Computable structure theory and infinitary logic

Part 1: Presentations and computations

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ASL NA Meeting 2026 Tutorial Part 1

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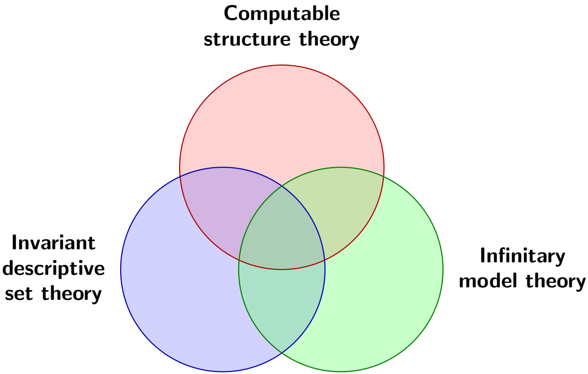
The goal of computable structure theory is to understand and measure this complexity.



Infinitary model theory: Model theory in the logic $\mathcal{L}_{\omega_1\omega}$ with countably infinite conjunctions and disjunctions.

Invariant descriptive set theory: Descriptive set theory of the action of S_∞ on the space of $\mathcal{L}$-structures.

There are three books on computable structure theory:

- Ash-Knight (2000): *Computable Structures and the Hyperarithmetical Hierarchy*.
- Montalbán (2021 and 2026): *Computable Structure Theory*, parts I and II.
- Downey-Melnikov (2026): *Computable Structure Theory: A Unified Approach*.

Outline

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Part 1: Presentations and computations

Part 2: Infinitary logic and forcing

Part 3: Back-and-forth and the Scott analysis

Computability-theoretic background

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There are computable pairing functions, so we can computably identify $\mathbb{N}^n$ with $\mathbb{N}$.

Computability-theoretic background

An important non-computable set is the Halting problem

$$0' = K = \{e \in \mathbb{N} : \varphi_e(e) \text{ halts}\}.$$

$0'$ is computably enumerable which means that there is a computable list of the elements of $0'$.

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A Turing degree is an equivalence class of Turing equivalence

$$A \equiv_T B \iff A \leq_T B \text{ and } B \leq_T A.$$

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We will see more of this in the later parts of the tutorial; for now, it is enough to know that the Turing degrees are a very rich structure, e.g., it is unknown if they have a non-trivial automorphism.

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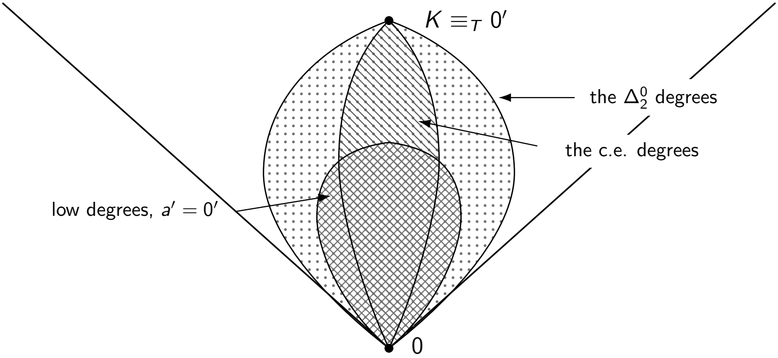

Background assumptions

Throughout this tutorial:

- All languages will be countable and computable.
- All structures will be countable.

Presentations of structures

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We often conflate $\mathcal{A}$ with its atomic diagram

$$\mathrm{Diag}(\mathcal{A}) = \bigoplus_{f \in \mathcal{L}} f^{\mathcal{A}} \oplus \bigoplus_{R \in \mathcal{L}} R^{\mathcal{A}} \oplus \bigoplus_{c \in \mathcal{L}} \{c\}^{\mathcal{A}}.$$

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Throughout the tutorial, all structures will have domain $\mathbb{N}$.

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Then $(\mathbb{N}, \prec) \cong (\mathbb{N}, <)$ is a computable copy.

If $0' = \{3, 0, 5, 2, \dots\}$,

$$0 \to 3 \to 2 \to 4 \to 7 \to 6 \to 1 \to 8 \to 10 \to 5 \to 12 \to \cdots$$

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In particular, $(\mathbb{N}, <)$ and $(\mathbb{N}, \prec)$ are not computably isomorphic.

