Computable structure theory and infinitary logic

Part 3: Back-and-forth and the Scott analysis

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Definition

$\mathcal{A}$ is relatively intrinsically Δ^0_α-categorical if whenever $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $g : \mathcal{B} \to \mathcal{A}$ which is Δ^0_α relative to $\mathcal{A}$ and $\mathcal{B}$.

Theorem (Ash-Knight-Manasse-Slaman, Chisholm)

$\mathcal{A}$ is relatively intrinsically Δ^0_α-categorical if and only if $\mathcal{A}$ has a c.e. Scott family of computable Σ_α formulas over finitely many parameters.

A Scott family is a collection of formulas S such that

- each tuple from $\mathcal{A}$ satisfies at least one formula from S, and
- if two tuples from $\mathcal{A}$ satisfy the same formula from S, then they are automorphic.

Definition

$\mathcal{A}$ is relatively intrinsically (boldface) $\mathbf{\Delta}^0_\alpha$-categorical if there is a set X such that whenever $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $g : \mathcal{B} \to \mathcal{A}$ which is Δ^0_α relative to X, $\mathcal{A}$, and $\mathcal{B}$.

Theorem (Ash–Knight–Manasse–Slaman, Chisholm)

$\mathcal{A}$ is relatively intrinsically (boldface) $\mathbf{\Delta}^0_\alpha$-categorical if and only if there is a set X such that $\mathcal{A}$ has an X-c.e. Scott family of X-computable Σ_α formulas over finitely many parameters.

$\mathcal{A}$ is relatively intrinsically (boldface) $\mathbf{\Delta}^0_\alpha$-categorical if and only if the automorphism orbits of $\mathcal{A}$ are Σ_α-definable over finitely many parameters.

It will be more convenient for now to use the uniform version, without parameters.

Theorem

$\mathcal{A}$ is uniformly relatively intrinsically (boldface) $\mathbf{\Delta}^0_\alpha$-categorical if and only if the automorphism orbits of $\mathcal{A}$ are Σ_α-definable without parameters.

The definability of automorphism orbits also shows up in the Scott analysis. Suppose that for $\bar{a} \in \mathcal{A}$ the orbit of $\bar{a}$ is definable by the Σ_α formula $\varphi_{\bar{a}}(\bar{x})$. Assume also that $\varphi_{\bar{a}}(\bar{x})$ includes the atomic type of $\bar{x}$.

Consider the sentence θ defined by

$$\theta := \varphi_{\varnothing} \;\wedge\; \bigwedge_{\bar{a} \in \mathcal{A}^{<\omega}} \forall \bar{x} \left(\varphi_{\bar{a}}(\bar{x}) \rightarrow \left[\bigwedge_{b \in \mathcal{A}} \exists y \, \varphi_{\bar{a}b}(\bar{x}, y) \;\wedge\; \forall y \bigvee\!\!\!\!\bigvee_{b \in \mathcal{A}} \varphi_{\bar{a}b}(\bar{x}, y) \right] \right).$$

This is $\Pi_{\alpha+1}$ and $\mathcal{A} \models \theta$.

We claim that for countable structure $\mathcal{B}$,

$$\mathcal{A} \cong \mathcal{B} \iff \mathcal{B} \models \theta.$$

$$\mathcal{B} \models \varphi_\varnothing \ \wedge \bigwedge_{\bar{a}\in\mathcal{A}^{<\omega}} \forall \bar{x} \left(\varphi_{\bar{a}}(\bar{x}) \to \right.$$

$$\left. \left[\bigwedge_{b\in\mathcal{A}} \exists y\, \varphi_{\bar{a}b}(\bar{x}, y) \ \wedge \ \forall y \bigvee_{b\in\mathcal{A}} \varphi_{\bar{a}b}(\bar{x}, y) \right] \right).$$

We argue that $\mathcal{B} \cong \mathcal{A}$ using a back-and-forth argument. Let $\mathcal{A} = \{a_0, a_1, \ldots\}$ and $\mathcal{B} = \{b_0, b_1, \ldots\}$.

