Computable structure theory and infinitary logic

Part 3: Back-and-forth and the Scott analysis

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ASL NA Meeting 2026 Tutorial Part 3

Definition

$\mathcal{A}$ is relatively intrinsically Δ^0_α-categorical if whenever $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $g : \mathcal{B} \to \mathcal{A}$ which is Δ^0_α relative to $\mathcal{A}$ and $\mathcal{B}$.

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Theorem (Ash-Knight-Manasse-Slaman, Chisholm)

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A Scott family is a collection of formulas S such that

- each tuple from $\mathcal{A}$ satisfies at least one formula from S, and
- if two tuples from $\mathcal{A}$ satisfy the same formula from S, then they are automorphic.

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Theorem (Ash–Knight–Manasse–Slaman, Chisholm)

$\mathcal{A}$ is relatively intrinsically (boldface) $\boldsymbol{\Delta}^0_\alpha$-categorical if and only if the automorphism orbits of $\mathcal{A}$ are Σ_α-definable over finitely many parameters.

It will be more convenient for now to use the uniform version, without parameters.

Theorem

$\mathcal{A}$ is uniformly relatively intrinsically (boldface) $\mathbf{\Delta}^0_\alpha$-categorical if and only if the automorphism orbits of $\mathcal{A}$ are Σ_α-definable without parameters.

The definability of automorphism orbits also shows up in the Scott analysis. Suppose that for $\bar{a} \in \mathcal{A}$ the orbit of $\bar{a}$ is definable by the Σ_α formula $\varphi_{\bar{a}}(\bar{x})$. Assume also that $\varphi_{\bar{a}}(\bar{x})$ includes the atomic type of $\bar{x}$.

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Consider the sentence θ defined by

$$\theta := \varphi_{\varnothing} \;\wedge \bigwedge_{\bar{a} \in \mathcal{A}^{<\omega}} \forall \bar{x} \Bigg(\varphi_{\bar{a}}(\bar{x}) \rightarrow$$

$$\left[\bigwedge_{b \in \mathcal{A}} \exists y \, \varphi_{\bar{a}b}(\bar{x}, y) \;\wedge\; \forall y \bigvee_{b \in \mathcal{A}} \varphi_{\bar{a}b}(\bar{x}, y) \right] \Bigg).$$

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This is $\Pi_{\alpha+1}$ and $\mathcal{A} \models \theta$.

We claim that for countable structure $\mathcal{B}$,

$$\mathcal{A} \cong \mathcal{B} \iff \mathcal{B} \models \theta.$$

$$\mathcal{B} \models \varphi_{\varnothing} \;\wedge \bigwedge_{\bar{a} \in \mathcal{A}^{<\omega}} \forall \bar{x} \left(\varphi_{\bar{a}}(\bar{x}) \to \right.$$

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We argue that $\mathcal{B} \cong \mathcal{A}$ using a back-and-forth argument. Let $\mathcal{A} = \{a_0, a_1, \ldots\}$ and $\mathcal{B} = \{b_0, b_1, \ldots\}$.

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Scott showed that every countable structure has a Scott sentence.

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- For $\alpha > 0$, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if for each $\beta < \alpha$ and $\bar{d} \in \mathcal{B}$ there is $\bar{c} \in \mathcal{A}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

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We also define the symmetric relations:

$$(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b}) \text{ if } \bar{a} \leq_\alpha \bar{b} \text{ and } \bar{b} \leq_\alpha \bar{a}.$$

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Each $\equiv_\alpha$ is an equivalence relation; we call its equivalence classes *back-and-forth types*.

Theorem (Karp)

$(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$

if and only if every Σ_α formula true of $\bar{b}$ is true of $\bar{a}$,

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In particular:

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In particular:

- $\bar{a} \leq_0 \bar{b}$ means that they satisfy the same quantifier-free formulas;
- $\bar{a} \leq_1 \bar{b}$ means that every existential formula true of $\bar{b}$ is true of $\bar{a}$.

Theorem (Ash and Knight)

Let $\mathcal{A}$ and $\mathcal{B}$ be countable $\mathcal{L}$-structures.

Then $\mathcal{A} \leq_\alpha \mathcal{B}$ if and only if for every $\mathbf{\Sigma}^0_\alpha$ set $S \subseteq 2^\omega$ there is a continuous map $x \mapsto \mathcal{C}_x$ such that for all x

$$x \in S \iff \mathcal{C}_x \cong \mathcal{A}$$

and

$$x \notin S \iff \mathcal{C}_x \cong \mathcal{B}.$$

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Ash and Knight proved an effective version of this theorem. This non-effective version can also be obtained by Louveau-Saint Raymond's separation theorem.

Let $\mathcal{A}$ be a countable structure.

