Computable structure theory and infinitary logic

Part 2: Infinitary logic and forcing

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ASL NA Meeting 2026 Tutorial Part 2

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Let's start with two simple examples.

Consider $(\mathbb{N}, <)$ and the adjacency relation $\mathrm{Adj} \subseteq \mathbb{N}^2$.

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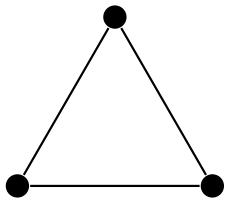
Why? Because Adj is definable by a universal formula:

$$\mathrm{Adj}(x, y) \iff x \neq y \ \wedge \ \forall z \, (z \geq x \wedge z \geq y) \vee (z \leq x \wedge z \leq y).$$

Let $X \subseteq \mathbb{N}$, and let $\mathcal{A}_X$ the graph with cycles of size $n+3$ for $n \in X$.



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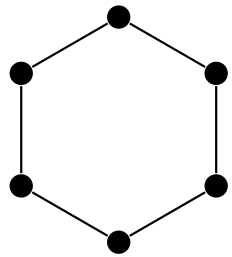

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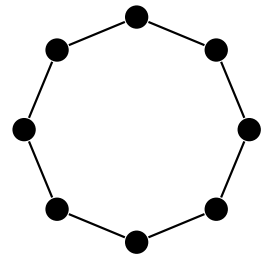

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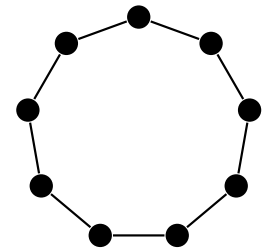

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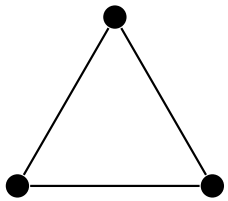

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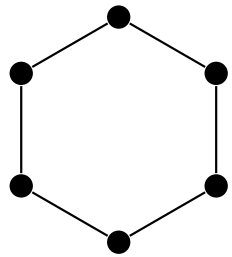

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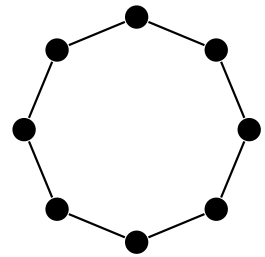

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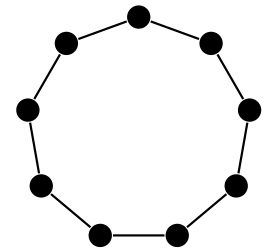

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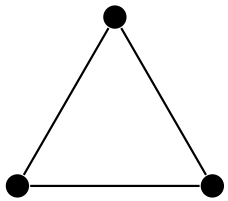

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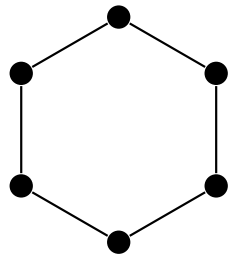

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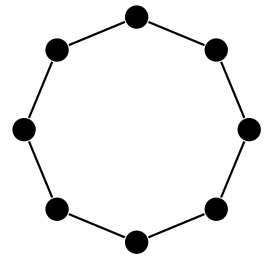

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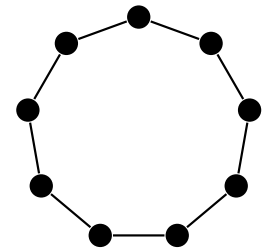

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For every $\mathcal{B} \cong \mathcal{A}_X$, $\mathcal{B}$ can computably enumerate X.

This is because there is a computable list of existential formulas φ_n such that

$$n \in X \iff \mathcal{A}_X \models \varphi_n.$$

Let $\mathcal{A}$ be a structure. Suppose that there were computable lists of existential formulas $\varphi_{n,m}(\bar{x})$ and a tuple $\bar{a}$ such that

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Thus: We should use infinitary logic.

Definition

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A $\mathbb{Q}$-vector space having infinite dimension can be expressed by:

$$\bigwedge_{n\geq 1} \exists x_1,\ldots,x_n \bigwedge_{\substack{(c_1,\ldots,c_n)\in\mathbb{Q}^n \\ \bar{c}\neq\bar{0}}} c_1x_1+\cdots+c_nx_n \neq 0.$$

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Let $X \subseteq \mathbb{N}$ be a set which is not c.e. Then the formula

$$\bigvee_{n \in X} \underbrace{x + \cdots + x}_{n \text{ times}} = 0$$

expressing that the order of x is in X is not computable.

The arithmetic hierarchy

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$$A, A', A'', A''' = A^{(3)}, A'''' = A^{(4)}, \ldots.$$

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Definition

A set A is arithmetic if $A \leq_T 0^{(n)}$ for some n if A is definable in $(\mathbb{N}, 0, 1, +, -, \cdot)$.

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This can all be relativized to an oracle, writing "Σ^0_n relative to A".