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For example, if $0 \notin A$, $1 \in A$, and $2 \notin A$, then

$$\underbrace{1 \longrightarrow 0}_{0 \notin A} \longrightarrow \underbrace{2 \longrightarrow 3}_{1 \in A} \longrightarrow \underbrace{5 \longrightarrow 4}_{2 \notin A} \longrightarrow \cdots$$

Presentations of structures

It is the fact that a structure can have many different presentations that makes computable structure theory interesting.

Today, we will explore some of the basic ideas, objects, and types of questions in computable structure theory.

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A $\mathbb{Q}$-vector space is computably categorical if and only if it has finite dimension.

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Theorem (Goncharov)

For each $n \in \mathbb{N}$ there is a computable structure of computable dimension n.

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Theorem (Goncharov)

If $\mathcal{A} \cong \mathcal{B}$ are computable structures which are $0'$-computably isomorphic but not computably isomorphic, then their computable dimension is ∞.

Computable dimension

Question (Hirschfeldt, Khoussainov)

Is there a locally finite graph of computable dimension 2?

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Example

$(\mathbb{N}, <)$ is $0'$-computably categorical. Given any two computable copies of $(\mathbb{N}, <)$, $0'$ can compute the adjacency relation on each:

$$\mathrm{Adj}(x, y) \text{ if there is no } z \text{ between } x \text{ and } y.$$

With the adjacency relation, we can compute the isomorphism.

Degrees of categoricity

Definition

$\mathcal{A}$ has degree of categoricity $\mathbf{d}$ if

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Question

Which degrees can be degrees of categoricity?

A set A is Δ^0_2 if it has a computable approximation $A(n, s)$ with finitely many mind changes:

$$A(n) = \lim_{s \to \infty} A(n, s).$$

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Which Δ^0_2 degrees are degrees of categoricity?

Example

Every c.e. degree is a degree of categoricity.

A set A is c.e. if it has an approximation $A(n,s)$ which starts with $A(n,0)=0$ and then has at most one mind change $0 \to 1$.

Example

Every c.e. degree is a degree of categoricity. Let D be a c.e. set.

$\mathcal{A}$ $\qquad$ $\boxed{n \notin D}$ $\qquad$ $\mathcal{B}$

$$\boxed{f(a_n) = c_n, \qquad f(b_n) = d_n}$$

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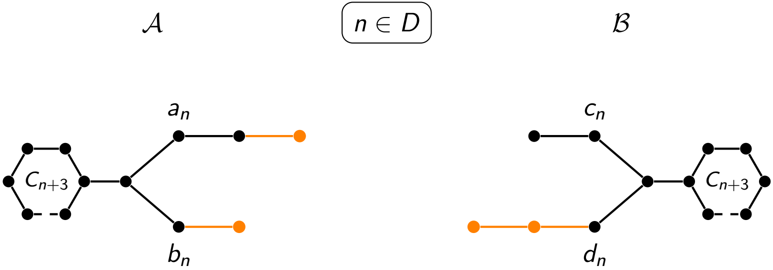

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Let $D = A - B$ be a c.e. set.

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Every 2-c.e. degree is a degree of categoricity.

Let $D = A - B$ be a c.e. set.

We'll build two isomorphic computable graphs $\mathcal{A}$ and $\mathcal{B}$ such that:

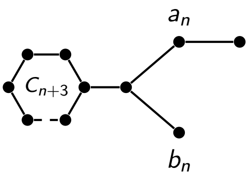
- any isomorphism between $\mathcal{A}$ and $\mathcal{B}$ computes D,
- for any other computable copy $\mathcal{C}$ of $\mathcal{A}$ and $\mathcal{B}$, D computes an isomorphism between $\mathcal{C}$ and $\mathcal{A}$.

