Big line or big convex polygon

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Abstract

Let $ES_\ell(n)$ be the minimum N such that every N-element point set in the plane contains either ℓ collinear members or n points in convex position. We prove that there is a constant $C > 0$ such that, for each $\ell, n \geq 3$,

$$(3\ell - 1) \cdot 2^{n-5} < ES_\ell(n) < \ell^2 \cdot 2^{n+C\sqrt{n \log n}}.$$

A similar extension of the well-known Erdős–Szekeres cups-caps theorem is also proved.

1 Introduction

Given an n-element point set P in the plane, we say that P is in *convex position* if P is the vertex set of a convex n-gon. We say that P is in *general position* if no three members of P are collinear. In 1935, addressing a problem raised by Klein, Erdős and Szekeres [4] proved that, for every integer $n \geq 3$, there is a minimal integer $ES(n)$ such that any set of $ES(n)$ points in the plane in general position contains n members in convex position. Moreover, they showed that $ES(n) \leq \binom{2n-4}{n-2} + 1 = 4^{n+o(n)}$. Many years later [5], they proved that $ES(n) \geq 2^{n-2} + 1$, a bound that they had already conjectured to be tight in their earlier paper. It remained an open problem for several decades to improve the bound $ES(n) \leq 4^{n+o(n)}$ by any significant factor. This was finally accomplished by Suk [13], who proved that $ES(n) = 2^{n+o(n)}$, coming close to matching Erdős and Szekeres' lower bound and proving their conjecture. The best explicit bound for the $o(n)$ term to date is due to Holmsen et al. [6], who optimized the argument in [13] and showed that $ES(n) \leq 2^{n+O(\sqrt{n \log n})}$.

In this paper, we extend these results to arbitrary point sets in the plane (see, for example, Exercise 3.1.3 in [8] for the corresponding qualitative result). Let $ES_\ell(n)$ be the minimum N such that every N-point set in the plane contains either ℓ collinear points or n points in convex position. Hence, $ES_3(n) = ES(n)$. For $\ell > 3$, a simple upper bound can be obtained as follows. Given an N-element point set P in the plane, if P contains ℓ collinear members, then we are done. Hence, we can assume that P does not contain ℓ collinear members. By the Szemerédi–Trotter theorem [14], there are at most $O(N^2 + \ell^2 N)$ collinear triples in P. A

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standard probabilistic argument shows that we can find a subset $P' \subset P$ of size $\Omega(N/\sqrt{N+\ell^2})$ such that P' is in general position. Therefore, as long as $|P'| \geq 2^{n+o(n)}$, we can apply the results above to find n members in convex position. Unfortunately, this only implies that $ES_\ell(n) \leq 2^{2n+O(\sqrt{n \log n})} + \ell \cdot 2^{n+O(\sqrt{n \log n})}$. Our main result is the following.

Theorem 1.1. *There exists $C > 0$ such that, for each $\ell, n \geq 3$, $ES_\ell(n) \leq \ell^2 \cdot 2^{n+C\sqrt{n \log n}}$.*

The proof of Theorem 1.1 is based on both a new cups-caps theorem for arbitrary point sets in the plane and a new positive fraction Erdős–Szekeres theorem. In the case where n is fixed and ℓ tends to infinity, our cups-caps theorem implies that $ES_\ell(n) = O(\ell)$, which is best possible up to constants. In turn, our lower bound for the cups-caps theorem implies the following lower bound for $ES_\ell(n)$, which agrees with the Erdős–Szekeres lower bound in the $\ell = 3$ case.

Theorem 1.2. *For each $\ell, n \geq 3$, $ES_\ell(n) \geq (3\ell - 1) \cdot 2^{n-5} + 1$.*

It remains an interesting open problem to determine the correct dependence of $ES_\ell(n)$ on ℓ.

The paper is organized as follows. In the next section, we prove our cups-caps theorem for arbitrary point sets and Theorem 1.2. In Section 3, we establish a positive fraction Erdős–Szekeres theorem for arbitrary point sets. Finally, in Section 4, we prove Theorem 1.1. For the sake of clarity, we omit floor and ceiling signs whenever they are not crucial. We assume throughout that our point sets have distinct x-coordinates, since we can slightly rotate the plane otherwise.

