possibilities is taken to be as large as possible.[3]

Definition 3.1. Let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. The *powerset possibilization* of $\mathcal{K}$ is the possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ where:

(1) $\mathcal{P} = \wp(\mathcal{W}) \setminus \{\varnothing\}$;
(2) for $X, Y \in \mathcal{P}$, $X \geq Y$ if and only if $X \subseteq Y$;
(3) $X\mathcal{R}Y$ if and only if $Y \subseteq \mathcal{S}[X] = \{w' : (\exists w \in X) w\mathcal{S}w'\}$;
(4) $V(X,p) = \mathtt{T}$ if for all $w \in X$, $U(w,p) = \mathtt{T}$; $V(X,p) = \mathtt{F}$ if for all $w \in X$, $U(w,p) = \mathtt{F}$; and otherwise $V(X,p) = ?$.

One can prove that for each $X \in \mathcal{P}$, and for each sentence φ, $\mathcal{M}, X \models \varphi$ if and only if for all $w \in X$, $\mathcal{K}, w \models \varphi$; and that for each world $w \in \mathcal{W}$, and each sentence φ, if $\mathcal{K}, w \models \varphi$ then there is a possibility $X \in \mathcal{P}$ with $w \in X$ and $\mathcal{M}, X \models \varphi$.

[3] Holliday [Hol15, Fact B.1] observes that a powerset possibilization might not satisfy Humberstone's stronger condition $\mathbf{R}^{++}$.

More generally, a possibilization of a Kripke model will be a possibility model where the possibilities are sets of worlds from the Kripke model. There must be a tight connection between the two models.

Definition 3.2. Let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model. A *possibilization* of $\mathcal{K}$ is a possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$, with $\mathcal{P}$ a non-empty collection of non-empty subsets of $\mathcal{W}$, such that:

(P1) for each world $w \in \mathcal{W}$, and each sentence φ, if $\mathcal{K}, w \models \varphi$ then there is a possibility $X \in \mathcal{P}$ with $w \in X$ and $\mathcal{M}, X \models \varphi$;
(P2) any two distinct worlds $v, w \in \mathcal{W}$ are separated by possibilities, that is, there is $X \in \mathcal{P}$ with $v \in X$, $w \notin X$;
(P3) if $X, Y \in \mathcal{P}$ are not disjoint, there is $Z \in \mathcal{P}$ with $Z \subseteq X \cap Y$;
(P4) for $X, Y \in \mathcal{P}$, $X \geq Y$ if and only if $X \subseteq Y$;
(P5) for $X, Y \in \mathcal{P}$, $X\mathcal{R}Y$ if and only if $Y \subseteq \mathcal{S}[X]$;
(P6) for each $X \in \mathcal{P}$, and for each sentence φ, $\mathcal{M}, X \models \varphi$ if and only if for all $w \in X$, $\mathcal{K}, x \models \varphi$.

(P1)-(P3) ensure that there are enough possibilities to distinguish between distinct worlds, to ensure that a sentence true at some world in the Kripke model is true somewhere in the possibility model, and so on. (P4) and (P5) are the same as in the definition of the powerset possibilization and are the natural ways to induce the refinement and accessibility relations on sets of worlds. (P6) says, as a special case, that the values of the propositional variables are defined as in the powerset possibilization, but it also says something about more complicated sentences, namely that what might be true at a possibility is exactly what might be true at the worlds that make up that possibility. One could replace (P6) by a few closure conditions on the set of possibilities. Essentially these conditions say that there are sufficiently many possibilities to distinguish the worlds of the Kripke model and to make truth in the possibilization correspond to truth in the Kripke model (with the accessibility and refinement relations defined in the natural way via (P4) and (P5)) without saying exactly what those possibilities should be.

The powerset possibilization of a Kripke model is a possibilization, and a Kripke model can be viewed as a possibilization of itself (using the singleton sets as possibilities and with the refinement relation being the identity).

Our goal is to show that every countable possibility model is the possibilization of a Kripke model. Our strategy will be to produce a worldification of the possibility model and then to view the possibility model as a possibilization of its worldification. There are two barriers to this. The first is that to produce a worldification of the right kind, we must have a possibility model which is *strong* as defined below. The second is that the possibility model may have redundant possibilities; this will be solved by assuming that our possibility model is *separative* as defined below.

3.2. **Separative possibility models.** The following natural class of possibility models is studied in Section 4.1 of [Hol15].[4]

Definition 3.3. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. $\mathcal{M}$ is *separative* if whenever $X \not\geq Y$, there is $X' \geq X$ such that X' and Y have no common refinements.

Define
$$X \geq_s Y \iff (\forall X' \geq X)(\exists X'' \geq X')\ X'' \geq Y.$$

[4]The terminology comes from set-theoretic forcing; see for example p. 204 of [Jec03].

Then a possibility model $\mathcal{M}$ is separative if and only if the refinement relation $\geq$ is equal to $\geq_s$. One can see that if a possibility model is separative, then, for example, any two possibilities which have exactly the same refinements must be equal.

Not every possibility model is separative, though as remarked above, every possibilization is separative. However, every possibility model embeds in a natural way into a separative quotient by identifying equivalent possibilities, such as duplicated possibilities.

Definition 3.4. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. Let $X \simeq_s Y$ if and only if $X \geq_s Y$ and $Y \geq_s X$; this is an equivalence relation. Write $[X]$ for the equivalence class of X under $\simeq_s$. Let:

(1) $\mathcal{P}'$ be the equivalence classes under $\simeq_s$,
(2) $[X]\mathcal{R}'[Y]$ if there are $X' \simeq_s X$ and $Y' \simeq_s Y$ with $X'\mathcal{R}Y'$,
(3) $V'([X], p) = \mathtt{T}$ if $V(X, p) = \mathtt{T}$ and $V'([X], p) = \mathtt{F}$ if $V(X, p) = \mathtt{F}$; otherwise $V'([X], p) = ?$.

$\mathcal{M}_s = (\mathcal{P}', \mathcal{R}', \preceq_s, V')$ is the *separative quotient* of $\mathcal{M}$.

This is well-defined. There is a natural embedding of a possibility model into its separative quotient, and this embedding maintains truth.

Proposition 3.5 (Proposition 4.10 of [Hol15])**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. $\mathcal{M}_s$ is a separative possibility model, and*

$$\mathcal{M}, X \models \varphi \Longleftrightarrow \mathcal{M}_s, [X] \models \varphi.$$

3.3. **Strong possibility models.** The following condition on possibility models, which is essentially a refinability condition on the accessibility relation, has been studied by Holliday:

Definition 3.6 (Section 2.3 of [Hol15])**.** Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. $\mathcal{M}$ is *strong* if whenever it is the case that for all $Y' \geq Y$ there is $Y'' \geq Y'$ such that $X\mathcal{R}Y''$, we already have $X\mathcal{R}Y$.

This condition implies, for example, that if every strict refinement of a possibility X sees, via the accessibility relation, a possibility Y, then X sees Y. Any possibilization is strong. Once again, every possibility model embeds in a natural way into a strong model (Proposition 2.37 of [Hol15]).

Proposition 3.7. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. Define a new accessibility relation $\mathcal{R}'$ by $X\mathcal{R}'Y$ if and only if for all $Y' \geq Y$ there is $Y'' \geq Y'$ with $X\mathcal{R}Y''$. Then $\mathcal{M}' = (\mathcal{P}, \mathcal{R}', \leq, V)$ is a strong possibility model and*

$$\mathcal{M}, X \models \varphi \Longleftrightarrow \mathcal{M}', X \models \varphi.$$

If $\mathcal{M}$ was separative, so is $\mathcal{M}'$, since we have not altered $\leq$.

4. Worldifications and the Proof of the Main Theorem

In this section, we will give the proof of our representation theorem. Given a countable possibility model $\mathcal{M}$, we show how to build a (weak) worldification $\mathcal{K}$ of that model. But this is not enough to show that $\mathcal{M}$ is a possibilization of $\mathcal{K}$; we need to know that $\mathcal{K}$ is a *strong* worldification of $\mathcal{M}$ (i.e., that $\mathcal{K}$ satisfies one additional condition), and to build such a $\mathcal{K}$ we need to know that $\mathcal{M}$ is a strong possibility model. Once we have a strong worldification of $\mathcal{M}$, if we also know that

$\mathcal{M}$ is separative, then we can show that $\mathcal{M}$ is isomorphic to a possibilization of $\mathcal{K}$, and thus $\mathcal{M}$ is isomorphic to the possibilization of a Kripke model.

To begin this section, we will first define weak and strong worldifications in Section 4.1. Then in Section 4.2 we will show that the existence of strong worldifications is enough to prove the representation theorem. Finally, in Section 4.3, we will first prove that every possibility model has a worldification. Then we will add an extra module to this proof to show that every strong possibility model has a strong worldification, completing the proof of the representation theorem.

4.1. **Worldifications.** We say that a Kripke model $\mathcal{K}$ is a *worldification* of a possibility model $\mathcal{M}$ if, informally speaking, each possibility in $\mathcal{M}$ is part of a total world from $\mathcal{K}$, and each total world in $\mathcal{K}$ is a limit of more and more refined possibilities. The definition is motivated in part by its use in the representation theorem, but it is also of independent interest.