$\mathcal{A}$		$\mathcal{B}$	$\mathcal{B} \models \varphi_\varnothing$
a_0	$\longmapsto$	b_5	$\mathcal{B} \models \varphi_{(a_0)}(b_5)$
a_3	$\longmapsto$	b_0	$\mathcal{B} \models \varphi_{(a_0,a_3)}(b_5, b_0)$
a_1	$\longmapsto$	b_8	$\mathcal{B} \models \varphi_{(a_0,a_3,a_1)}(b_5, b_0, b_8)$
a_7	$\longmapsto$	b_1	$\mathcal{B} \models \varphi_{(a_0,a_3,a_1,a_7)}(b_5, b_0, b_8, b_1)$
$\vdots$		$\vdots$	$\vdots$

Definition

We say that θ is a Scott sentence for $\mathcal{A}$ if for all countable structures $\mathcal{B}$,

$$\mathcal{A} \cong \mathcal{B} \Longleftrightarrow \mathcal{B} \models \theta.$$

If the orbits of $\mathcal{A}$ are Σ_α-definable, then $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.

Scott showed that every countable structure has a Scott sentence.

Definition

The *standard asymmetric back-and-forth relations* $\leq_\alpha$, for $\alpha < \omega_1$, are defined by:

- $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas.
- For $\alpha > 0$, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if for each $\beta < \alpha$ and $\bar{d} \in \mathcal{B}$ there is $\bar{c} \in \mathcal{A}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

Often we write $\bar{a} \leq_\alpha \bar{b}$, dropping the reference to $\mathcal{A}$ and $\mathcal{B}$.

We also define the symmetric relations:

$$(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b}) \text{ if } \bar{a} \leq_\alpha \bar{b} \text{ and } \bar{b} \leq_\alpha \bar{a}.$$

Each $\equiv_\alpha$ is an equivalence relation; we call its equivalence classes *back-and-forth types*.

Theorem (Karp)
$(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$

if and only if every Σ_α formula true of $\bar{b}$ is true of $\bar{a}$,
if and only if every Π_α formula true of $\bar{a}$ is true of $\bar{b}$.

In particular:

- $\bar{a} \leq_0 \bar{b}$ means that they satisfy the same quantifier-free formulas;
- $\bar{a} \leq_1 \bar{b}$ means that every existential formula true of $\bar{b}$ is true of $\bar{a}$.

Theorem (Ash and Knight)

Let $\mathcal{A}$ and $\mathcal{B}$ be countable $\mathcal{L}$-structures.

Then $\mathcal{A} \leq_\alpha \mathcal{B}$ if and only if for every $\mathbf{\Sigma}^0_\alpha$ set $S \subseteq 2^\omega$ there is a continuous map $x \mapsto \mathcal{C}_x$ such that for all x

$$x \in S \Longleftrightarrow \mathcal{C}_x \cong \mathcal{A}$$

and

$$x \notin S \Longleftrightarrow \mathcal{C}_x \cong \mathcal{B}.$$

Ash and Knight proved an effective version of this theorem. This non-effective version can also be obtained by Louveau-Saint Raymond's separation theorem.

Let $\mathcal{A}$ be a countable structure.

There are only countably many ordinals α such that for some $\bar{a}, \bar{b} \in \mathcal{A}$,

$$\bar{a} \leq_\alpha \bar{b} \text{ but } \bar{a} \not\equiv_{\alpha+1} \bar{b}.$$

Thus there is some $\alpha < \omega_1$ such that for all $\bar{a}, \bar{b} \in \mathcal{A}$,

$$\bar{a} \leq_\alpha \bar{b} \Longrightarrow \bar{a} \equiv_{\alpha+1} \bar{b}.$$

Claim: If $\bar{a} \equiv_\alpha \bar{b}$ then there is an automorphism taking $\bar{a}$ to $\bar{b}$.

