A Note on Dividing of NTP$_2$ Formulas

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June 2025

1 Question

Throughout, T is a complete first-order theory, $\mathcal{U}$ is a monster model of T, and M denotes an elementary substructure of $\mathcal{U}$. $\varphi(x, y)$ denotes a partitioned formula and $S_\varphi(A)$ denotes the space of complete φ-types (in the free variable tuple x) over A. A φ-formula means a positive or negative instance of φ, i.e. $\varphi(x, a)$ or $\neg\varphi(x, a)$ for some $a \in \mathcal{U}^{|y|}$.

We wish to answer the following question: suppose $\{\varphi_i(x, y_i)\}_{i<n}$ are NTP$_2$ formulas over a model M and $\{b_i\}_{i<n}$ are parameter tuples of lengths $|y_i|$, respectively. Assuming the formulas $\varphi_i(x, b_i)$ each divide over M, does the disjunction $\bigvee_{i<n} \varphi_i(x, b_i)$ also divide over M? By introducing dummy variables to the tuples y_i we may assume that all tuples feature the same variables and in fact that $b_i = b_j$ for all i, j. So now we have $\{\varphi_i(x, y)\}_{i<n}$ NTP$_2$ formulas over M with $\varphi_i(x, b)$ dividing over M for each i, and wish to show that $\bigvee_i \varphi_i(x, b)$ divides over M.

Suppose that there is a single indiscernible sequence $(b_j)_{j\in\omega}$ in the type $\mathrm{tp}(b/M)$ such that for all $i < n$, $\{\varphi_i(x, b_j)\}_{j<\omega}$ is inconsistent. Then by pigeonholing and indiscernibility, the set $\{\bigvee_{i<n} \varphi_i(x, b_j)\}_{j<\omega}$ must be inconsistent, and since $(b_j \dots b_j)_{j<\omega}$ is an indiscernible sequence in the type $\mathrm{tp}(b \dots b/M)$, this shows that the disjunction $\bigvee_{i<n} \varphi_i(x, b)$ divides.

It will be shown in the rest of this note, with no assumptions on the ambient theory, there is an indiscernible sequence $(b_i)_{i<\omega}$ in the type of b such that for any NTP$_2$ formula $\varphi(x, y)$ such that $\varphi(x, b)$ divides over M, in fact $\varphi(x, b)$ divides along $(b_i)_{i<\omega}$.

The existence of such a witness follows from results in [1]. The key fact needed is [1, Theorem 2.26] which guarantees the existence of Kim-strict invariant extensions of types over models in an arbitrary theory. Morley sequences in such types will prove to be the witnesses required above, in light of [2, Lemma 3.12] and a slight strengthening of [2, Lemma 3.14] which is given as Lemma 3.4 below. For a quick deduction of Lemma 3.4 from results of [1] concerning NBTP, see Remark 3.5. Otherwise, the remainder of this note will isolate just what is necessary to answer the above question.

Our affirmative answer to the above question generalizes two previously known facts: first, the result in [2, Corollary 3.22] that forking and dividing over models coincide in an NTP_2 theory, and second (though only partially), the result of local stability theory that a disjunction of A-dividing *stable* formulas is also A-dividing, where A can be any parameter set in an arbitrary theory (see, for example, [5, Lemma 2.21]). This latter fact seems to be the extent of previous "forking equals dividing" results formulated in a local way, i.e. for formulas rather than theories.

2 Definitions

Definition 2.1. The invariant ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{i}$ on subsets of $\mathcal{U}$ is defined by:

$$a \mathop{\smile\!\!\!\!\!\!|}^{i}_{A} b \text{ iff } \operatorname{tp}(a/Ab) \text{ can be extended to a global } A\text{-invariant type}$$

$\mathop{\smile\!\!\!\!\!\!|}^{i}$ satisfies several properties of an abstract independence relation (see, for example, [2, Section 2]), but only two properties will be needed in the proof below:

Fact 2.2. *$\mathop{\smile\!\!\!\!\!\!|}^{i}$ satisfies base monotonicity, i.e. $a \mathop{\smile\!\!\!\!\!\!|}^{i}_{A} bc$ implies $a \mathop{\smile\!\!\!\!\!\!|}^{i}_{Ab} c$*