Definition 4.1. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model and let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model. $\mathcal{K}$ is a *weak worldification* of $\mathcal{M}$ via an embedding $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ which assigns to each total world $w \in \mathcal{W}$ a non-empty set of possibilities $\Phi(w) \subseteq \mathcal{P}$ if:

- (W1) for each world $w \in \mathcal{W}$, $\Phi(w)$ is a *maximal order ideal* in the poset $(\mathcal{P}, \leq)$, i.e.,
  - (a) $\Phi(w)$ is downwards-closed under refinement,
  - (b) any two elements of $\Phi(w)$ have a common refinement in $\Phi(w)$, and
  - (c) $\Phi(w)$ is maximal with these two properties;
- (W2) for each possibility $X \in \mathcal{P}$, there is a world $w \in \mathcal{W}$ such that $X \in \Phi(w)$;
- (W3) any two distinct total worlds $v, w \in \mathcal{W}$ are separated by possibilities, that is, there is $X \in \Phi(v) \setminus \Phi(w)$;
- (W4) for each world $w \in \mathcal{W}$, and for each sentence φ, $\mathcal{K}, w \models \varphi$ if and only if there is some $X \in \Phi(w)$ such that $\mathcal{M}, X \models \varphi$; and
- (W5) for each pair of worlds $w, v \in \mathcal{W}$, $w\mathcal{S}v$ if and only if for each $X \in \Phi(w)$ there is $Y \in \Phi(v)$ such that $X\mathcal{R}Y$.

We say that $\mathcal{K}$ is a worldification of $\mathcal{M}$ if the Φ which makes $\mathcal{K}$ a worldification is understood from the context.

When our models are countable, $\Phi(w)$ is determined by some increasing chain in $\mathcal{M}$. Note that if $X, Y \in \Phi(w)$, then they have a common refinement, so we cannot have $\mathcal{M}, X \models \varphi$ and $\mathcal{M}, Y \models \neg\varphi$.

While the conditions (W1)-(W4) look like the corresponding conditions in the definition of a possibilization, (W5) differs greatly from (P5). (P5) seems like the natural way to induce an accessibility relation on possibilities, given such a relation on worlds, while (W5) seems like the natural way to induce an accessibility relation on worlds given one on possibilities. Unfortunately, the two do not agree. To rectify this, we introduce strong worldifications, which place an additional condition on the accessibility relation.

Definition 4.2. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model and let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model. $\mathcal{K}$ is a *strong worldification* of $\mathcal{M}$ via an embedding $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ which assigns to each total world $w \in \mathcal{W}$ a non-empty set of possibilities $\Phi(w) \subseteq \mathcal{P}$ if, in addition to (W1)-(W5):

- (W6) $X\mathcal{R}Y$ if and only if for all w with $Y \in \Phi(w)$ there is v with $X \in \Phi(v)$ and $v\mathcal{S}w$.

Remark 4.3. The notion of worldification bears some similarity with what one would obtain by taking the Jónsson-Tarski ultrafilter frame (see Section 5.3 of [BdRV01]) of the modal algebra associated with a possibility frame (for details on the duality between possibility frames and modal algebras, see [Hol15]). However, such a construction does not seem like it can be used to prove our representation theorem. For example, suppose that we start with a possibility model which has countably many distinct possibilities $X_1, X_2, \ldots$, none of which are refinements of any of the others. The Kripke model which we might try to build using the Jónsson-Tarski construction has a world for each ultrafilter in the Boolean algebra associated with the possibility frame, which in the case of the possibility model just described is simply the powerset algebra. The principal ultrafilters are generated by the singleton sets $\{X_i\}$ and correspond to treating $X_1, X_2, \ldots$ as total worlds. But the non-principal ultrafilters would not be contained in any of the possibilities $X_1, X_2, \ldots$, and so (P1) of Definition 3.2 would not be satisfied. We might try to not include these non-principal ultrafilters in the Kripke model, but it is not clear how to do this without having problems with the accessibility relation; for example, if $\Diamond\varphi$ is satisfied at some world as witnessed by a non-principal ultrafilter, it is not clear that $\Diamond\varphi$ will still hold after removing that non-principal ultrafilter. One can think of the construction of a worldification given below as providing a way of making the choice of which ultrafilters to include in order to both satisfy (P1) of Definition 3.2 and to satisfy the correct formulas.

4.2. **Proof of the Main Theorem.** We will take as a black box the existence of strong worldifications, which will be proved in Section 4.3:

Theorem 4.4. *Every strong countable possibility model in a countable language has a strong worldification.*

We will now prove the representation theorem: up to isomorphism of possibility models, every countable, separative, and strong possibility model in a countable language is the possibilization of a Kripke model.

Proof of Theorem 1.1. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a countable, separative, and strong possibility model. Using Theorem 4.4, let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a strong worldification of $\mathcal{M}$ via $\Phi\colon \mathcal{P} \to \wp(\mathcal{W})$.

Given $X \in \mathcal{P}$, let $S_X = \{w \in \mathcal{W}\colon X \in \Phi(w)\}$. We claim that $S_X = S_Y$ if and only if $X = Y$; this is where we use the fact that $\mathcal{M}$ is separative. Suppose that $S_X = S_Y$. We claim that $X \simeq_s Y$ where $\simeq_s$ is defined as in Definition 3.4. If $X' \geq X$, then by (W2) there is v such that $X' \in \Phi(v)$. So $X \in \Phi(v)$ and since $S_X = S_Y$, $Y \in \Phi(v)$. So there is $X'' \in \Phi(v)$ with $X'' \geq X', Y$. Thus $X \geq_s Y$. By interchanging X and Y, we see that $X \simeq_s Y$. Since $\mathcal{M}$ is separative, $X = Y$ as desired.

Identify $X \in \mathcal{P}$ with S_X. We can interpret $\mathcal{R}$, $\leq$, and V as acting on the sets S_X. Let $\mathcal{M}'$ be the possibility model with possibilities S_X. We claim that $\mathcal{M}'$ is a possibilization of $\mathcal{K}$. In verifying properties (P1)-(P6) of a possibilization, we will write X interchangeably with S_X (so that we write $v \in X$ for $v \in S_X$).

- **(P1):** This follows from (W4).
- **(P2):** This follows from (W3).
- **(P3):** Suppose that $v \in X \cap Y$. Then by (W1), there is Z with $v \in Z$ and $Z \geq X, Y$, i.e., $Z \subseteq X \cap Y$.

(P4): If $X \geq Y$, then by (W1), whenever $v \in X$, $v \in Y$. So $X \subseteq Y$.

(P5): This follows from (W6).

(P6): If $\mathcal{M}, X \models \varphi$, then by (W4), $\mathcal{K}, w \models \varphi$ for all $w \in X$. On the other hand, if, $\mathcal{M}, X \not\models \varphi$, then there is $Y \geq X$ such that $\mathcal{M}, Y \models \neg\varphi$. Then, picking $w \in Y$, by (W4) we have that $\mathcal{K}, w \models \neg\varphi$. But $w \in X$, and so it is not the case that for all $w \in X$, $\mathcal{K}, w \models \varphi$. □

4.3. **Constructing worldifications.** We will prove Theorem 1.2, which says that every countable possibility model in a language with countably many propositional variables has a worldification, and Theorem 4.4, which says that every countable strong possibility model in a countable language has a strong worldification. The proof is essentially to construct infinite ascending chains while managing the accessibility relation to get the appropriate properties. Doing this is surprisingly complicated. We will begin with a warmup in which we use the stronger condition $\mathbf{R}^{++}$ from Section 2.1.

Theorem 4.5. *Let $\mathcal{M}$ be a countable possibility model in a countable language, satisfying $\boldsymbol{R}^{++}$. Then there is a Kripke model $\mathcal{K}$ which is a worldification of $\mathcal{M}$.*

Proof. For each $X \in \mathcal{M}$, we will define an increasing chain of possibilities $A_X = (A_X(n))_{n\in\omega}$. These chains will form the total worlds of the model $\mathcal{K}$. Let $(X_n)_{n\in\omega}$ be an enumeration of the possibilities in $\mathcal{M}$ and $\varphi_0, \varphi_1, \ldots$ an enumeration of the sentences in the language. For simplicity, we occasionally write A_i for A_{X_i}. We define the chains A_X using a recursive construction. Begin with $A_X(0) = X$ for each X.

The idea is that we need to extend the chains in such a way that every formula is decided at some point in each chain, and also so that if the chain does not satisfy $\Box\varphi$ at some point X, there is a witnessing possibility Y at which $\neg\varphi$ holds so that any refinement of X is still related via the accessibility relation to Y. As we extend the chains, we alternate between these two requirements, at each step either deciding some new formula using the refinability property, or using $\mathbf{R}^{++}$ to lock in a witness to $\Diamond\neg\varphi$.

Suppose that we have defined $A_X(0), \ldots, A_X(n)$ for each X. We will now define $A_X(n+1)$ for each X. We have two cases, depending on whether n is odd or even.

n **is even:** Write $n = 2k$. For each $X \in \mathcal{M}$, choose $X' \geq A_X(n)$ such that $\mathcal{M}, X' \models \varphi_k$ or $\mathcal{M}, X' \models \neg\varphi_k$. Now, if $X_k \geq X'$, set $A_X(n+1) = X_k$, and otherwise set $A_X(n+1) = X'$.

n **is odd:** Write $n = 2\langle k, i\rangle + 1$ where $\langle\cdot,\cdot\rangle : \omega^2 \to \omega$ is bijective. If there is Y such that $A_i(n)\mathcal{R}Y$ and $\mathcal{M}, Y \models \varphi_k$, then we also have $A_i(n)\mathcal{R}A_Y(n)$ since $A_Y(n) \geq Y$. Using $\mathbf{R}^{++}$, choose $X \geq A_i(n)$ such that for all $X' \geq X$, $X'\mathcal{R}A_Y(n)$. Set $A_i(n+1) = X$. For each other possibility Z, set $A_Z(n+1) = A_Z(n)$. If no such Y exists, set $A_Z(n+1) = A_Z(n)$ for all Z.