2 A cups-caps theorem for arbitrary point sets

Let X be a k-element point set in the plane with distinct x-coordinates. We say that X forms a *k-cup* (*k-cap*) if X is in convex position and its convex hull is bounded above (below) by a single segment. The *length* of a k-cup (k-cap) is $k-1$. Write $f_\ell(m,n)$ for the minimum N such that every N-point set in the plane contains either ℓ collinear members, an m-cup or an n-cap. Erdős and Szekeres [4] proved that

$$f_3(m,n) = \binom{m+n-4}{n-2} + 1. \tag{1}$$

For $\ell \geq 3$, we prove the following.

Theorem 2.1. *There is an absolute constant $c > 1$ such that, for $m, n, \ell \geq 3$,*

$$f_\ell(m,n) \leq c(\min\{m-1, n-1\} + \ell) \cdot \binom{m+n-4}{n-2}.$$

The proof of this theorem is based on a connection between down-sets and (1) discovered by Moshkovitz and Shapira [9]. We will also need the following lemma due to Beck (Theorem 1.2 in [1]).

Lemma 2.2. *There is an absolute constant $\varepsilon > 0$ such that every t-element point set in the plane contains either εt collinear points or determines at least $\varepsilon\binom{t}{2}$ distinct lines.*

Proof of Theorem 2.1. Let $\varepsilon > 0$ be the absolute constant from Lemma 2.2 and set $c = 10/\varepsilon$. Let P be an N-element point set in the plane where $N = c \cdot (\min\{m-1, n-1\} + \ell) \cdot \binom{m+n-4}{n-2}$. We may assume that P does not contain ℓ collinear members, since otherwise we would be done. Given points $p, q \in P$, we write $p < q$ if the x-coordinate of p is less than the x-coordinate of q. For the sake of contradiction, suppose P contains neither an m-cup nor an n-cap. Hence, the longest cup in P has length at most $m-2$ and the longest cap in P has length at most $n-2$.

Let $p, q \in P$ be such that $p < q$. We label the pair pq with the ordered pair (x_{pq}, y_{pq}), where x_{pq} is the length of the longest cup ending at pq and y_{pq} is the length of the longest cap ending at pq. Let $L(m-2, n-2)$ be the poset on $[m-2] \times [n-2]$ where $(x, y) \preceq (x', y')$ iff $x \leq x'$ and $y \leq y'$. Then a set $D \subset [m-2] \times [n-2]$ is called a *down-set* if $(x, y) \in D$ implies that $(x', y') \in D$ for all $(x', y') \prec (x, y)$. For each $q \in P$, let $S(q) = \{(x_{pq}, y_{pq}) : p \in P,\, p < q\}$. Let

$$D(q) = \{(x, y) \in L(m-2, n-2) : \exists (x_{pq}, y_{pq}) \in S(q),\, (x, y) \preceq (x_{pq}, y_{pq})\}$$

be the *down-set* in $L(m-2, n-2)$ generated by $S(q)$.

The number of down-sets in $L(m-2, n-2)$ is $\binom{m+n-4}{n-2}$ (see, e.g., [9, Observation 2.1]). Hence, by the pigeonhole principle, there are points $q_1 < q_2 < \cdots < q_t$ in P with $t \geq c \cdot (\min\{m-1, n-1\} + \ell)$ such that $D(q_i) = D(q_j)$ for all $i < j$. Set $Q = \{q_1, \ldots, q_t\}$. By Lemma 2.2, Q contains either εt collinear members or determines at least $\varepsilon\binom{t}{2}$ distinct lines. In the former case, we have $\varepsilon t > \ell$ collinear points, which is a contradiction. Hence, Q determines at least $\varepsilon\binom{t}{2}$ distinct lines. By averaging, there is a point $p \in Q$ and a subset $Q' \subset Q$ of size at least $\varepsilon t/2 > \min\{m-1, n-1\}$ such that $p < q$ for each $q \in Q'$ and there are $|Q'|$ distinct lines between p and Q'. Consider the labels on pq for each $q \in Q'$. Since the maximum size of an antichain in $L(m-2, n-2)$ is $\min\{m-1, n-1\}$, by the pigeonhole principle, we obtain points p, q, q' such that $p < q < q'$ and

1. $D(p) = D(q) = D(q')$ and
2. $(x_{pq}, y_{pq}) \preceq (x_{pq'}, y_{pq'})$ or $(x_{pq}, y_{pq}) \succeq (x_{pq'}, y_{pq'})$.