There are only countably many ordinals α such that for some $\bar{a}, \bar{b} \in \mathcal{A}$,

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Thus there is some $\alpha < \omega_1$ such that for all $\bar{a}, \bar{b} \in \mathcal{A}$,

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$$\bar{a} \leq_\alpha \bar{b} \Longrightarrow \bar{a} \equiv_{\alpha+1} \bar{b}.$$

Claim: If $\bar{a} \equiv_\alpha \bar{b}$ then there is an automorphism taking $\bar{a}$ to $\bar{b}$.

Assume that $\equiv_\alpha$ implies $\equiv_{\alpha+1}$. If $\bar{a} \equiv_\alpha \bar{b}$ then we construct an automorphism taking $\bar{a}$ to $\bar{b}$.

$$\begin{array}{ccc} \mathcal{A} & & \mathcal{A} \\ \hline \bar{a} & \longmapsto & \bar{b} \\ \color{red}{c_0} & & \end{array}$$

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\mathcal{A} & & \mathcal{A} \\
\hline
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c_0 & \longmapsto & c_5 \\
 & & {\color{red} c_0}
\end{array}
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Thus the orbits are $\Pi_{2\alpha}$, hence $\Sigma_{2\alpha+1}$, definable.

Theorem (Scott)

Every countable structure has a Scott sentence.

For every countable $\mathcal{L}$-structure $\mathcal{A}$ there is an $\mathcal{L}_{\omega_1\omega}$ sentence φ such that for all countable $\mathcal{L}$-structures $\mathcal{B}$,

$$\mathcal{B} \cong \mathcal{A} \iff \mathcal{B} \models \varphi.$$

Theorem (Morley)

Let φ *be a sentence of* $\mathcal{L}_{\omega_1\omega}$*. Then* φ *has either at most* $\aleph_1$ *countable models or* $2^{\aleph_0}$ *countable models.*

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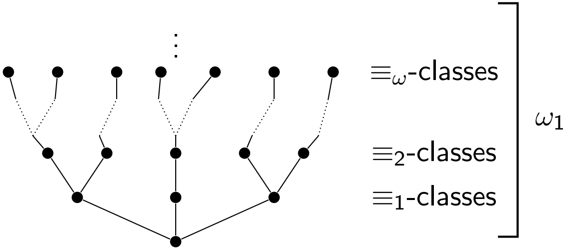
The proof uses the *Vaught tree* of φ: level α consists of the $\equiv_\alpha$-classes of models of φ.



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  There are $\aleph_1$-many levels of the tree, so at most $\aleph_1$-many models.

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$\mathcal{A}$ has parametrized Scott rank α if and only if there is $\bar{a} \in \mathcal{A}$ such that $(\mathcal{A}, \bar{a})$ has unparametrized Scott rank α.

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$$\begin{tikzcd}
\Sigma_1 \arrow[dr] & & \Sigma_2 \arrow[dr] & & \Sigma_3 \arrow[dr] & & \\
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\Pi_1 \arrow[ur] & & \Pi_2 \arrow[ur] & & \Pi_3 \arrow[ur] & &
\end{tikzcd}$$

Example

The group $\mathbb{Z}$ has Scott complexity d-Σ_2: it is the unique structure which is a torsion-free abelian group of rank 1 (Π_2) with an element having no proper divisors (Σ_2).

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Question (Knight)

What is the Scott complexity of $\mathbb{Q}(t_1, t_2, t_3, \ldots)$?

Theorem (Gonzalez, Montalbán)

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1. *There are continuum-many $\equiv_{\alpha+\omega}$-back-and-forth types of models of φ, or*
2. *There are countably many models of φ and each of them has a $\Pi_{\alpha+n}$ Scott sentence for some n.*

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Question

Is this true for Boolean algebras?

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Even when we are not in this idealized situation, it should still inform how we think. I'll give two examples of different character.

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A set $X \subseteq \mathbb{N}$ is low$_n$ if $X^{(n)} \equiv_T 0^{(n)}$.

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Question

Is every low$_5$ Boolean algebra isomorphic to a computable one?

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If $\mathcal{B}$ is a low, then $0' \geq_T \mathcal{B}'$. $\mathcal{B}'$ can find the $\equiv_1$-types of elements in $\mathcal{B}$. So we have a Δ^0_2 approximation of not just $\mathcal{B}$ but its $\equiv_1$-types.

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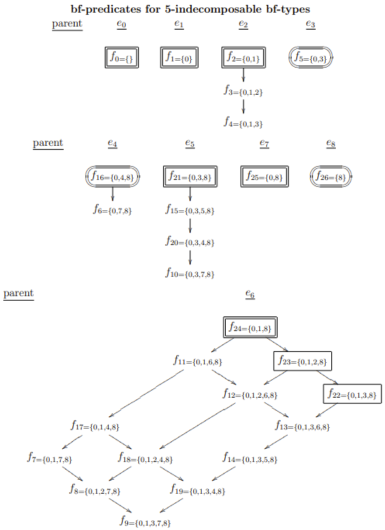
This Δ^0_2 approximation can make mistakes but Boolean algebras are very "flexible" so we can recover.