This completes the construction of the sequences A_X. Let $\widehat{A}_X$ be the order ideal which is the downwards closure of A_X. At even stages, we ensure that for each Y, either Y is part of the chain A_X or there is some n such that Y is not a refinement of $A_X(n)$. So $\widehat{A}_X$ is maximal. Now let $\mathcal{W}$ be the set of these order ideals and note that there may be possibilities X and Y such that $\widehat{A}_X = \widehat{A}_Y$. Such an order ideal is included in $\mathcal{W}$ only once. We will define a total world model $\mathcal{K}$ with domain $\mathcal{W}$ which is a worldification of $\mathcal{M}$ via the identity function. The accessibility relation

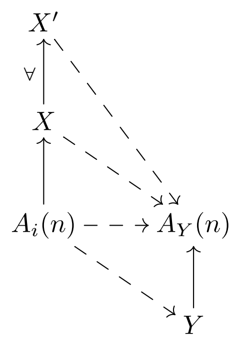


Figure 1. The extensions of possibilities in the construction. The dotted line shows the relation $\mathcal{R}$. For all X' as shown, $\mathcal{R}$ relates X' and $A_Y(n)$.

will be $\mathcal{S}$. For $I, J \in \mathcal{W}$, define $I\mathcal{S}J$ if and only if there is a $Y \in J$ such that for all $X \in I$, $X\mathcal{R}Y$. Have a propositional variable p hold at $I \in \mathcal{W}$ if and only if for some $X \in I$, $\mathcal{M}, X \models p$. We make p false at $I \in \mathcal{W}$ if and only if for some $X \in I$, $\mathcal{M}, X \models \neg p$. By construction, for each formula φ and $I \in \mathcal{W}$, there is $X \in I$ such that either $\mathcal{M}, X \models \varphi$ or $\mathcal{M}, X \models \neg\varphi$. Also, if for some $Y \in I = \widehat{A}_X$, $\mathcal{M}, Y \models \varphi$, then there is some n such that $A_X(n) \geq Y$ and so $\mathcal{M}, A_X(n) \models \varphi$.

Properties (W1), (W2), (W3), and (W5) of a worldification are immediate. To complete the proof, we check (W4) from the definition of worldification. The proof is by induction on the complexity of formulas. For a propositional variable p, let $I \in \mathcal{W}$ and let $X \in \mathcal{M}$ be such that $\widehat{A}_X = I$. Then

$$\mathcal{K}, I \models p \Leftrightarrow (\exists n)\mathcal{M}, A_X(n) \models p$$

since, for some n, either $\mathcal{M}, A_X(n) \models p$ or $\mathcal{M}, A_X(n) \models \neg p$. For $\varphi \wedge \psi$,

$$\begin{aligned} \mathcal{K}, I \models \varphi \wedge \psi &\Leftrightarrow \mathcal{K}, I \models \varphi \text{ and } \mathcal{K}, I \models \psi \\ &\Leftrightarrow (\exists X \in I)\mathcal{M}, X \models \varphi \text{ and } (\exists Y \in I)\mathcal{M}, Y \models \psi \\ &\Leftrightarrow (\exists Z \in I)\mathcal{M}, Z \models \varphi \wedge \psi \end{aligned}$$

where, given X and Y witnesses to the second line, the witness Z to the third line is a common refinement of X and Y. For $\neg\varphi$,

$$\begin{aligned} \mathcal{K}, I \models \neg\varphi &\Leftrightarrow \mathcal{K}, I \not\models \varphi \\ &\Leftrightarrow (\forall X \in I)\mathcal{M}, X \not\models \varphi \\ &\Leftrightarrow (\exists X \in I)\mathcal{M}, X \models \neg\varphi \end{aligned}$$

since for some $X \in I$, either $\mathcal{M}, X \models \varphi$ or $\mathcal{M}, X \models \neg\varphi$.

Finally, we have the case $\Box\varphi$. Suppose that for all $X \in I$, $\mathcal{M}, X \not\models \Box\varphi$. Let X be such that $I = \widehat{A}_X$ where $X = X_i$ and let k be such that $\neg\varphi = \varphi_k$. Then at stage $n = 2\langle k, i\rangle + 1$ of the construction, we have $A_X(n) \not\models \Box\varphi$, so there is some $Y \in \mathcal{M}$ with $X\mathcal{R}Y$ such that $\mathcal{M}, Y \not\models \varphi$; refining Y if necessary, we may assume that $\mathcal{M}, Y \models \neg\varphi$ while still maintaining $X\mathcal{R}Y$ by **P2**. Then (possibly for some different Y such that $Y \models \neg\varphi$ and $X\mathcal{R}Y$) we have $A_Y(n+1) \geq Y$ and for all $Z \geq A_X(n+1)$, $Z\mathcal{R}A_Y(n+1)$. Hence, for each $\ell \geq n+1$ and $m \geq n+1$,

$A_X(\ell)\mathcal{R}A_Y(m)$. Thus $\widehat{A}_X\mathcal{S}\widehat{A}_Y$. Since $\mathcal{K}, \widehat{A}_Y \models \neg\varphi$, $\mathcal{K}, \widehat{A}_X \not\models \Box\varphi$. Thus we have shown that if $\mathcal{K}, I \models \Box\varphi$, then for some $X \in I$, $\mathcal{M}, X \models \Box\varphi$.

Now suppose that for some $Y \in \widehat{A}_X$, $\mathcal{M}, Y \models \Box\varphi$. Then by persistence and the fact that $\widehat{A}_X$ is the downwards closure of the chain A_X, $\mathcal{M}, A_X(n) \models \Box\varphi$ for some n. Let Z be such that $\widehat{A}_X\mathcal{S}\widehat{A}_Z$. Then there is some m such that $A_X(n)\mathcal{R}A_Z(m)$, and so $\mathcal{M}, A_Z(m) \models \varphi$. Hence $\mathcal{K}, \widehat{A}_Z \models \varphi$. Since Z was arbitrary, $\mathcal{K}, \widehat{A}_X \models \Box\varphi$. $\Box$

Now for Theorem 1.2, we must use $\mathbf{R}$ which is weaker than $\mathbf{R}^{++}$. While using $\mathbf{R}^{++}$ we were able to lock in the witness to $\Diamond\neg\varphi$ in a single step, this is no longer possible with $\mathbf{R}$. Instead, we have to constantly make sure that we maintain the same witness for each chain. We will keep track of the witnesses in a tree, so that there are no circular witness requirements. (By a circular witness requirement, we mean for example that Y is a witness for X, Z is a witness for Y, and X is a witness for Z.) This makes the proof somewhat complicated.

Proof of Theorem 1.2. For each $X \in \mathcal{P}$, we will define infinitely many increasing chains of possibilities $A^s_X = (A^s_X(n))_{n\in\omega}$ with $A^s_X(0) = X$. Let $(X_n)_{n\in\omega}$ be an enumeration of the worlds in $\mathcal{P}$ and $\varphi_0, \varphi_1, \ldots$ an enumeration of the sentences in the language $\mathcal{L}$. The chains A^s_X will be defined using a recursive construction. First, we must define an auxiliary object that we will build during the construction.

A *tree* is a graph such that between any two edges there is a unique path. A *rooted tree* is a tree with a distinguished node. Each edge in a rooted tree has a natural direction, towards or away from the root. Thus a rooted tree can be viewed as a *directed tree*, a tree in which each edge has a specified direction pointing away from the root. Our trees will always be directed and rooted. A *forest* is the disjoint union of trees. Let T be a forest. We denote the edge relation of T by T as well. We say that b is a *child* of a if $T(a, b)$. We say that a node a is a *leaf* if it has no children. A *connected component* of a forest is a maximal set of nodes which are pairwise connected by an undirected path; each connected component of a forest is a tree.

At each stage n of the construction, we will have a forest T_n with domain $\omega \times \mathcal{P}$, representing the pairs $\langle s, X\rangle$ corresponding to some chain A^s_X via some bijection. Each T_n will have only finitely many edges and the T_n will be nested; that is, if $m < n$, and $\langle s, X\rangle$ is a child of $\langle t, Y\rangle$ in T_m, then the same is true in T_n (but not necessarily vice versa). If there is an edge in T_n involving $\langle s, X\rangle$, then after stage n, we will only add edges outward from $\langle s, X\rangle$, and never inward. Thus the roots of any non-trivial connected components in T_n will remain the roots of their connected components. We will satisfy the requirement:

$(*)$: If $\langle s, X\rangle$ is a child of $\langle t, Y\rangle$ in T_n, then for all $Y' \geq A^t_Y(n)$, there is $X' \geq A^s_X(n)$ such that $Y'\mathcal{R}X'$.

Begin the construction with $A^s_X(0) = X$ for each $X \in \mathcal{P}$ and $s \in \omega$. Suppose that we have defined $A^s_X(0), \ldots, A^s_X(n)$ for each $\langle s, X\rangle$. Write $n+1 = 2\langle s, i, k\rangle + \epsilon$ where ϵ is 0 or 1. Let $X = X_i$ and $\varphi = \varphi_k$. Let $\langle t_0, Y_0\rangle, \langle t_1, Y_1\rangle, \ldots, \langle t_\ell, Y_\ell\rangle, \langle s, X\rangle$ be a path from the root $\langle t_0, Y_0\rangle$ of the connected component of $\langle s, X\rangle$ in T_n. Essentially what we want to do is to extend $A^s_X(n)$ as we did in the warm-up proof. But to maintain $(*)$, we first need to "prepare" the path $\langle t_0, Y_0\rangle, \langle t_1, Y_1\rangle, \ldots, \langle t_\ell, Y_\ell\rangle, \langle s, X\rangle$ by extending each of those chains using $(*)$ (and losing $(*)$ in the process), then extend $A^s_X(n)$, and then use **R** to recover the property $(*)$. See Figure 2 for a diagram showing how we do these extensions.