Proof. Clear. □

Fact 2.3. *$\mathop{\smile\!\!\!\!\!\!|}^{i}$ preserves indiscernibility; that is, if I is an A-indiscernible sequence and $a \mathop{\smile\!\!\!\!\!\!|}^{i}_{A} I$, then I is also Aa-indiscernible.*

Proof. It suffices to suppose $I = (b_i)_{i<\omega}$ has order type ω. Let $\varphi(x, y_0, \ldots, y_n)$ be a formula over A and suppose that $\models \varphi(a, b_{i_0}, \ldots, b_{i_n})$ for some $i_0 < \cdots < i_n$. Let $p(x)$ be a global A-invariant extension of $\operatorname{tp}(a/IA)$. Then it must be that $\varphi(x, b_{i_0}, \ldots, b_{i_n}) \in p(x)$. For any other $j_0 < \cdots < j_n$, since $b_{i_0} \ldots b_{i_n} \equiv_A b_{j_0} \ldots b_{j_n}$ and $p(x)$ is A-invariant, it must be that $\varphi(x, b_{j_0}, \ldots, b_{j_n}) \in p(x)$. Since $a \models p(x) |_{AI}$, conclude $\models \varphi(a, b_{j_0}, \ldots, b_{j_n})$. □

The remaining terminology will follow the example of [1]. In addition to the usual definition of dividing we need the notions of Kim-dividing (and Kim-forking):

Definition 2.4. Say that a formula $\varphi(x, b)$ *Kim-divides* over A if there is a global A-invariant type $p(y) \supseteq \operatorname{tp}(b/A)$ such that $\varphi(x, b)$ divides along some (equivalently, any) Morley sequence in p over A. The "equivalently" is justified by the fact that the type over A of a Morley sequence of length ω in the A-invariant type p is determined uniquely by p. The type of this Morley sequence is denoted $p^{\otimes\omega}|_A$

Say that a formula *Kim-forks* over A if it implies a disjunction of formulas each Kim-dividing over A.

This gives rise to another independence notion:

Definition 2.5. The ternary invariant relation $\mathop{\smile\!\!\!\!\!\!\;\mid}^{K}$ on subsets of $\mathcal{U}$ is defined by:

$$a \mathop{\smile\!\!\!\!\!\!\;\mid}_A^K b \text{ iff } \operatorname{tp}(a/Ab) \text{ does not imply any formula Kim-forking over } A$$

We will also need a characterization of non-Kim-dividing in terms of indiscernible sequences, similar to another standard lemma for non-dividing (this is [3, Lemma 3.18]):

Fact 2.6. *The following are equivalent:*

1. $\operatorname{tp}(a/Ab)$ *does not Kim-divide over* A.
2. *For any global* A*-invariant* $p \supseteq \operatorname{tp}(b/A)$ *and* I *a Morley sequence in* p *over* A *starting with* b*, there is* $a' \equiv_{Ab} a$ *such that* I *is* Aa'*-indiscernible.*
3. *For any global* A*-invariant* $p \supseteq \operatorname{tp}(b/A)$ *and* I *a Morley sequence in* p *over* A *starting with* b*, there is* $I' \equiv_{Ab} I$ *which is* Aa*-indiscernible.*

Next, two stronger notions of invariant types:

Definition 2.7. Say that a global A-invariant type p is strictly (resp. Kim-strictly) A-invariant iff for any set B and realization $a \models p\restriction_{AB}$, $B \mathop{\smile\!\!\!\!\!\!\;\mid}_A^f a$ (resp. $B \mathop{\smile\!\!\!\!\!\!\;\mid}_A^K a$).

This gives rise to two notions further restricting Kim-dividing:

Definition 2.8. Say that $\varphi(x, b)$ strictly (resp Kim-strictly) divides over A if there is a global strictly (resp Kim-strictly) A-invariant type $p \supseteq \operatorname{tp}(b/A)$ such that some $\varphi(x, b)$ divides along some (equivalently any) Morley sequence in p over A.