Assume that $\equiv_\alpha$ implies $\equiv_{\alpha+1}$. If $\bar{a} \equiv_\alpha \bar{b}$ then we construct an automorphism taking $\bar{a}$ to $\bar{b}$.

$$
\begin{array}{ccc}
\mathcal{A} & & \mathcal{A} \\
\hline
\bar{a} & \longmapsto & \bar{b} \\
c_0 & \longmapsto & c_5 \\
c_3 & \longmapsto & c_0 \\
c_1 & \longmapsto & c_8 \\
c_7 & \longmapsto & c_1 \\
\vdots & & \vdots
\end{array}
$$

$$\bar{a} \equiv_{\alpha+1} \bar{b} \quad (\bar{a}, c_0) \leq_\alpha (\bar{b}, c_5) \quad (\bar{a}, c_0) \leq_\alpha (\bar{b}, c_5) \implies (\bar{a}, c_0) \equiv_{\alpha+1} (\bar{b}, c_5)$$

$$(\bar{a}, c_0) \equiv_{\alpha+1} (\bar{b}, c_5)$$

$$(\bar{a}, c_0, c_3) \geq_\alpha (\bar{b}, c_5, c_0)$$

$$(\bar{a}, c_0, c_3) \geq_\alpha (\bar{b}, c_5, c_0) \implies (\bar{a}, c_0, c_3) \equiv_{\alpha+1} (\bar{b}, c_5, c_0)$$

There are $\Pi_{2\alpha}$ formulas $\varphi_{\bar{a}}(\bar{x})$ such that

$$\bar{b} \equiv_\alpha \bar{a} \iff \mathcal{A} \models \varphi_{\bar{a}}(\bar{b}).$$

For example,

$$\bar{x} \geq_1 \bar{a} \iff \forall \bar{y} \bigvee\!\!\!\!\bigvee_{\bar{a}'} \mathrm{Diag}(\bar{x}\bar{y}) = \mathrm{Diag}(\bar{a}\bar{a}')$$

and

$$\bar{x} \leq_1 \bar{a} \iff \bigwedge\!\!\!\!\bigwedge_{\bar{a}'} \exists \bar{y}\, \mathrm{Diag}(\bar{x}\bar{y}) = \mathrm{Diag}(\bar{a}\bar{a}').$$

Thus the orbits are $\Pi_{2\alpha}$, hence $\Sigma_{2\alpha+1}$, definable.

Theorem (Scott)

Every countable structure has a Scott sentence.

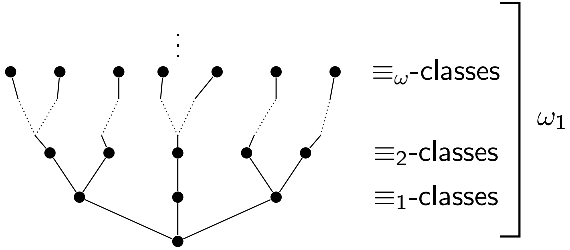
For every countable $\mathcal{L}$-structure $\mathcal{A}$ there is an $\mathcal{L}_{\omega_1\omega}$ sentence φ such that for all countable $\mathcal{L}$-structures $\mathcal{B}$,

$$\mathcal{B} \cong \mathcal{A} \iff \mathcal{B} \models \varphi.$$

Theorem (Morley)

Let φ be a sentence of $\mathcal{L}_{\omega_1\omega}$. Then φ has either at most $\aleph_1$ countable models or $2^{\aleph_0}$ countable models.

The proof uses the *Vaught tree* of φ: level α consists of the $\equiv_\alpha$-classes of models of φ.



Theorem (Silver)

A co-analytic equivalence relation has either countably-many or continuum-many equivalence classes.

$\equiv_\alpha$ is Borel. Thus each level of the tree is either countable or continuum-sized.

- If some level of the Vaught tree has continuum-many classes, then φ has continuum many countable models.
- Otherwise, every level of the tree is countable.

  Each model is isolated on the tree; if it has a Π_α Scott sentence then it is isolated at level α.

  There are $\aleph_1$-many levels of the tree, so at most $\aleph_1$-many models.

We connected definability of orbits to constructing a Scott sentence. Actually, this gives an equivalence.

Theorem (Montalbán)

Let $\mathcal{A}$ be a countable structure and let α be a countable ordinal. The following are equivalent:

- *$\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.*
- $\mathrm{Copies}(\mathcal{A}) = \{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{B} \cong \mathcal{A}\}$ *is* $\mathbf{\Pi}^0_{\alpha+1}$.
- *Every automorphism orbit in $\mathcal{A}$ is Σ_α-definable without parameters.*
- *$\mathcal{A}$ is uniformly relatively intrinsically (boldface) $\mathbf{\Delta}^0_\alpha$-categorical.*

We can also do this with parameters.