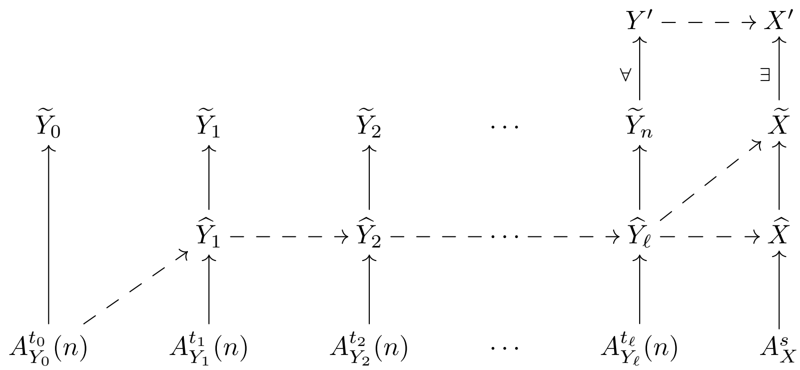


Figure 2. The extensions of possibilities in the construction. The dotted line shows the relation $\mathcal{R}$. For all Y' as shown, there is an X' filling in the diagram.

Let $\widehat{Y}_0 = A^{t_0}_{Y_0}(n)$. By $(*)$, there is $\widehat{Y}_1 \geq A^{t_1}_{Y_1}(n)$ such that $\widehat{Y}_0 \mathcal{R} \widehat{Y}_1$. Then $\widehat{Y}_1 \geq A^{t_1}_{Y_1}(n)$, so again by $(*)$ there is $\widehat{Y}_2 \geq A^{t_2}_{Y_2}(n)$ such that $\widehat{Y}_1 \mathcal{R} \widehat{Y}_2$. Continuing in this way, we get that $\mathcal{R}$ relates $\widehat{Y}_0$ to $\widehat{Y}_1$, $\widehat{Y}_1$ to $\widehat{Y}_2$, and so on until $\widehat{Y}_\ell$ is related to $\widehat{X} \geq A^s_X(n)$. This completes the "preparation."

Now in each case $\epsilon = 0$ or $\epsilon = 1$, we will define $\widetilde{X} \geq \widehat{X}$.

$\epsilon = 0$: Choose $\widetilde{X} \geq \widehat{X}$ such that either $\widetilde{X} \models \varphi_k$ or $\widetilde{X} \models \neg\varphi_k$, and so that either $\widetilde{X} \geq X_k$, or $\widetilde{X}$ and X_k have no common refinement.

$\epsilon = 1$: If there is $Z \in \mathcal{P}$ such that $\widehat{X} \mathcal{R} Z$ and $Z \models \varphi_k$, choose u such that $\langle u, Z\rangle$ has no edge in T_n, and is greater than any other pair connected to any edge in T_n. Let T_{n+1} be T_n with an additional edge from $\langle s, X\rangle$ to $\langle u, Z\rangle$. Using **R**, choose $\widetilde{X} \geq \widehat{X}$ such that for all $X' \geq \widetilde{X}$, there is $Z' \geq A^u_Z(n)$ with $X' \mathcal{R} Z'$ (this is to satisfy $(*)$).

Now we need to recover $(*)$. Note that $\mathcal{R}$ relates $\widehat{Y}_\ell$ to $\widetilde{X}$ by **P2**. Using **R**, choose $\widetilde{Y}_\ell \geq \widehat{Y}_\ell$ such that for all $Y''_\ell \geq \widetilde{Y}_n$, there is $X'' \geq \widetilde{X}$ with $Y'' \mathcal{R} X''$. Then using **R** again, choose $\widetilde{Y}_{\ell-1} \geq \widehat{Y}_{\ell-1}$ such that for all $Y''_{\ell-1} \geq \widetilde{Y}_{\ell-1}$ there is $Y''_\ell \geq \widetilde{Y}_\ell$ with $Y''_{\ell-1} \mathcal{R} Y''_\ell$. Continue in this way to define $\widetilde{Y}_0, \ldots, \widetilde{Y}_\ell$. Set $A^{t_i}_{Y_i}(n+1) = \widetilde{Y}_i$. Set $A^s_X(n+1) = \widetilde{X}$. For each other $\langle u, Z\rangle$, set $A^u_Z(n+1) = A^u_Z(n)$. It is easy to see that $(*)$ remains satisfied. Also, $A^{t_\ell}_{Y_\ell}(n)$ is related by $\mathcal{R}$ to $A^s_X(n+1)$.

This completes the construction. Let T be the union of the T_n (i.e., all of the edges which were in any of the T_n).

Claim 1. *For each $X \in \mathcal{P}$, $s \in \omega$, and formula φ, there is an n such that $A^s_X(n) \models \varphi$ or $A^s_X(n) \models \neg\varphi$. Similarly, for each $X \in \mathcal{P}$, $s \in \omega$, and possibility Y, there is an n such that $A^s_X(n) \geq Y$, or $A^s_X(n)$ and Y have no common refinements.*

Proof. Let k be such that $\varphi = \varphi_k$ and i be such that $X = X_i$. Let $n+1 = 2\langle s, i, k\rangle$; then at stage $n+1$ of the construction, we set $A^s_X(n+1)$ to be a refinement of a possibility X' with $X' \models \varphi$ or $X' \models \neg\varphi$; by persistence, either $A^s_X(n+1) \models \varphi$ or $A^s_X(n+1) \models \neg\varphi$. The proof of the second claim is similar. □

Claim 2. *For each $X, Y \in \mathcal{P}$ and $s, t \in \omega$ with an edge from $\langle s, X\rangle$ to $\langle t, Y\rangle$ in T, and for every n, there is m such that $A^s_X(n)\mathcal{R}A^t_Y(m)$.*

Proof. Recall that if i is the index of Y, then at each stage $n+1 = 2\langle t, i, k\rangle + \epsilon$ for any k and ϵ, we ensured that $A^s_X(n)\mathcal{R}A^t_Y(n+1)$. Thus for infinitely many n, there is m such that $A^s_X(n)\mathcal{R}A^t_Y(m)$. So each n has some $n' \geq n$ and m such that $A^s_X(n')\mathcal{R}A^t_Y(m)$; by **P1**, $A^s_X(n)\mathcal{R}A^t_Y(m)$. This suffices to prove the claim. □

Claim 3. *Let $X \in \mathcal{P}$, $s \in \omega$, and let φ be a formula. Suppose that for each m, there is Y_m with $Y_m \models \varphi$ and $A^s_X(m)\mathcal{R}Y_m$. Then there are Y and t, with $Y \models \varphi$, such that $T(\langle s, X\rangle, \langle t, Y\rangle)$.*

Proof. Let i be such that $X = X_i$ and k such that $\varphi = \varphi_k$. Let n be such that $n+1 = 2\langle s, i, k\rangle + 1$. Let Y be such that $Y \models \varphi$ and $A^s_X(n+1)\mathcal{R}Y$; then, for the $X' \leq A^s_X(n+1)$ defined at stage $n+1$, $X'\mathcal{R}Y$. So at stage $n+1$ of the construction, we find such a Y and t, and we put an edge between $\langle s, X\rangle$ and $\langle t, Y\rangle$ in T_{n+1}. □

We are now ready to define our Kripke model $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. For each $\langle s, X\rangle$, let $\widehat{A}^s_X$ be the downwards closure of the chain A^s_X. By Claim 1, this is a maximal order ideal. Let $\mathcal{W} = \{\widehat{A}^s_X : X \in \mathcal{P} \text{ and } s \in \omega\}$. Define $I\mathcal{S}J$ if for each $X \in I$, there is $Y \in J$ with $X\mathcal{R}Y$. Claim 2 implies that if in T there is an edge from $\langle s, X\rangle$ to $\langle t, Y\rangle$, then $\widehat{A}^s_X\mathcal{S}\widehat{A}^s_Y$. Define $U(I, p) = \mathtt{T}$ if, for some $X \in I$, $V(X, p) = \mathtt{T}$; similarly, define $U(I, p) = \mathtt{F}$ if, for some $X \in I$, $V(X, p) = \mathtt{F}$. By Claim 1, we are in exactly one of these two cases.