Lastly, the notion of universal (Kim, strict, Kim-strict)-dividing:

Definition 2.9. Say that $\varphi(x, b)$ *universally* Kim (resp. strictly, Kim-strictly) divides over A iff for *any* global A-invariant (resp. strictly invariant, Kim-strictly invariant) extension p of $\operatorname{tp}(b/A)$, $\varphi(x, b)$ divides along some (equivalently any) Morley sequence in p over A.

Note that these definitions may be vacuously satisfied (even by a non-dividing formula) if invariant global extensions fail to exist. In everything that follows, the base is taken to be a model, so invariant extensions will exist (e.g. coheirs). Still, in arbitrary theories, strictly invariant extensions may not exist. But [1, Theorem 2.26], quoted as Theorem 3.3 below, guarantees that *Kim-strict* invariant extensions do always exist (over a model).

3 Main ideas

The context of [2, Lemma 3.12] assumes NTP_2 of the ambient theory but the proof only uses this hypothesis of the single formula in question. Hence restricting our attention to dividing over models, and taking $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^i$ for the independence relation in the hypothesis, that result can be phrased as follows:

Lemma 3.1. *(T arbitrary) Any NTP_2 formula which divides over a model M, Kim-divides over M.*

Similarly, [2, Lemma 3.14], after noting that the proof is local, can be restated as:

Lemma 3.2. *(T arbitrary) Any NTP_2 formula which divides over M, universally strictly divides over M.*

The difficulty with using this lemma in the setting of an arbitrary theory is that strictly invariant global extensions of a given type over a model may not exist. However, [1, Theorem 2.26] is the following:

Theorem 3.3. *(T arbitrary) Every type over a model M has a global Kim-strict M-invariant global extension.*

The main observation to make now is that the proof of [2, Lemma 3.14] can be easily modified to strengthen the conclusion from "universal strict dividing" to "universal Kim-strict dividing". The proof below is identical to that in [2] except for the single place noted, but we include it in full for completeness' sake:

Lemma 3.4. *(T arbitrary) An NTP_2 formula which divides over a model M, universally Kim-strictly divides over M.*

Proof. Let $\varphi(x, y)$ be NTP_2. Suppose $\varphi(x, a)$ divides over M. By Lemma 3.1 above there is $p(y) \supseteq \mathrm{tp}(a/M)$ a global M-invariant type such that φ divides along every Morley sequence generated by p over M. Let $I = (a_i)_{i\in\omega} \models p^{\otimes\omega}|_M$, so that $\{\varphi(x, a_i)\}_{i\in\omega}$ is inconsistent (in fact, k-inconsistent for some $k < \omega$). Suppose towards contradiction that there is a global Kim-strict M-invariant type $q(y) \supseteq \mathrm{tp}(a/M)$ such that $\varphi(x, a)$ does not divide along some (equivalently, any) Morley sequence in q over M. Then there is $(b_i)_{i\in\omega} \models q^{\otimes\omega}|_M$ such that $\{\varphi(x, b_i)\}_{i\in\omega}$ is consistent. We seek to build an array of parameters witnessing TP_2 to yield contradiction.

It will suffice to inductively construct rows $(I_i)_{i<n} = ((a_{i,j})_{j\in\omega})_{i<n}$ such that the first term of I_i is b_i, each $I_i \equiv_M I$, and each I_i is indiscernible over $MI_{<i}b_{>i}$: for once this induction is established, it will follow by compactness that there exists an infinite array whose rows have the same properties. Then for each i, $\{\varphi(x, a_{i,j})\}_{j<\omega}$ must be inconsistent (and hence k-inconsistent) by $I_i \equiv_M I$. But $\{\varphi(x, a_{i,0})\}_{i<\omega} = \{\varphi(x, b_i)\}_{i<\omega}$ is inconsistent, so if it can be established that all vertical paths through the array have the same type over M (and hence all corresponding sequences of instances of φ are consistent), then TP_2 is established. For this, it will suffice to show that all vertical paths of length n have the same type over M. So let $\sigma : n \to \omega$ be a sequence of length n. We will show by decreasing induction on $i \leq n$ that

$$a_{0,\sigma(0)} \dots a_{n-1,\sigma(n-1)} \equiv_M a_{0,\sigma(0)} \dots a_{i-1,\sigma(i-1)}, b_i, \dots, b_{n-1}$$