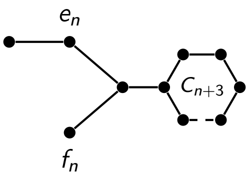
Degrees of categoricity

nth component of $\mathcal{A}$

$$\boxed{n \notin A \Rightarrow n \notin D}$$

nth component of $\mathcal{B}$

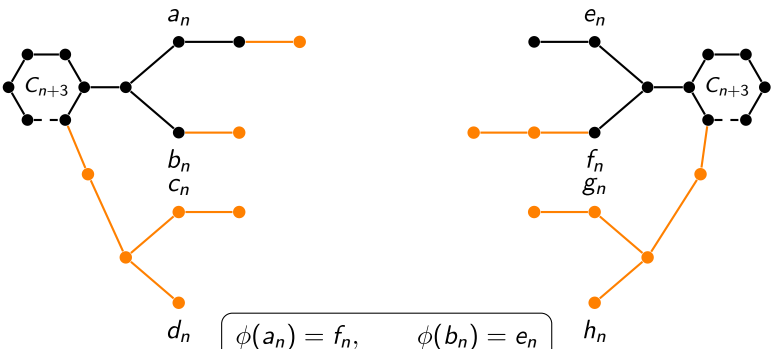




$$\boxed{\phi(a_n) = e_n, \qquad \phi(b_n) = f_n}$$

Degrees of categoricity

nth component of $\mathcal{A}$ $\quad\boxed{n \in A \quad n \notin B \Rightarrow n \in D}\quad$ nth component of $\mathcal{B}$



$$\boxed{\begin{array}{ll} \phi(a_n) = f_n, & \phi(b_n) = e_n \\ \phi(c_n) = g_n, & \phi(d_n) = h_n \end{array}}$$

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$\boxed{n \in A \quad n \in B \Rightarrow n \notin D}$

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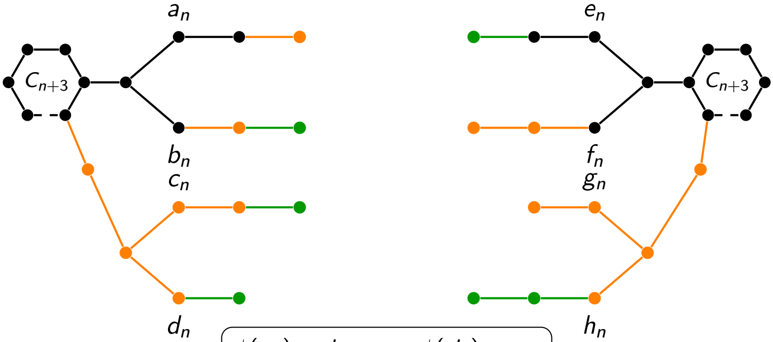

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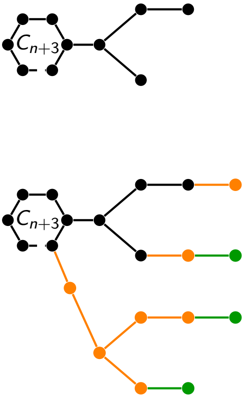
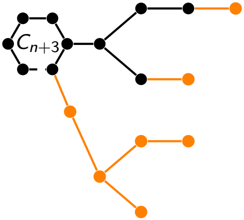


$\boxed{n \in D}$



Degrees of categoricity

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Every Δ_2^0 degree is a degree of categoricity.

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Every Δ^0_2 degree is a degree of categoricity.

Theorem (Anderson, Csima)

There is a Σ^0_2 degree which is not a degree of categoricity.

Degree spectra

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The degree spectrum of $\mathcal{A}$ is

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Theorem (Knight)

Except for automorphically trivial structures, if X computes a copy of $\mathcal{A}$, then there is a copy of $\mathcal{A}$ whose atomic diagram is Turing equivalent to X.

$$\mathrm{DgSp}(\mathcal{A}) = \{\deg_T(\mathcal{B}) : \mathcal{A} \cong \mathcal{B}\}.$$

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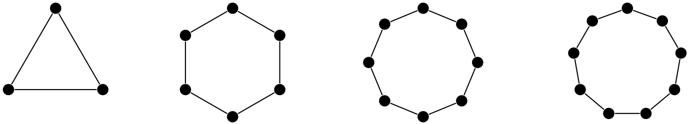

$2(0) + 3 = 3$ $\quad$ $2(1) + 4 = 6$ $\quad$ $2(2) + 4 = 8$ $\quad$ $2(3) + 3 = 9$

$0 \in X$ $\quad$ $1 \notin X$ $\quad$ $2 \notin X$ $\quad$ $3 \in X$

Degree spectra

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What are the possible degree spectra?

Theorem (Slaman, Wehner)

There is a structure $\mathcal{A}$ which has no computable copy, but which has a copy computable from each non-computable degree.

Thus the non-computable degrees are a degree spectrum:

$$\mathrm{DgSp}(\mathcal{A}) = \{X \subseteq \mathbb{N} : X \textit{ non-computable}\}.$$

Degree spectra

List out all of the computable structures as $(\mathcal{C}_e)_{e \in \mathbb{N}}$.