Harris and Montalbán carried out a systematic analysis of the $\equiv_n$ back-and-forth types in Boolean algebras.



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Theorem (Downey, Jockusch; Thurber; Knight, Stob)

For $n = 1, 2, 3, 4$ every low$_n$ Boolean algebra is $0^{(n+2)}$-computably isomorphic to any computable one.

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Is every low_5 Boolean algebra isomorphic to a computable one?

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Question

Are the non-arithmetic degrees a degree spectrum?

I've already told you the answer, but I want to tell you about how I came to the answer because it is a great illustration of the power of infinitary logic in understanding computation.

We saw earlier that there are Π_{2n} formulas $\varphi_{\mathcal{A},\bar{a}}(\bar{x})$ such that

$$(\mathcal{A}, \bar{a}) \leq_n (\mathcal{B}, \bar{b}) \iff \mathcal{B} \models \varphi_{\mathcal{A},\bar{a}}(\bar{b}).$$

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For example,

$$(\mathcal{B}, \bar{x}) \geq_1 (\mathcal{A}, \bar{a}) \iff \forall \bar{y} \bigvee_{\bar{a}' \in \mathcal{A}} \operatorname{Diag}(\bar{x}\bar{y}) = \operatorname{Diag}(\bar{a}\bar{a}')$$

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If $\mathcal{A}$ is computable these formulas are not just Π_{2n}, but computable Π_{2n} if n is computable.

If $\mathcal{A}$ is computable and has a Π_n Scott sentence, then

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But there is a computable Π_{2n} sentence φ such that

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Following work with Gonzalez and Chen, I expected that this cannot be improved.

Theorem (Alvir, Csima, Harrison-Trainor)

There is a computable structure $\mathcal{A}$ such that $\mathcal{A}$ has a Π_2 Scott sentence but does not have a computable Σ_4 Scott sentence.

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The main technique for doing this is called a Marker extension, but the best this can obtain is:

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Corollary (Alvir, Csima, Harrison-Trainor)

There is a computable structure $\mathcal{A}$ such that $\mathcal{A}$ has a Π_n Scott sentence but does not have a computable Σ_{n+2} Scott sentence.

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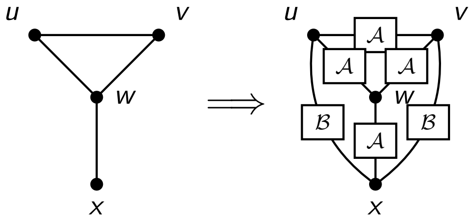
Consider graphs with edge relation E. Fix two structures $\mathcal{A}$ and $\mathcal{B}$. Given a graph $\mathcal{G}$, we can replace each edge between u and v by a copy of $\mathcal{A}$ attached to u and v, and each non-edge by a copy of $\mathcal{B}$ attached to u and v.



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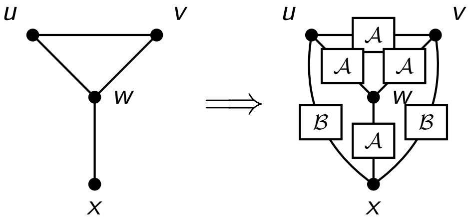

A common choice for $\mathcal{A}$ and $\mathcal{B}$ is ω and ω^*.

The edge relation becomes a definable relation in the new structure. The more similar $\mathcal{A}$ and $\mathcal{B}$ are, the harder it is to define. Generally, for some fixed n, we just make the edge relation into a Δ_{n+1}-definable relation in the new structure.

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Thus a Π_2 Scott sentence for $\mathcal{G}$ becomes a Π_{n+2} Scott sentence for the jump inversion.

Similarly, if $\mathcal{G}$ has no computable Σ_4 Scott sentence, then the jump inversion has no computable Σ_{n+4} Scott sentence.

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Essentially we make the edge relation definable by a non-computable Δ_{n+1}-formula, but only definable by a computable Δ_{2n+1} formula.

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Theorem

There is a computable structure $\mathcal{A}$ such that $\mathcal{A}$ has a Π_n Scott sentence but does not have a computable Σ_{2n} Scott sentence.

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Let's sketch the proof.

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- We have access to a non-arithmetic set.
- We are diagonalizing against an arithmetic structure, say $0^{(n)}$-computable.

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- We started by considering computational properties, like computing isomorphisms and degree spectra.
- We saw that robust relative computational properties for all copies of a structure corresponded to definability in infinitary logic.
- Infinitary logic led us to a robust way of understanding structural complexity.
- Our understanding of a structure in infinitary logic helped us understand the computational properties of structures, both directly as with Boolean algebras and indirectly with unfriendly jump inversions.