The base case $i = n$ (by which we mean the claim $a_{0,\sigma(0)} \dots a_{n-1,\sigma(n-1)} \equiv_M a_{0,\sigma(0)} \dots a_{n-1,\sigma(n-1)}$) is immediate. Now suppose the claim for i. By the inductive hypothesis, there exists an automorphism $\tau \in \mathrm{Aut}(\mathcal{U}/M)$ taking $a_{0,\sigma(0)} \dots a_{n-1,\sigma(n-1)}$ to $a_{0,\sigma(0)} \dots a_{i-1,\sigma(i-1)}, b_i, \dots, b_{n-1}$. By the assumption that I_{i-1} is $MI_{<(i-1)}b_{>(i-1)}$-indiscernible, there exists an automorphism $\rho \in \mathrm{Aut}(\mathcal{U}/MI_{<(i-1)}b_{>(i-1)})$ taking $a_{i-1,\sigma(i-1)}$ to b_{i-1}. In particular, ρ fixes $a_{0,\sigma(0)} \dots a_{i-2,\sigma(i-2)}, b_i, \dots, b_{n-1}$. Therefore $\rho \circ \tau$ takes

$$a_{0,\sigma(0)} \dots a_{n-1,\sigma(n-1)} \mapsto a_{0,\sigma(0)} \dots a_{i-2,\sigma(i-2)}, b_{i-1}, \dots, b_{n-1}$$

Which proves the induction.

Now to inductively construct the rows as stipulated. For the base case, the first row, take an M-conjugate of I whose first element is b_0; this is possible since $b_0 \equiv_M a_0$.

For the inductive step, suppose that rows $I_{<n}$ have been built as claimed. Let $b'_n \models q\restriction_{MI_{<n}}$. Then in particular, $b'_n \equiv_{Mb_{<n}} b_n$, so conjugate to find $J_{<n} \equiv_{Mb_{<n}} I_{<n}$ such that $b_n \models q\restriction_{MJ_{<n}}$. Explicitly, let $\sigma \in \mathrm{Aut}(\mathcal{U}/Mb_{<n})$ take $b'_n \mapsto b_n$. Then for any $c \in MI_{<n}$ and formula $\psi(y,z)$ we have $\psi(y, \sigma(c)) \in q$ iff (by M-invariance) $\psi(y,c) \in q$ iff $\models \psi(b'_n, c)$ iff $\models \psi(b_n, \sigma(c))$, so $b_n \models q\restriction_{M\sigma(I_{<n})}$, so we take $J_{<n} := \sigma(I_{<n})$.

Note that at this stage $J_{<n}$ still satisfies the hypotheses put on $I_{<n}$. The rest of this paragraph is the only modification of the original proof: by Kim-strictness of q, $J_{<n} \mathop{\smile\!\!\!\!\!\mid}^K_M b_n$ and in particular $\mathrm{tp}(J_{<n}/Mb_n)$ does not Kim-divide over M. As in the base case take $I' \equiv_M I$ such that I' starts with b_n. Then $I' \models p^{\otimes\omega}\restriction_M$ with p a global M-invariant extension of $\mathrm{tp}(b_n/M)$, so by Fact 2.6(3), there exists $J_n \equiv_{Mb_n} I'$ such that J_n is $MJ_{<n}$-indiscernible. Note that in particular J_n begins with b_n; we take this as our new nth row.

It remains to verify that the array $J_{\leq n}$ has all the required properties. M-equivalence of the rows and correct first entries are immediate. As for the indiscernibility: J_n is indiscernible over $MJ_{<n}$ by the above construction. For $i < n$, J_i is indiscernible over $MJ_{<i}b_{i+1} \dots b_{n-1}$ by the induction hypothesis. Since $b_n \models q\restriction_{MJ_{<n}}$, and q is a global M-invariant type, we get $b_n \mathop{\smile\!\!\!\!\!\mid}^i_M J_{<n}$ and in particular (by base monotonicity), $b_n \mathop{\smile\!\!\!\!\!\mid}^i_{MJ_{<i}b_{i+1}\dots b_{n-1}} J_i$. Since $\mathop{\smile\!\!\!\!\!\mid}^i$ preserves indiscernibility, this implies that J_i is indiscernible over $MJ_{<i}b_{i+1} \dots b_{n-1}b_n$ as required. $\square$