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Then our structure $\mathcal{A}$ has as its eth sort $\mathcal{A}_e$.

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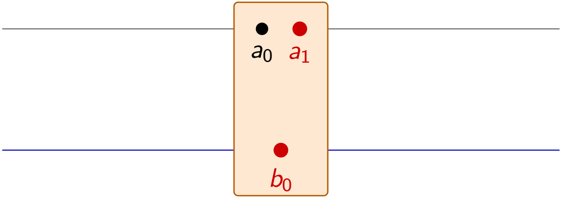

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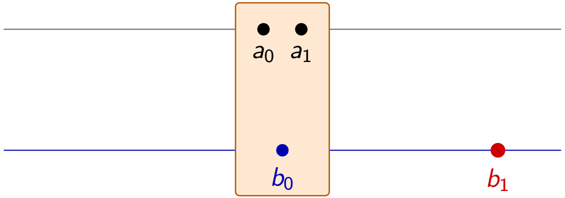

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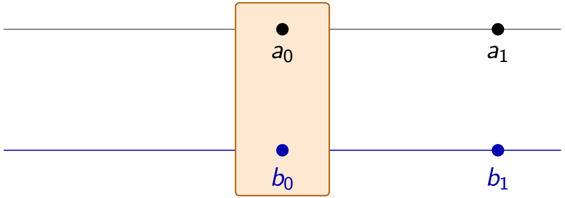

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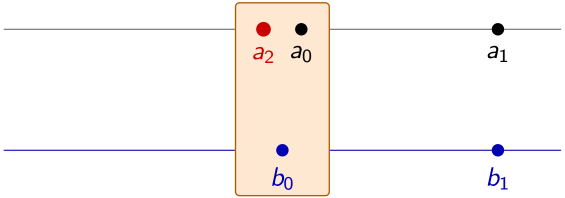

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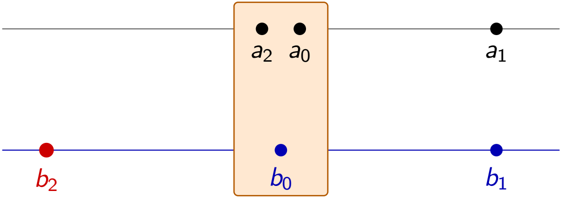

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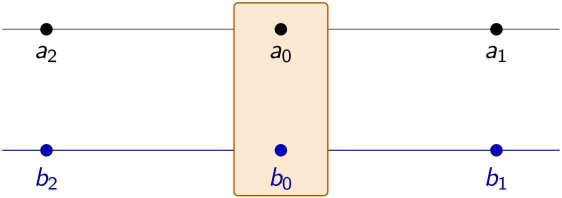

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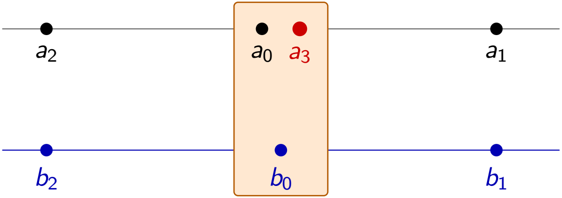

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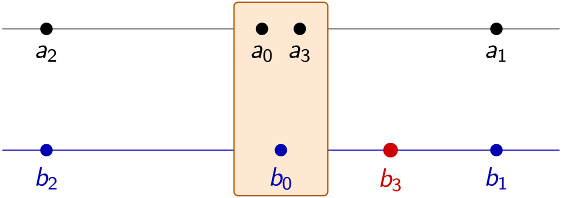

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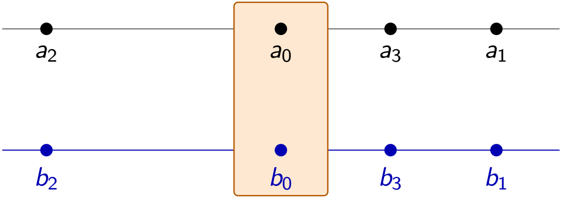

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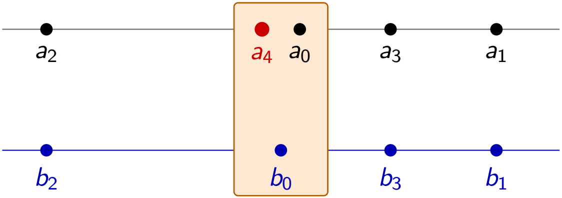