Claim 4. *For each sentence φ, $\mathcal{K}, I \models \varphi$ if and only if for some $X \in I$, $\mathcal{M}, X \models \varphi$.*

Proof. Let $\langle s, X\rangle$ be such that $I = \widehat{A}^s_X$. Then for some $Y \in I$, $\mathcal{M}, Y \models \varphi$ if and only if for some n, $\mathcal{M}, A^s_X(n) \models \varphi$. The proof is by induction on the complexity of the formula φ. If φ is p, then this follows from the definition of U. For a sentence $\varphi \wedge \psi$,

$$\begin{aligned}
\mathcal{K}, \widehat{A}^s_X \models \varphi \wedge \psi \quad &\Leftrightarrow \quad \mathcal{K}, \widehat{A}^s_X \models \varphi \text{ and } \mathcal{K}, \widehat{A}^s_X \models \psi \\
&\Leftrightarrow \quad (\exists m)\mathcal{M}, A^s_X(m) \models \varphi \text{ and } (\exists n)\mathcal{M}, A^s_X(n) \models \psi \\
&\Leftrightarrow \quad (\exists n)\mathcal{M}, A^s_X(n) \models \varphi \text{ and } \mathcal{M}, A^s_X(n) \models \psi \\
&\Leftrightarrow \quad (\exists n)\mathcal{M}, A^s_X(n) \models \varphi \wedge \psi
\end{aligned}$$

using persistence on the third line. For $\neg\varphi$,

$$\begin{aligned}
\mathcal{K}, \widehat{A}^s_X \models \neg\varphi \quad &\Leftrightarrow \quad \mathcal{K}, \widehat{A}^s_X \not\models \varphi \\
&\Leftrightarrow \quad (\forall n)\mathcal{M}, A^s_X(n) \not\models \varphi \\
&\Leftrightarrow \quad (\exists n)\mathcal{M}, A^s_X(n) \models \neg\varphi
\end{aligned}$$

since by Claim 1, for some n $\mathcal{M}, A^s_X(n) \models \varphi$ or $\mathcal{M}, A^s_X(n) \models \neg\varphi$.

For $\Box\varphi$, suppose that for all n, $\mathcal{M}, A^s_X(n) \not\models \Box\varphi$. Then, for each n, there is a Y such that $A^s_X(n)\mathcal{R}Y$ and $\mathcal{M}, Y \not\models \varphi$; refining Y if necessary, we may assume that $\mathcal{M}, Y \models \neg\varphi$. Then by Claim 3, there is some such Y and $t \in \omega$ with an edge between $\langle s, X\rangle$ and $\langle t, Y\rangle$ in T. By Claim 2, $\widehat{A}^s_X\mathcal{S}\widehat{A}^t_Y$. Now $\mathcal{K}, \widehat{A}^t_Y \models \neg\varphi$, so $\mathcal{K}, \widehat{A}^s_X \not\models \Box\varphi$. Thus we have shown that if $\mathcal{K}, \widehat{A}^s_X \models \Box\varphi$, then $\mathcal{M}, A^s_X(n) \models \Box\varphi$ for some n.

Now suppose that for some n, $\mathcal{M}, A^s_X(n) \models \Box\varphi$. Let $\langle t, Y\rangle$ be such that $\widehat{A}^s_X \mathcal{S} \widehat{A}^t_Y$. Then there is m such that $A^s_X(n) \mathcal{R} A^t_Y(m)$, and so $\mathcal{M}, A^t_Y(m) \models \varphi$. Hence $\mathcal{M}, \widehat{A}^t_Y \models \varphi$. Since $\langle t, Y\rangle$ was arbitrary, $\mathcal{K}, \widehat{A}^s_X \models \Box\varphi$. □

Now we will prove Theorem 4.4. We now assume that our possibility model is strong, and we have to produce a strong worldification. To meet the requirements for (W6) we must add new edges to our tree, except that they behave differently from the edges we added previously. To deal with this, we use the colours red and blue to distinguish between the edges of different types.

Proof of Theorem 4.4. Let $\mathcal{M}$ be a strong countable possibility model in a countable language. We modify the construction from Theorem 1.2. We will make a small modification to the trees from that theorem. In T_n we will now have two types of edges, red and blue. The edges we added in Theorem 1.2 will be the red edges, and the blue edges will be added for the sake of the extra condition in the statement of this theorem. We call $\langle s, X\rangle$ a *red* child of $\langle t, Y\rangle$ if there is a red edge from $\langle t, Y\rangle$ to $\langle s, X\rangle$, and a *blue* child if there is a blue edge. $(*)$ from Theorem 1.2 will hold for the red edges:

$(*)$: If $\langle s, X\rangle$ is a red child of $\langle t, Y\rangle$ in T_n, then for all $Y' \geq A^t_Y(n)$, there is $X' \geq A^s_X(n)$ such that $Y' \mathcal{R} X'$.

We have a new property $(\dagger)$ for the blue edges:

$(\dagger)$: If $\langle s, X\rangle$ is a blue child of $\langle t, Y\rangle$ in T_n, then $A^s_X(n) \mathcal{R} A^t_Y(n)$.

Note that the direction of the accessibility relation here is the opposite of that in $(*)$.

The construction begins in the same way with $A^s_X(0) = X$ for each $X \in \mathcal{P}$ and $s \in \omega$. Suppose that we have defined $A^s_X(0), \ldots, A^s_X(n)$ for each $\langle s, X\rangle$. Write $n+1 = 3\langle s, i, k\rangle + \epsilon$ where ϵ is 0, 1, or 2. Let $X = X_i$. Let $\langle t_0, Y_0\rangle, \ldots, \langle t_\ell, Y_\ell\rangle, \langle s, X\rangle$ be a path from the root $\langle t_0, Y_0\rangle$ of the connected component of $\langle s, X\rangle$ in T_n; some of the edges in this path may be red, and others may be blue. Choose $\widehat{Y}_0 = A^{t_0}_{Y_0}(n)$. Now, if $\langle t_1, Y_1\rangle$ is a red child of $\langle t_0, Y_0\rangle$, using $(*)$ choose $\widehat{Y}_1 \geq A^{t_1}_{Y_1}(n)$ such that $\widehat{Y}_0 \mathcal{R} \widehat{Y}_1$. If $\langle t_1, Y_1\rangle$ is a blue child of $\langle t_0, Y_0\rangle$, using **R** choose $\widehat{Y}_1 \geq A^{t_1}_{Y_1}(n)$ such that for all $\widehat{Y}'_1 \geq \widehat{Y}_1$, there is $\widehat{Y}'_0 \geq \widehat{Y}_0$ with $\widehat{Y}'_1 \mathcal{R} \widehat{Y}'_0$. Continuing in this way, we get $\widehat{Y}_0 \geq A^{t_0}_{Y_0}(n), \widehat{Y}_1 \geq A^{t_1}_{Y_1}(n), \ldots, \widehat{Y}_\ell \geq A^{t_\ell}_{Y_\ell}(n)$, and $\widehat{X} \geq A^s_X(n)$ such that if $\langle t_{i+1}, Y_{i+1}\rangle$ is a red child of $\langle t_i, Y_i\rangle$, then $\widehat{Y}_i \mathcal{R} \widehat{Y}_{i+1}$, and if $\langle t_{i+1}, Y_{i+1}\rangle$ is a blue child of $\langle t_i, Y_i\rangle$, then for all $\widehat{Y}'_{i+1} \geq \widehat{Y}_{i+1}$, there is $\widehat{Y}'_i \geq \widehat{Y}_i$ with $\widehat{Y}'_{i+1} \mathcal{R} \widehat{Y}'_i$.

Recall that $n+1 = 3\langle s, i, k\rangle + \epsilon$. Now for each ϵ, we will define $\widetilde{X} \geq \widehat{X}$.

$\epsilon = 0$: Same as Theorem 1.2. Choose $\widetilde{X} \geq \widehat{X}$ such that either $\widetilde{X} \models \varphi_k$ or $\widetilde{X} \models \neg\varphi_k$, and so that either $\widetilde{X} \geq X_k$, or $\widetilde{X}$ and X_k have no common refinement.

$\epsilon = 1$: Same as Theorem 1.2, adding a red edge. If there is $Z \in \mathcal{P}$ such that $\widehat{X} \mathcal{R} Z$ and $Z \models \varphi_k$, choose u such that $\langle u, Z\rangle$ has no edge in T_n and is greater than any other pair connected to any edge in T_n. Let T_{n+1} be T_n with an additional red edge from $\langle s, X\rangle$ to $\langle u, Z\rangle$. Using **R**, choose $\widetilde{X} \geq \widehat{X}$ such that for all $X' \geq \widetilde{X}$, there is $Z' \geq A^u_Z(n)$ with $X' \mathcal{R} Z'$ (this is to satisfy $(*)$).

$\epsilon = 2$: Let $Z = X_k$. If $Z\mathcal{R}\widehat{X}$, then choose u such that $\langle u, Z\rangle$ has no edge in T_n and is greater than any other pair connected to any edge in T_n. Let T_{n+1} be T_n with an additional blue edge from $\langle s, X\rangle$ to $\langle u, Z\rangle$.

Now we need to recover $(*)$ and $(\dagger)$. If $\langle s, X\rangle$ is a red child of $\langle t_\ell, Y_\ell\rangle$, then note that $\mathcal{R}$ relates $\widehat{Y}_\ell$ to $\widehat{X}$ by **P2**. Using **R**, choose $\widetilde{Y}_\ell \geq \widehat{Y}_\ell$ such that for all $Y''_\ell \geq \widetilde{Y}_n$, there is $X'' \geq \widetilde{X}$ with $Y''\mathcal{R}X''$. Thus we have recovered $(*)$ between $\langle s, X\rangle$ and $\langle t_\ell, Y_\ell\rangle$. If $\langle s, X\rangle$ is a blue child of $\langle t_\ell, Y_\ell\rangle$, then by choice of $\widehat{Y}_\ell$, there is $\widetilde{Y}_\ell \geq \widehat{Y}_\ell$ such that $\widetilde{X}\mathcal{R}\widetilde{Y}_\ell$. Thus we have recovered $(\dagger)$ between $\langle s, X\rangle$ and $\langle t_\ell, Y_\ell\rangle$. Continue in this way to define $\widetilde{Y}_0, \ldots, \widetilde{Y}_\ell$. Set $A^{t_i}_{Y_i}(n+1) = \widetilde{Y}_i$. Set $A^s_X(n+1) = \widetilde{X}$. For each other $\langle u, Z\rangle$, set $A^u_Z(n+1) = A^u_Z(n)$. Note that both $(*)$ and $(\dagger)$ have the property that if they held between $\langle u, Z\rangle$ and its child $\langle u', Z'\rangle$ at stage n, and if $A^u_Z(n+1) \geq A^u_Z(n)$ and $A^{u'}_{Z'}(n+1) = A^{u'}_{Z'}(n)$, then $(*)$ and $(\dagger)$ hold between $\langle u, Z\rangle$ and $\langle u', Z'\rangle$ at stage $n+1$. Thus $(*)$ and $(\dagger)$ both hold for T_{n+1}.