Remark 3.5. [1, Theorem 3.5] remarks that, in light of the equivalence between forking and Kim-forking over models in NTP_2 theories, Kim-strictly invariant types are the same as strictly invariant types in NTP_2 theories. Hence, if the whole theory is assumed NTP_2, one immediately upgrades the conclusion of [2, Lemma 3.14] from "universally strictly divides" to "universally Kim-strictly divides" as above, without having to revisit the proof of the lemma. The argument above has been included to show one way that this can be done without assuming NTP_2 of the whole theory.

Lemma 3.4 could also be deduced more directly from other results in [1]. There the authors first introduce what they call New Kim's Lemma as a common generalization of existing variants of Kim's Lemma for NTP_2 and NSOP_1 theories. Namely, a theory T is said to satisfy New Kim's Lemma if for any formula $\varphi(x,b)$ and model M, if $\varphi(x,b)$ Kim-divides over M then it universally Kim-strictly divides over M. They further define a syntactic property of formulas called the k-bizarre tree property (k-BTP) and say a theory T is NBTP iff modulo T no formula has k-BTP for any k (one could just as well define BTP and NBTP for formulas in this way). [1, Theorem 5.2] is that all NBTP theories satisfy New Kim's Lemma. The proof shows specifically that any NBTP formula satisfies New Kim's Lemma (in the appropriate local sense). Moreover, [1, Proposition 5.3] proves that all NTP_2 formulas are NBTP. Altogether, we have that an NTP_2 formula dividing over a model M in fact Kim-divides (by Lemma 3.1), which (by New Kim's Lemma) further implies that it universally Kim-strictly divides.

4 Conclusions

Returning to the setting of several NTP_2 formulas $\{\varphi_i(x,b)\}_{i<\omega}$ each dividing over a model M, Theorem 3.3 guarantees the existence of a global Kim-strict M-invariant type p extending $\mathrm{tp}(b/M)$, and Lemma 3.4 guarantees that each $\varphi_i(x,b)$ divides along any $(b_i)_{i<\omega} \models p^{\otimes\omega}\restriction_M$, which is all that was needed to conclude:

Theorem 4.1. *(T arbitrary) Let $\{\varphi_i(x,y_i)\}_{i<n}$ be a collection of NTP_2 formulas over M. Suppose $(b_i)_{i<n}$ are such that $\varphi_i(x,b_i)$ divides over M for every $i<n$. Then*

$$\bigvee_{i<n} \varphi_i(x,b_i)$$

divides over M.

Applying this result to draw conclusions about φ-types for an arbitrary NTP_2 formula φ is complicated by the fact that NTP_2 is not in general preserved under boolean combinations. For example, if one wishes to take a nondividing extension of a φ-type over one model to a complete φ-type over another, one must exclude all dividing boolean combinations of instances of φ. Theorem 4.1 offers no help if said combinations do not happen also to be NTP_2.

However, we can carry out the argument just hinted at in the case that φ is NIP, since NIP formulas are closed under boolean combinations. For the remainder of this section, then, we will restrict attention to NIP formulas, obtaining a few conclusions which generalize existing facts from local stability theory.

Proposition 4.2. *(T arbitrary) Let $\varphi(x, y)$ be NIP and let $\pi(x)$ be a partial φ-type over a set A which does not divide over a model $M \subseteq A$. Then there exists a global extension $p(x) \in S_\varphi(\mathcal{U})$ of $\pi(x)$ which does not divide over M.*

Proof. Let $\Psi(x)$ denote the collection of M-dividing boolean combinations of instances of φ over $\mathcal{U}$. It will suffice to show that the partial type $\pi(x) \cup \{\neg\psi(x) : \psi(x) \in \Psi(x)\}$ is consistent, since dividing of a partial type can always be witnessed by dividing of some conjunction of formulas from that type. Suppose towards contradiction that $\pi(x) \cup \{\neg\psi(x) : \psi(x) \in \Psi(x)\}$ is not consistent. Then by compactness, it must be that $\pi(x) \vdash \bigvee_{i<n} \psi_i(x)$ for some $\{\psi_i(x)\}_{i<n} \subseteq \Psi(x)$. But each $\psi_i(x)$ is NIP, being a boolean combination of NIP formulas; then since each $\psi_i(x)$ divides over M, so does the disjunction $\bigvee_{i<n} \psi_i(x)$ by Theorem 4.1. This contradicts the assumption that $\pi(x)$ does not divide over M. $\square$