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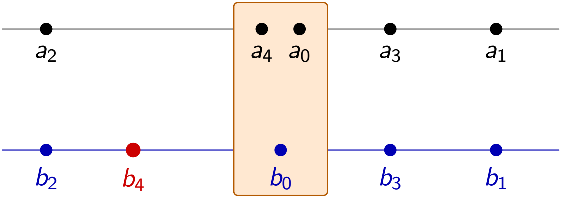

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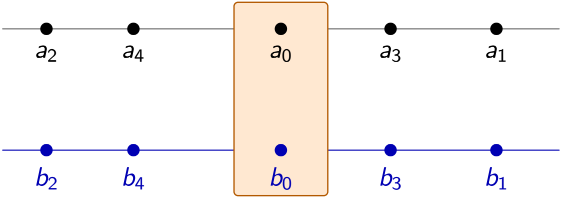

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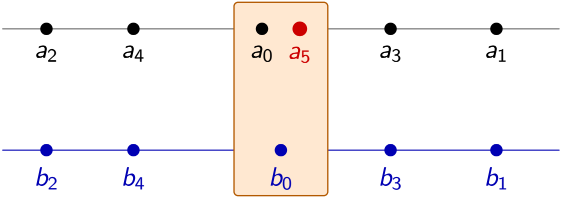

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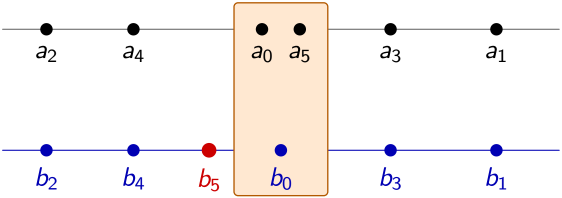

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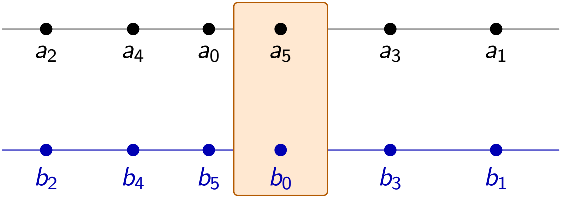

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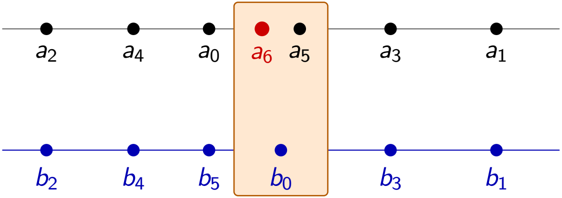

$X(5) = 1$

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This is what we wanted:

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Is there a linear order with Slaman-Wehner degree spectrum?

[1] I think with my REU student Crittenden we can show the answer is yes.

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Question (Harrison-Trainor)

Is every degree spectrum the degree spectrum of a simple rooted tree?

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The non-Δ^0_2 degrees are a degree spectrum.

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Harris and Montalbán: There is a change in behaviour starting at level 5.

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Similarly we consider the isomorphism problem for a class $\mathcal{C}$ of structures,

$$\{(i, j) : \mathcal{A}_i \text{ and } \mathcal{A}_j \text{ are in } \mathcal{C} \text{ and } \mathcal{A}_i \cong \mathcal{A}_j\}.$$

Index sets

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Cases where it is different include:

- there is a difference between looking at only computable copies and looking at all copies
- the class of objects we are considering is countable.

Index sets

Theorem (Downey, Kach, Lempp, Lewis-Pye, Montalbán, Turetsky)

The set

$$\{i : \mathcal{A}_i \text{ is computably categorical}\}$$

is Π^1_1*-complete.*

Thus there is no characterization of the computably categorical structures.

Index sets

Theorem (Harrison-Trainor)

The set

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is Σ^1_1-complete.

Thus there is no characterization of when a computable structure has a decidable copy with a computable elementary diagram.

Index sets

Let G_e be the eth finitely presented group.

Theorem (Lempp)

The set

$$\{e : G_e \text{ is torsion-free}\}$$

is Π^0_2-complete.

Index sets

In part 2 on Tuesday, we will introduce infinitary logic and see what it has to do with computation.

We will also learn about computability-theoretic forcing with countable structures.