Define the model $\mathcal{K}$ in the same way as before. The proofs of the claims in Theorem 1.2 still hold for the red edges and so $\mathcal{K}$ is a worldification of $\mathcal{M}$ via Φ. Also, if there is a blue edge from $\langle s, X\rangle$ to $\langle t, Y\rangle$ in T, then $(\dagger)$ implies that $\widehat{A}^t_Y \mathcal{S} \widehat{A}^s_X$. We now have a new claim.

Claim 5. *$X\mathcal{R}Y$ if and only if for all w with $Y \in \Phi(w)$ there is v with $X \in \Phi(v)$ and $v\mathcal{S}w$.*

Proof. Suppose that $X\mathcal{R}Y$. Let $w = \widehat{A}^s_Z$ be such that $Y \in \Phi(w)$. Then, for sufficiently large n, $X\mathcal{R}A^s_Z(n)$. So at some stage, we put a blue edge from $\langle s, Z\rangle$ to $\langle u, X\rangle$. Then $\widehat{A}^u_X \mathcal{S} w$. Note that $X \in \Phi(\widehat{A}^u_X)$.

Suppose that $\neg X\mathcal{R}Y$. Then since $\mathcal{M}$ is strong, there is $Y' \geq Y$ such that for all $Y'' \geq Y'$, $\neg X\mathcal{R}Y''$. Fix w with $Y' \in \Phi(w)$. Then it follows from (W5) that for all v with $X \in \Phi(v)$, $\neg v\mathcal{S}w$. □

This completes the proof. □

5. Uncountable models

The representation theorem, Theorem 1.1, and the theorem on worldifications, Theorem 1.2, required that the possibility model and the language were countable. We will now construct two possibility models, one which shows that the assumption that the language was countable was required to prove both theorems, and a second that shows that there are uncountable models with no worldifications. This second model can easily be made to be separative, but the example seems to rely in an essential way on not being strong. We do not know whether the representation theorem can be extended to uncountable models.

Proposition 5.1. *There is a separative and strong possibility model $\mathcal{M}$ with countably many possibilities in a language with uncountably many propositional variables which does not have any worldifications and which is not isomorphic to the possibilization of any Kripke model.*

In fact, $\mathcal{M}$ will not even use the refinability relation.

Proof. Let $2^{<\omega}$ be the infinite binary tree, that is, the elements of $2^{<\omega}$ are the finite string of 0's and 1's. Let 2^ω be the set of infinite binary strings, which we view as paths through $2^{<\omega}$. Let P be a set of continuum-many propositional variables.

Let $f\colon P \to 2^\omega$ be a bijection between P and 2^ω. Let $\mathcal{P} = 2^{<\omega}$. The refinement relation $\leq$ is the natural extension relation on strings. The accessibility relation $\mathcal{R}$ is trivially empty. Define $V(\sigma, p)$ as follows. Let $\pi = f(p)$. Either σ is an initial segment of π, in which case we set $V(\sigma, p) = ?$, or there is some smallest initial segment τ of σ which is not an initial segment of π (so every proper initial segment of τ is an initial segment of π, but the last entry of τ differs from the corresponding entry of π). If $\tau = \sigma$, i.e., if σ differs only from π at its last entry, then set $V(\sigma, p) = ?$. Otherwise, if σ extends $\tau\hat{\ }1$, set $V(\sigma, p) = \mathtt{T}$, and otherwise if σ extends $\tau\hat{\ }0$, set $V(\sigma, p) = \mathtt{F}$. Then $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ is a possibility model; it also trivially strong as the accessibility relation is trivial, and it is not hard to see that it is also separative. Every ascending chain in $\mathcal{P}$ corresponds to a path π through 2^ω, and this path does not decide whether $p = f^{-1}(\pi)$ is true or false. Thus by (W4) none of the ascending chains in $\mathcal{P}$ can be in a worldification. By (W2), there are no worldifications of this model.

One sees from the definitions that if the possibilization of a Kripke model has empty accessibility relation, then the associated Kripke model is a worldification of the possibilization. Thus $\mathcal{M}$ is not isomorphic to a possibilization. □

Now we will give the second example, which is more complicated than the first. While this model is not separative, it can easily be transformed into a separative model which is still a counterexample. However, the fact that it is not strong seems to be used in an essential way.

Proposition 5.2. *There is a possibility model $\mathcal{M}$ with uncountably many possibilities in a language with countably many propositional variables which does not have any worldifications, and which is not isomorphic to the possibilization of any Kripke model.*

Proof. By a tree, we now mean a poset $(T, \precsim)$ such that $\{b : b \prec a\}$ is well-ordered for each a. We call the order type of $\{b : b \prec a\}$ the height of a, height(a). The height of a tree is the supremum of the heights of its elements. A path through a tree is a linearly ordered set in the tree closed under predecessor. Let $(T, \precsim)$ be a well-pruned Aronszajn tree, that is, a tree with:

1. (1) height ω_1,
2. (2) every element of T has countable height,
3. (3) every path in T is countable,
4. (4) for each element a of height α, and each β with $\omega_1 > \beta > \alpha$, there is an element $b \succsim a$ of height β.

The first three properties are what it means to be an Aronszajn tree, and the last says that the tree is well-pruned (see [Kun80, pp. 69-72]). Let $\mathcal{P}$ be the disjoint union of ω_1 and T.[5] Define the refinement relation $\leq$ on $\mathcal{P}$ by making it the natural ordering on ω_1, and the tree ordering on T, but having elements of ω_1 and of T be incomparable. Set $\alpha\mathcal{R}\sigma$ if $\alpha \in \omega_1$ and $\sigma \in T$ and height$(\sigma) \geq \alpha$. We will have one propositional variable p. Set $V(X, p) = \mathtt{T}$ for all $X \in \mathcal{P}$. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$.

For **P1**, if $\alpha\mathcal{R}\sigma$ and $\beta \leq \alpha$, then height$(\sigma) \geq \alpha \geq \beta$ and so $\beta\mathcal{R}\sigma$. For **P2**, if $\alpha\mathcal{R}\sigma$ and $\tau \succsim \sigma$, then height$(\tau) \geq$ height$(\sigma) \geq \alpha$ and so $\alpha\mathcal{R}\tau$. For **R**, suppose that $\alpha\mathcal{R}\sigma$ so that height$(\sigma) \geq \alpha$. Then for all $\beta \geq \alpha$, since T is well-pruned there

[5] One can make the model separative by replacing ω_1 with the tree $2^{<\omega_1}$, using the ordinal length of a string in $2^{<\omega_1}$ to define the accessibility relation.

is $\tau \gtrsim \sigma$ of height at least β, and hence $\beta\mathcal{R}\tau$. **Refinement** and **Persistence** are clear. Thus $\mathcal{M}$ is a possibility model.

Now we claim that there is no worldification of $\mathcal{M}$. Suppose that there was, say $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. Then $U(w,p) = \mathtt{T}$ for all w by (W4). By (W2), let w be such that $0 \in \Phi(w)$ (where $0 \in \omega_1$); in fact, by (W1) we get $\Phi(w) = \omega_1$. Then, since $0\mathcal{R}\varnothing$ (where $\varnothing \in T$ is the empty string) and $V(\varnothing, p) = \mathtt{T}$, $\mathcal{M}, 0 \models \Diamond p$. By (W4), $\mathcal{K}, w \models \Diamond p$. Let v be such that $w\mathcal{S}v$ and $\mathcal{K}, v \models p$. By (W1), $\Phi(v) \subseteq T$ is a path through T. Since T is an Aronszajn tree, there is a countable bound on the height of the elements of $\Phi(v)$. On the other hand, by (W5), for each $\alpha \in \omega_1$, there is $\sigma \in \Phi(v)$ with $\alpha\mathcal{R}\sigma$ and hence height$(\sigma) \geq \alpha$, so that the heights of elements of $\Phi(v)$ are unbounded below ω_1. This is a contradiction. So $\mathcal{M}$ has no worldification.

A similar argument will show that $\mathcal{M}$ is not isomorphic to the possibilization of a Kripke model. Suppose that it was isomorphic to the possibilization of a Kripke model $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$, and identify the possibilities of $\mathcal{M}$ with the possibilities of this Kripke model. By (P6), $U(w,p) = \mathtt{T}$ for all $w \in \mathcal{W}$. Now the possibility 0 is identified with a non-empty set of total worlds; fix an element w from this set. □

6. Frames

In this last section we will discuss what one might do to extend the worldification construction to frames. One can of course consider the representation theorem for frames, but our goal here is just to give a little bit of the flavour and to leave the rest for future work.