This non-dividing extension is, moreover, M-invariant, by the following proposition:

Proposition 4.3. *Let $\varphi(x, y)$ be an NIP formula and let $p(x) \in S_\varphi(\mathcal{U})$ be a global φ-type, nondividing over a model M. Then $p(x)$ is also M-invariant.*

Proof. This is essentially the argument given in [4, Proposition 5.21], which we restate just to show that it is local: Suppose $p(x)$ is not M-invariant. Then there must be $a \equiv_M b$ such that $p(x) \vdash \varphi(x, a) \wedge \neg\varphi(x, b)$. Since $a \equiv_M b$, there exists some parameter c such that a, c and c, b each begin M-indiscernible sequences, say I, J respectively. Either $p(x) \vdash \neg\varphi(x, c)$ or $p(x) \vdash \varphi(x, c)$ so by taking I or J respectively, we obtain an M-indiscernible sequence $(c_i)_{i<\omega}$ such that $p(x) \vdash \varphi(x, c_0) \wedge \neg\varphi(x, c_1)$. The collection $\{\varphi(x, c_{2i}) \wedge \neg\varphi(x, c_{2i+1})\}_{i<\omega}$ must be inconsistent by finiteness of alternation rank for the NIP formula $\varphi(x, y)$. But the sequence $(c_{2i}c_{2i+1})_{i<\omega}$ is an M-indiscernible sequence in $\mathrm{tp}(c_0c_1/M)$. This shows that $\varphi(x, c_0) \wedge \neg\varphi(x, c_1)$ divides over M, so $p(x)$ divides over M. $\square$

Summarizing:

Corollary 4.4. *If $\pi(x)$ is a a partial φ-type for φ an NIP formula and $\pi(x)$ does not divide over a model M, then $\pi(x)$ can be extended to a complete global M-invariant φ-type.*

As another conclusion we have the following:

Corollary 4.5. *Let $\varphi(x, y)$ be an NIP formula and suppose that $\varphi(x, b)$ does not divide over a model M. Then the collection $\{\varphi(x, b') : b' \equiv_M b\}$ does not divide over M.*

Proof. By Proposition 4.2 extend $\{\varphi(x, b)\}$ to $p(x) \in S_\varphi(\mathcal{U})$ nondividing over M. By Proposition 4.3, $p(x)$ is invariant over M and so contains the collection $\{\varphi(x, b') : b' \equiv_M b\}$ which therefore does not divide over M. $\square$

Corollary 4.5 also grants the equivalence of dividing, Kim-dividing, and Kim-forking over models for NIP formulas in an arbitrary theory, via another theorem of [1], which we mention here along with the definition of quasi-dividing:

Definition 4.6. A formula $\varphi(x, b)$ *quasi-divides* over a set A if there exist $(b_i)_{i<n}$ such that each $b_i \equiv_A b$ and $\bigwedge_{i<n} \varphi(x, b_i)$ is inconsistent.

Fact 4.7. *[1, Corollary 2.25] Any formula which Kim-forks over a model M quasi-divides over M.*

Now we can prove:

Proposition 4.8. *Let $\varphi(x, y)$ be an NIP formula, M a model, and b a parameter tuple of length $|y|$. Then $\varphi(x, b)$ divides over M iff it Kim-divides over M iff it Kim-forks over M iff it quasi-divides over M.*

Proof. Dividing $\implies$ Kim-dividing is Lemma 3.1. Kim-dividing $\implies$ Kim-forking is immediate from definitions. Kim-forking $\implies$ quasi-dividing is Fact 4.7. Lastly, if $\varphi(x, b)$ quasi-divides over M, then $\{\varphi(x, b') : b' \equiv_M b\}$ is inconsistent, so by Corollary 4.5, $\varphi(x, b)$ divides over M. $\square$

5 References

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