Theorem (Montalbán)

Let $\mathcal{A}$ be a countable structure and let α be a countable ordinal. The following are equivalent:

- *$\mathcal{A}$ has a $\Sigma_{\alpha+2}$ Scott sentence.*
- $\mathrm{Copies}(\mathcal{A}) = \{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{B} \cong \mathcal{A}\}$ *is* $\boldsymbol{\Sigma}^0_{\alpha+2}$.
- *Every automorphism orbit in $\mathcal{A}$ is Σ_α-definable over finitely many parameters.*
- *$\mathcal{A}$ is relatively intrinsically (boldface) $\boldsymbol{\Delta}^0_\alpha$-categorical.*

There have been many inequivalent definitions of Scott rank. These definitions are robust in the sense that they have many different characterizations.

Definition (Montalbán)

The (unparametrized) Scott rank of $\mathcal{A}$ is the least ordinal $\mathrm{SR}(\mathcal{A}) = \alpha$ such that $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.

Definition (Montalbán)

The (parametrized) Scott rank of $\mathcal{A}$ is the least ordinal $\mathrm{pSR}(\mathcal{A}) = \alpha$ such that $\mathcal{A}$ has a $\Sigma_{\alpha+2}$ Scott sentence.

$\mathcal{A}$ has parametrized Scott rank α if and only if there is $\bar{a} \in \mathcal{A}$ such that $(\mathcal{A}, \bar{a})$ has unparametrized Scott rank α.

A d-Σ_α formula is the conjunction of a Σ_α formula and a Π_α formula.

Definition

The Scott complexity of $\mathcal{A}$ is the least complexity, from among Σ_α, Π_α, and d-Σ_α, of a Scott sentence for $\mathcal{A}$ — equivalently, the Wadge degree of Copies($\mathcal{A}$).

$$\begin{tikzcd}
\Sigma_1 \arrow[dr] & & \Sigma_2 \arrow[dr] & & \Sigma_3 \arrow[dr] & & \\
& \text{d-}\Sigma_1 \arrow[ur] \arrow[dr] & & \text{d-}\Sigma_2 \arrow[ur] \arrow[dr] & & \text{d-}\Sigma_3 \arrow[r] & \cdots \\
\Pi_1 \arrow[ur] & & \Pi_2 \arrow[ur] & & \Pi_3 \arrow[ur] & &
\end{tikzcd}$$

Example

The group $\mathbb{Z}$ has Scott complexity d-Σ_2: it is the unique structure which is a torsion-free abelian group of rank 1 (Π_2) with an element having no proper divisors (Σ_2).

Example (Harrison-Trainor, Ho)

There is a finitely generated group with Scott complexity Σ_3, equiv., with a non-trivial 1-elementary self-embedding $G <_1 G$.

Example (Carson, Harizanov, Knight, Lange, McCoy, Morozov, Quinn, Safranski, Wallbaum)

The free group F_ω on countably many generators has Scott complexity Π_4.

Question (Knight)

What is the Scott complexity of $\mathbb{Q}(t_1, t_2, t_3, \ldots)$?

Theorem (Gonzalez, Montalbán)

Linear orders satisfy the ω-Vaught's conjecture:

Let φ be a Π_α sentence in the language of linear orders extending the linear order axioms. Then either

1. *There are continuum-many $\equiv_{\alpha+\omega}$-back-and-forth types of models of φ, or*
2. *There are countably many models of φ and each of them has a $\Pi_{\alpha+n}$ Scott sentence for some n.*

Theorem (Harrison-Trainor)

For any ordinal $\alpha < \omega_1$, there is a satisfiable Π_2 sentence all of whose models have Scott rank $\geq \alpha$.

Theorem (Gonzalez, Harrison-Trainor)

Let φ be a satisfiable Π_α sentence in the language of linear orders extending the linear order axioms. Then φ has a model with a $\Pi_{\alpha+5}$ Scott sentence, and hence of Scott rank at most $\alpha + 4$.

Question

Is this true for Boolean algebras?