Theorem (Knight)

Let $\mathcal{A}$ be a structure and $X \subseteq \mathbb{N}$. The following are equivalent:

(1) *X is Σ^0_n relative to every copy $\mathcal{B} \cong \mathcal{A}$;*

(2) *there is $\bar{a} \in \mathcal{A}$ and a computable list of computable Σ_n formulas φ_i such that*

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(2)$\Rightarrow$(1) is easy: Given $\mathcal{B} \cong \mathcal{A}$, we can fix $\bar{b} \in \mathcal{B}$ the image of $\bar{a}$. Hard-coding $\bar{b}$, evaluating a Σ_n formula is Σ^0_n.

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Model theorist: Omitting types?

The forcing conditions

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Such a p is really just a tuple $\bar{p}$ from $\mathcal{A}$. We write $\mathcal{A}^*$ for the set of finite tuples from $\mathcal{A}$ without repetition. These are the forcing conditions.

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We can also define what it means for a formula to be computable.

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- We can then take the disjunction of all of these formulas.

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- If $\varphi \equiv \neg R^{\dot{\mathcal{G}}}(a_1, \ldots, a_n)$, then $p \Vdash \varphi$ if $\bar{p}(a_1), \ldots, \bar{p}(a_n)$ are defined and $\mathcal{A} \models \neg R(\bar{p}(a_1), \ldots, \bar{p}(a_n))$.

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Note that the negation of a formula is not directly a formula, but is equivalent to a formula. We call this the formal negation.

Forcing lemmas

Lemma

If $p \Vdash \varphi$ *and* $q \supseteq p$, *then* $q \Vdash \varphi$.

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It is not the case that $p \Vdash \varphi$ and $p \Vdash \neg\varphi$.

Generics

Given a countable collection of formulas of the forcing language,

$$g : \mathbb{N} \to \mathcal{A}$$

is generic (for that collection) if it decides each formula, i.e., for each such φ there is $p \subseteq g$ such that p decides φ.

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Lemma

If g is generic and $\mathcal{G} = g^{-1}(\mathcal{A})$, then $\varphi(\mathcal{G})$ is true if and only if there is $p \subseteq g$ such that $p \Vdash \varphi(\dot{\mathcal{G}})$.

Definability of forcing

Lemma

Given a computable Σ_n formula φ in the forcing language there are computable formulas $\mathrm{Force}_{\varphi,n}(x_1, \ldots, x_n)$ *such that*

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Theorem (Knight)

Let $\mathcal{A}$ be a structure and $X \subseteq \mathbb{N}$. The following are equivalent:

(1) *X is Σ^0_n relative to every copy $\mathcal{B} \cong \mathcal{A}$;*

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The right-hand-side is a computable sequence of Σ_n formulas with paramters p.

Definition

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R is relatively intrinsically Σ^0_n if whenever $g : \mathcal{A} \cong \mathcal{B}$, $g(R) \subseteq \mathcal{B}^n$ is Σ^0_n relative to $\mathcal{B}$.

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Theorem (Ash-Knight-Manasse-Slaman, Chisholm)

R is relatively intrinsically Σ^0_n if and only if it is definable in $\mathcal{A}$ by a computable Σ_n formula over a finite tuple of parameters.

Definition

$\mathcal{A}$ is relatively intrinsically Δ^0_n-categorical if whenever $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $g : \mathcal{B} \to \mathcal{A}$ which is Δ^0_n relative to $\mathcal{A}$ and $\mathcal{B}$.

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The basic clopen sets are given by atomic or negated atomic facts,

$$[\varphi(n_1, \ldots, n_k)] = \{\mathcal{A} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{A} \models \varphi(n_1, \ldots, n_k)\}.$$

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A set $A \subseteq \mathrm{Mod}(\mathcal{L})$ is *invariant* if it is closed under isomorphism.

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This characterization does not extend to incomplete theories.

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Let T be an $\mathcal{L}$-sentence and let T' be an $\mathcal{L}'$-sentence.

Every Borel functor from models of T' to models of T is Borel naturally isomorphic to the functor induced by an interpretation of $(\mathcal{L}, T)$ in $(\mathcal{L}', T')$.

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Infinitary interpretations can use tuples of all sizes as part of the domain.

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There is no characterization of the computable structures which are computably categorical.

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So relative computable categoricity has a characterization, but (plain) computable categoricity does not.

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- *Every automorphism of $\mathcal{A}$ is computable.*
- *There is a finite set of parameters $\bar{a}$ such that every automorphism f of $\mathcal{A}$ is Σ_1-definable over $\bar{a}, f(\bar{a})$.*

An automorphism f of $\mathcal{A}$ is Σ_1-definable over $\bar{b}$ if there is a computable Σ_1 formulas $\varphi(x, y, \bar{b})$ such that $f(x) = y$ if and only if $\mathcal{A} \models \varphi(x, y, \bar{b})$.

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(This follows from the fact that there is a structure whose tree of tuples is computable but the structure has no computable copy.)

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At the end of Part 3, we'll see how this all ties back to computation.