6.1. No worldifications of Basic Possibility Frames.

Recall from Definition 2.1 the definition of a basic possibility frame. In this section we will consider worldifications on the level of frames. By a frame-worldification of a possibility frame $\mathcal{F}$, we mean a Kripke frame $\mathcal{K}$ satisfying (W1)-(W3) and (W5) of the definition of a worldification and such that any possibility model based on $\mathcal{F}$ gives rise to a model-worldification based on $\mathcal{K}$.

Definition 6.1. Let $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq)$ be a (basic) possibility frame and let $\mathcal{F} = (\mathcal{W}, \mathcal{S})$ be a Kripke model. $\mathcal{F}$ is a *frame-worldification* of $\mathcal{G}$ via an embedding $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ which assigns to each total world $w \in \mathcal{W}$ a non-empty set of possibilities $\Phi(w) \subseteq \mathcal{P}$ if:

- (W1) for each world $w \in \mathcal{W}$, $\Phi(w)$ is a *maximal order ideal* in the poset $(\mathcal{P}, \leq)$, i.e.,
  - (a) $\Phi(w)$ is downwards-closed under refinement,
  - (b) any two elements of $\Phi(w)$ have a common refinement in $\Phi(w)$, and
  - (c) $\Phi(w)$ is maximal with these two properties;
- (W2) for each possibility $X \in \mathcal{P}$, there is a world $w \in \mathcal{W}$ such that $X \in \Phi(w)$;
- (W3) any two distinct total worlds $v, w \in \mathcal{W}$ are separated by possibilities, that is, there is $X \in \Phi(v) \setminus \Phi(w)$; and
- (W5) for each pair of worlds $w, v \in \mathcal{W}$, $w\mathcal{S}v$ if and only if for each $X \in \Phi(w)$ there is $Y \in \Phi(v)$ such that $X\mathcal{R}Y$;

and such that:

- for any possibility model $\mathcal{M}$ based on $\mathcal{G}$, there is a Kripke model $\mathcal{K}$ based on $\mathcal{F}$ such that $\mathcal{K}$ is a worldification of $\mathcal{M}$ via Φ, and
- for any Kripke model $\mathcal{K}$ based on $\mathcal{F}$, there is a possibility model $\mathcal{M}$ based on $\mathcal{G}$ such that $\mathcal{K}$ is a worldification of $\mathcal{M}$ via Φ.

There are basic possibility frames without a worldification. The issue is that in the construction of Theorem 1.2, at some stages we extended a possibility X to a further refinement X' which decided some formula φ. This required us to have countably many definable sets of possibilities; but there may be uncountably many sets of possibilities which are definable in some model based on a countable frame.

Proposition 6.2. *There is a countable basic frame $\mathcal{F}$ with no frame-worldification.*

Proof. Consider the following example of a basic possibility frame $\mathcal{G}$ which is similar to Proposition 5.1. Let $\mathcal{P}$ be the infinite binary tree $2^{<\omega}$. The accessibility relation $\mathcal{R}$ is trivially empty, and $\leq$ is the natural relation on extension of strings. We claim that there cannot possibly be a frame-worldification of $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq)$.

Given a frame-worldification $\mathcal{F}$ of $\mathcal{G}$, pick a world w in $\mathcal{F}$; w corresponds to some infinite path π through the binary tree. Now, using a single propositional variable, we can define a valuation V to get a possibility model $\mathcal{M}$ based on $\mathcal{G}$. Define $V(\sigma, p)$ as in Proposition 5.1, as follows. Either σ is an initial segment of π, in which case we set $V(\sigma, p) = ?$, or there is some smallest initial segment τ of σ which is not an initial segment of π (so every proper initial segment of τ is an initial segment of π, but the last entry of τ differs from the corresponding entry of π). If $\tau = \sigma$, i.e., if σ differs only from π at its last entry, then set $V(\sigma, p) = ?$. Otherwise, if σ extends $\tau\hat{}1$, set $V(\sigma, p) = \mathtt{T}$, and otherwise if σ extends $\tau\hat{}0$, set $V(\sigma, p) = \mathtt{F}$.

Then V satisfies **Persistence** and **Refinability**. However, the ascending chain π never decides p, and so there is no model-worldification of $\mathcal{M}$ based on $\mathcal{F}$. Thus $\mathcal{F}$ is not a frame-worldification of $\mathcal{G}$. $\square$

Note that while we know that, for every valuation V on $\mathcal{G}$, the model obtained from this valuation admits a worldification, the proof shows that there is no single frame-worldification $\mathcal{F}$ of $\mathcal{G}$ that works for every valuation.

6.2. **Worldifications of General Possibility Frames.** If we are willing to work with general frames, then we can make a worldification construction. Holliday [Hol15, Definition 2.21] has a natural definition of a general possibility frame.

Definition 6.3. $\mathcal{F} = \langle \mathcal{P}, \mathcal{R}, \leq, \mathcal{A} \rangle$ is a (general) possibility frame if $\langle \mathcal{P}, \mathcal{R}, \leq \rangle$ is a basic possibility frame and $\mathcal{A} \subseteq \wp(\mathcal{P})$, the set of admissible propositions, satisfies:

(1) $\varnothing, \mathcal{P} \in \mathcal{A}$;
(2) Given $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$;
(3) Given $A \in \mathcal{A}$, $A^* = \{X \in \mathcal{P} : \forall Y \geq X, Y \notin A\} \in \mathcal{A}$;
(4) Given $A \in \mathcal{A}$, $\Box A = \{X \in \mathcal{P} : (\forall Y) X \mathcal{R} Y \Rightarrow Y \in A\} \in \mathcal{A}$;
(5) Each $A \in \mathcal{A}$ is regular open in the upset topology of $(\mathcal{P}, \leq)$.

A set A is regular open set in the upset topology if and only if it satisfies the following conditions of persistence and refinability for sets:

(i) for each $X \in A$ and $X' \geq X$, $X' \in A$, and
(ii) for each $X \in \mathcal{P}$, if $X \notin A$, then there is $X' \geq X$ such that for all $X'' \geq X'$, $X'' \notin A$.

A possibility model $\mathcal{M} = \langle \mathcal{P}, \mathcal{R}, \leq, V \rangle$ is based on $\mathcal{F} = \langle \mathcal{P}, \mathcal{R}, \leq, \mathcal{A} \rangle$ if $\{X : V(X, p) = \mathtt{T}\} \in \mathcal{A}$ for each X and p.

Condition (3) corresponds to the usual condition (for general Kripke frames) of closure under complements. (For a review of general Kripke frames, see Section 5.5

of [BdRV01].) If $\mathcal{M}$ is a possibility model based on a general frame $\mathcal{F}$, then the sets of possibilities definable in $\mathcal{M}$ are all admissible in $\mathcal{F}$

If $\mathcal{F}$ is a general possibility frame, $(p_i)_{i\in I}$ are propositional variables, and $(A_i)_{i\in I}$ are admissible sets, then setting $V(X, p_i) = \mathtt{T}$ if $X \in A_i$, $V(X, p_i) = \mathtt{F}$ if $X \in A_i^*$, and $V(X, p_i) = ?$ otherwise determines a possibility model based on $\mathcal{F}$. The requirement that each admissible set be regular open ensures that **Persistence** and **Refinability** are satisfied.

We can define frame-worldifications of general possibility frames as follows.

Definition 6.4. Let $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq, \mathcal{A})$ be a general possibility frame and let $\mathcal{F} = (\mathcal{W}, \mathcal{S}, \mathcal{B})$ be a general Kripke frame. $\mathcal{F}$ is a *frame-worldification* of $\mathcal{G}$ via an embedding $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ which assigns to each total world $w \in \mathcal{W}$ a non-empty set of possibilities $\Phi(w) \subseteq \mathcal{P}$ if:

(W1) for each world $w \in \mathcal{W}$, $\Phi(w)$ is a *maximal order ideal* in the poset $(\mathcal{P}, \leq)$, i.e.,
  (a) $\Phi(w)$ is downwards-closed under refinement,
  (b) any two elements of $\Phi(w)$ have a common refinement in $\Phi(w)$, and
  (c) $\Phi(w)$ is maximal with these two properties;

(W2) for each possibility $X \in \mathcal{P}$, there is a world $w \in \mathcal{W}$ such that $X \in \Phi(w)$;

(W3) any two distinct total worlds $v, w \in \mathcal{W}$ are separated by possibilities, that is, there is $X \in \Phi(v) \setminus \Phi(w)$; and

(W5) for each pair of worlds $w, v \in \mathcal{W}$, $w\mathcal{S}v$ if and only if for each $X \in \Phi(w)$ there is $Y \in \Phi(v)$ such that $X\mathcal{R}Y$;

and such that:

- for any possibility model $\mathcal{M}$ based on $\mathcal{G}$, there is a Kripke model $\mathcal{K}$ based on $\mathcal{F}$ such that $\mathcal{K}$ is a worldification of $\mathcal{M}$ via Φ, and
- for any Kripke model $\mathcal{K}$ based on $\mathcal{F}$, there is a possibility model $\mathcal{M}$ based on $\mathcal{G}$ such that $\mathcal{K}$ is a worldification of $\mathcal{M}$ via Φ.

(Note this is almost word-for-word the same definition as for basic frames, though some of the words, such as "based on", now have a different meaning.)

We now prove Theorem 1.3, which says that a countable possibility frame with countably many admissible sets has a frame-worldification.

Proof of Theorem 1.3. Let $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq, \mathcal{A})$ be a countable general possibility frame with countably many admissible sets. For each $A \in \mathcal{A}$, we will have a propositional variable p_A. Define $\mathcal{M}$ a possibility model based on $\mathcal{G}$ with valuation $V(X, p_A) = \mathtt{T}$ if $X \in A$, $V(X, p_A) = \mathtt{F}$ if $X \in A^*$, and $V(X, p_A) = ?$ otherwise.