To finish off the tutorial, I want to complete the circle by bringing everything back to the computational.

We have seen that robust computational phenomena are connected with definability in infinitary logic, which lead us to the back-and-forth relations and Scott analysis.

To make these connections, we needed to consider all (possibly non-computable) presentations of a structure.

If we wanted to work in pure infinitary logic (not just with computable sentences) we also needed access to an oracle.

Even when we are not in this idealized situation, it should still inform how we think. I'll give two examples of different character.

Definition

A set $X \subseteq \mathbb{N}$ is low$_n$ if $X^{(n)} \equiv_T 0^{(n)}$.

Theorem (Downey, Jockusch; Thurber; Knight, Stob)

If a Boolean algebra has a low$_4$ copy then it has a computable copy.

In particular, Boolean algebras cannot have the Slaman-Wehner degree spectrum of all non-computable degrees.

Question

Is every low$_5$ Boolean algebra isomorphic to a computable one?

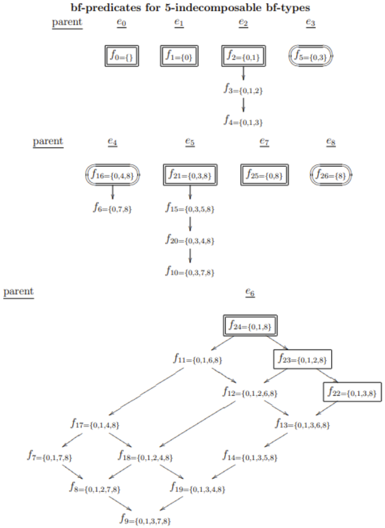
To solve the low_n case, you have to understand the $\equiv_n$-types of elements in Boolean algebras. Though this is not true in general, the back-and-forth types exactly capture what is going on computability-theoretically.

For $n = 1$ the basic building blocks are atoms and non-atoms.

If $\mathcal{B}$ is a low, then $0' \geq_T \mathcal{B}'$. $\mathcal{B}'$ can find the $\equiv_1$-types of elements in $\mathcal{B}$. So we have a Δ^0_2 approximation of not just $\mathcal{B}$ but its $\equiv_1$-types.

This Δ^0_2 approximation can make mistakes but Boolean algebras are very "flexible" so we can recover.

Harris and Montalbán carried out a systematic analysis of the $\equiv_n$ back-and-forth types in Boolean algebras.



Though they did not resolve the problem, for $n = 5$ Harris and Montalbán show that there is a new type of behaviour that shows up that makes everything more complicated.

Theorem (Downey, Jockusch; Thurber; Knight, Stob)

For $n = 1, 2, 3, 4$ every low$_n$ Boolean algebra is $0^{(n+2)}$-computably isomorphic to any computable one.

Theorem (Harris, Montalbán)

There is a low$_5$ Boolean algebra that is not $0^{(5+2)}$-computably isomorphic to any computable one.

Question

Is every low$_5$ Boolean algebra isomorphic to a computable one?

Recall from Part 1 the following two theorems.

Theorem (Slaman, Wehner)

The non-computable degrees are a degree spectrum.

Theorem (Greenberg, Montalbán, Slaman)

The non-hyperarithmetic degrees are a degree spectrum.

Question

Are the non-arithmetic degrees a degree spectrum?

I've already told you the answer, but I want to tell you about how I came to the answer because it is a great illustration of the power of infinitary logic in understanding computation.

We saw earlier that there are Π_{2n} formulas $\varphi_{\mathcal{A},\bar{a}}(\bar{x})$ such that

$$(\mathcal{A},\bar{a}) \leq_n (\mathcal{B},\bar{b}) \iff \mathcal{B} \models \varphi_{\mathcal{A},\bar{a}}(\bar{b}).$$

For example,

$$(\mathcal{B},\bar{x}) \geq_1 (\mathcal{A},\bar{a}) \iff \forall \bar{y} \bigvee\!\!\!\bigvee_{\bar{a}' \in \mathcal{A}} \mathrm{Diag}(\bar{x}\bar{y}) = \mathrm{Diag}(\bar{a}\bar{a}')$$

and

$$(\mathcal{B},\bar{x}) \leq_1 (\mathcal{A},\bar{a}) \iff \bigwedge\!\!\!\bigwedge_{\bar{a}' \in \mathcal{A}} \exists \bar{y}\, \mathrm{Diag}(\bar{x}\bar{y}) = \mathrm{Diag}(\bar{a}\bar{a}').$$

If $\mathcal{A}$ is computable these formulas are not just Π_{2n}, but computable Π_{2n} if n is computable.

If $\mathcal{A}$ is computable and has a Π_n Scott sentence, then

$$\mathcal{A} \leq_n \mathcal{B} \iff \mathcal{A} \cong \mathcal{B}.$$

But there is a computable Π_{2n} sentence φ such that

$$\mathcal{A} \leq_n \mathcal{B} \iff \mathcal{B} \models \varphi.$$

This φ is a computable Π_{2n} Scott sentence for $\mathcal{A}$.

Following work with Gonzalez and Chen, I expected that this cannot be improved.

Theorem (Alvir, Csima, Harrison-Trainor)

There is a computable structure $\mathcal{A}$ such that $\mathcal{A}$ has a Π_2 Scott sentence but does not have a computable Σ_4 Scott sentence.

We'd like to extend this to arbitrary n, but there are difficulties.

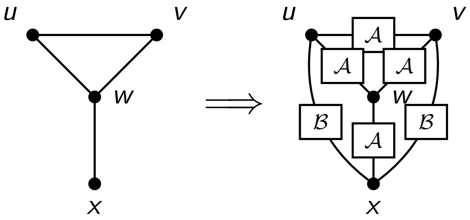
The main technique for doing this is called a Marker extension, but the best this can obtain is:

Corollary (Alvir, Csima, Harrison-Trainor)

There is a computable structure $\mathcal{A}$ such that $\mathcal{A}$ has a Π_n Scott sentence but does not have a computable Σ_{n+2} Scott sentence.

In a Marker extension, we take a symbol R from the language and make it more complicated.

Consider graphs with edge relation E. Fix two structures $\mathcal{A}$ and $\mathcal{B}$. Given a graph $\mathcal{G}$, we can replace each edge between u and v by a copy of $\mathcal{A}$ attached to u and v, and each non-edge by a copy of $\mathcal{B}$ attached to u and v.



A common choice for $\mathcal{A}$ and $\mathcal{B}$ is ω and ω^*.

The edge relation becomes a definable relation in the new structure. The more similar $\mathcal{A}$ and $\mathcal{B}$ are, the harder it is to define. Generally, for some fixed n, we just make the edge relation into a Δ_{n+1}-definable relation in the new structure.

Thus a Π_2 Scott sentence for $\mathcal{G}$ becomes a Π_{n+2} Scott sentence for the jump inversion.

Similarly, if $\mathcal{G}$ has no computable Σ_4 Scott sentence, then the jump inversion has no computable Σ_{n+4} Scott sentence.

This led to unfriendly jump inversions: We choose structures $\mathcal{A}$ and $\mathcal{B}$ for which the infinitary logic is as far from capturing what happens as possible. (Most of the work is in building these structures.)

Essentially we make the edge relation definable by a non-computable Δ_{n+1}-formula, but only definable by a computable Δ_{2n+1} formula.

Theorem

There is a computable structure $\mathcal{A}$ such that $\mathcal{A}$ has a Π_n Scott sentence but does not have a computable Σ_{2n} Scott sentence.

There have been many applications of these unfriendly jump inversions, but in my opinion the most interesting is:

Theorem (Harrison-Trainor)

The non-arithmetic degrees are a degree spectrum.

Let's sketch the proof.

We want to apply something similar to the Slaman-Wehner argument we did in Part 1, but with two differences:

- We have access to a non-arithmetic set.
- We are diagonalizing against an arithmetic structure, say $0^{(n)}$-computable.

We have now come full circle.

- We started by considering computational properties, like computing isomorphisms and degree spectra.
- We saw that robust relative computational properties for all copies of a structure corresponded to definability in infinitary logic.
- Infinitary logic led us to a robust way of understanding structural complexity.
- Our understanding of a structure in infinitary logic helped us understand the computational properties of structures, both directly as with Boolean algebras and indirectly with unfriendly jump inversions.