$\mathcal{M}$ is a countable model in a countable language. By Theorem 1.2, $\mathcal{M}$ has a worldification $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$, say via Φ. Let $\mathcal{B}$ be the collection of sets

$$B_A = \{w \in \mathcal{W} : \mathcal{K}, w \models p_A\} = \{w \in \mathcal{W} : (\exists X \in \Phi(w)) X \in A\}.$$

Claim 1. *$\mathcal{F} = (\mathcal{W}, \mathcal{S}, \mathcal{B})$ is a general Kripke frame.*

Proof. $U(w, p_\varnothing) = \mathtt{F}$ for all $w \in \mathcal{W}$, so $B_\varnothing = \varnothing \in \mathcal{B}$.

To see that $\mathcal{B}$ is closed under complements, we show that for $A \in \mathcal{A}$, the complement of B_A is B_A^*. We must show that for $w \in \mathcal{W}$, if $\mathcal{K}, w \not\models p_A$, then $\mathcal{K}, w \models p_{A^*}$. Since $\mathcal{K}, w \models \neg p_A$, there is some $X \in \Phi(w)$ such that $\mathcal{M}, X \models \neg p_A$. So for all $Y \geq X$, $Y \notin A$. Thus $X \in A^*$, and so $\mathcal{M}, X \models p_{A^*}$. But then $\mathcal{K}, w \models p_{A^*}$.

Now we will see that $\mathcal{B}$ is closed under intersections. Given $A, A' \in \mathcal{A}$, we will show that $B_A \cap B_{A'} = B_{A\cap A'}$. Suppose that $w \in B_A \cap B_{A'}$. Then $\mathcal{K}, w \models p_A \wedge p'_A$, and so there are $X \in \Phi(w)$ with $X \in A$ and $X' \in \Phi(w)$ with $X' \in A$. But then there is $X'' \in \Phi(w)$ with $X'' \geq X, X'$, and so $X'' \in A \cap A'$. Hence $\mathcal{M}, X \models p_{A\cap A'}$, and so $\mathcal{K}, w \models p_{A\cap A'}$. Thus $x \in B_{A\cap A'}$. The other direction is similar.

Finally, given $A \in \mathcal{A}$, we will show that $\Box B_A = \{w \in \mathcal{W} : (\forall v) w\mathcal{S}v \Rightarrow v \in B_A\}$ is equal to $B_{\Box A}$. First, suppose that for all v with $w\mathcal{S}v$, $v \in B_A$. Thus for all such v, $\mathcal{K}, v \models p_A$. So $\mathcal{K}, w \models \Box p_A$. There must be some $X \in \Phi(w)$ with $\mathcal{M}, X \models \Box p_A$. So for all Y with $X\mathcal{R}Y$, $\mathcal{M}, Y \models p_A$ and so $X \in \Box A$. Thus $\mathcal{M}, X \models p_{\Box A}$ and so $\mathcal{K}, w \models p_{\Box A}$. The other direction is similar. □

Finally, we want to check that $\mathcal{F}$ is a frame-worldification of $\mathcal{G}$ via Φ. Since $\mathcal{K}$ is a worldification of $\mathcal{M}$ via Φ, it suffices to check that for each possibility model $\mathcal{M}'$ based on $\mathcal{G}$ there is a Kripke model $\mathcal{K}'$ based on $\mathcal{F}$ such that $\mathcal{K}'$ is a worldification of $\mathcal{M}'$ via Φ, and that for each Kripke model $\mathcal{K}'$ based on $\mathcal{F}$ there is a possibility model $\mathcal{M}'$ based on $\mathcal{G}$ such that $\mathcal{K}'$ is a worldification of $\mathcal{M}'$ via Φ.

Claim 2. *For each possibility model $\mathcal{M}'$ based on $\mathcal{G}$, there a Kripke model $\mathcal{K}'$ based on $\mathcal{F}$ such that $\mathcal{K}'$ is a worldification of $\mathcal{M}'$ via Φ.*

Proof. Let $\mathcal{M}' = (\mathcal{P}, \mathcal{R}, \leq, V')$ be a possibility model based on $\mathcal{G}$. Define a valuation U' on $\mathcal{F}$ as follows. For each propositional variable q, let $A_q \in \mathcal{A}$ be such that $A_q = \{X : \mathcal{M}', X \models q\}$. Then define $U'(w, q) = \mathtt{T}$ if $w \in B_{A_q}$, and $U'(w, q) = \mathtt{F}$ if $w \notin B_{A_q}$. So $\mathcal{K}' = (\mathcal{W}, \mathcal{S}, U')$ a Kripke model based on $\mathcal{F}$.

Note that we have both that $\mathcal{M}', X \models q$ if and only if $\mathcal{M}, X \models p_{A_q}$, and that $\mathcal{K}', w \models q$ if and only if $\mathcal{K}, w \models p_{A_q}$. Given a formula φ in the language of $\mathcal{M}'$, we can translate φ to a formula φ^* in the language of $\mathcal{M}$ by replacing each variable q with p_{A_q}. Then $\mathcal{M}', X \models \varphi$ if and only if $\mathcal{M}, X \models \varphi^*$, and $\mathcal{K}', w \models \varphi$ if and only if $\mathcal{K}, w \models \varphi^*$. Since $\mathcal{K}$ is a worldification of $\mathcal{M}$, it follows that $\mathcal{K}'$ is a worldification of $\mathcal{M}'$. □

Claim 3. *For each Kripke model $\mathcal{K}'$ based on $\mathcal{F}$, there a possibility model $\mathcal{M}'$ based on $\mathcal{G}$ such that $\mathcal{K}'$ is a worldification of $\mathcal{M}'$ via Φ.*

Proof. Let $\mathcal{K}' = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model based on $\mathcal{F}$. Define a valuation V' on $\mathcal{G}$ as follows. For each propositional variable q, let $A_q \in \mathcal{A}$ be such that $B_{A_q} = \{w : \mathcal{K}', w \models q\}$. Then define $V'(X, q) = \mathtt{T}$ if $X \in A_q$, and $V'(X, q) = \mathtt{F}$ if $X \notin A_q$. The rest of the argument is similar to the previous claim. □

So we have shown that $\mathcal{F}$ is a frame-worldification of $\mathcal{G}$, completing the proof of the theorem. □

Though we used Theorem 1.2, on worldifications of countable possibility models in a countable language, to prove Theorem 1.3, the latter should be viewed as a generalization of the former. We can see this as follows. A possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ for a countable language induces a general possibility frame $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq, \mathcal{A})$ with a countable set $\mathcal{A}$ of admissible sets, namely the sets definable by formulas in $\mathcal{M}$. Then, applying Theorem 1.3 to $\mathcal{G}$, we get a frame-worldification $\mathcal{F}$ of $\mathcal{G}$, and by the last sentence of Definition 6.4, there is a worldification of $\mathcal{M}$ based on $\mathcal{F}$.

References

[BBH17] Johan van Benthem, Nick Bezhanishvili, and Wesley H. Holliday. A bimodal perspective on possibility semantics. *Journal of Logic and Computation*, 27(5):1353–1389, 2017.

[BdRV01] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2001.

[BH16] Guram Bezhanishvili and Wesley H. Holliday. Locales, nuclei, and Dragalin frames. In *Advances in Modal Logic. Vol. 11*, pages 177–196. King's Coll. Publ., London, 2016.

[Coh66] Paul J. Cohen. *Set theory and the continuum hypothesis*. W. A. Benjamin, Inc., New York-Amsterdam, 1966.

[Cre04] M. Cresswell. Possibility semantics for intuitionistic logic. *The Australasian Journal of Logic*, 2(0), 2004.

[Edg85] Dorothy Edgington. The paradox of knowability. *Mind*, 94(376):557–568, 1985.

[Gar13] James Garson. *What Logics Mean: From Proof Theory to Model-Theoretic Semantics*. Cambridge University Press, 2013.

[Hal13] Bob Hale. *Necessary Beings: An Essay on Ontology, Modality, and the Relations Between Them*. Oxford University Press, USA, 2013.

[Hol14] Wesley H. Holliday. Partiality and adjointness in modal logic. In Rajeev Goré, Barteld Kooi, and Agi Kurucz, editors, *Advances in Modal Logic*, pages 313–332. College Publications, London, 2014.

[Hol15] Wesley H. Holliday. Possibility frames and forcing for modal logic. UC Berkeley Working Paper in Logic and the Methodology of Science, available at http://escholarship.org/uc/item/5462j5b6, 2015.

[HT16] Matthew Harrison-Trainor. First-order possibility models and finitary completeness proofs. UC Berkeley Working Paper in Logic and the Methodology of Science, available at http://escholarship.org/uc/item/8ht6w3kk, 2016.

[Hum81] I. L. Humberstone. From worlds to possibilities. *J. Philos. Logic*, 10(3):313–399, 1981.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[Kun80] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980.

[Lit05] Tadeusz Litak. On notions of completeness weaker than Kripke completeness. In *Advances in modal logic. Vol. 5*, pages 149–169. King's Coll. Publ., London, 2005.

[Rum15] Ian Rumfitt. *The Boundary Stones of Thought: An Essay in the Philosophy of Logic*. Oxford University Press, Oxford, 2015.

[Yam17] Kentaro Yamamoto. Results in modal correspondence theory for possibility semantics. *Journal of Logic and Computation*, 2017. to appear.

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