Let us assume by symmetry that $(x_{pq}, y_{pq}) \succeq (x_{pq'}, y_{pq'})$. Since $D(p) = D(q)$, there exists $(x, y) \in S(p) \subset D(p)$ such that $(x, y) \succeq (x_{pq}, y_{pq})$ and, by transitivity, $(x, y) \succeq (x_{pq'}, y_{pq'})$. By the definition of $S(p)$, there exists $p' < p$ such that $x = x_{p'p}$ and $y = y_{p'p}$. Since p, q, q' are not collinear, one of $p'pq, p'pq'$ is not collinear. Without loss of generality, we can assume that $p'pq$ is not collinear, since the other case is symmetric. Then the triple $p'pq$ is either a cup or a cap. In the former case, the longest cup ending at $p'p$ with length $x_{p'p}$ can be extended to end at pq, which is a contradiction. If instead $p'pq$ is a cap, then the longest cap ending at $p'p$ with length $y_{p'p}$ can be extended to end at pq, again a contradiction. □

In the other direction, we prove the following.

Theorem 2.3.

$$f_\ell(m, n) \geq \frac{\ell-1}{2}\binom{m+n-4}{n-2} - \frac{\ell-3}{2}\binom{m+n-6}{n-3} + 1.$$

Proof. Set

$$h_\ell(m, n) := \frac{\ell-1}{2}\binom{m+n-4}{n-2} - \frac{\ell-3}{2}\binom{m+n-6}{n-3}.$$

In what follows, we will recursively construct planar point sets $X_{\ell,m,n}$ with $|X_{\ell,m,n}| \geq h_\ell(m, n)$ that contain neither ℓ collinear points, m-cups nor n-caps. Consider a regular m-gon, whose vertices are located at $(\cos(2\pi i/m), \sin(2\pi i/m)) \in \mathbb{R}^2$, for $i = 1, \ldots, m$. We refer to the sides of the m-gon as segments. For $\ell, m \geq 3$, we construct $X_{\ell,m,3}$ by taking the lower half of the regular m-gon that lies below the x-axis and, on $\lfloor (m-1)/2 \rfloor$ of these non-vertical segments (sides of the m-gon), placing $\ell - 1$ collinear points in the interior of the segment. If $m-1$ is odd, then add another point on an empty segment by itself (adding more than one point to this segment would create an m-cup). Hence, we have no ℓ collinear points, no m-cup and no 3-cap. Moreover,

$$|X_{\ell,m,3}| = \begin{cases} (\ell-1)\frac{m-1}{2} & \text{if } m-1 \text{ is even} \\ (\ell-1)\frac{m-2}{2} + 1 & \text{if } m-1 \text{ is odd.} \end{cases}$$

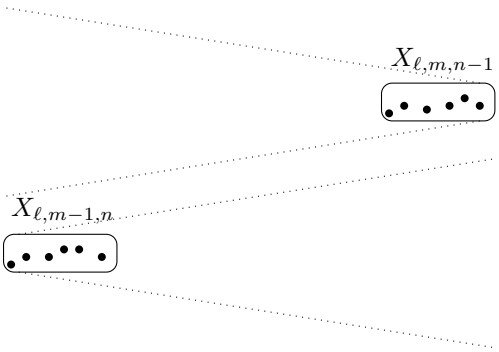

Figure 1: Construction of $X_{\ell,m,n}$ from $X_{\ell,m-1,n}$ and $X_{\ell,m,n-1}$.

Hence, for all $m \geq 3$,

$$|X_{\ell,m,3}| \geq \frac{\ell-1}{2}(m-1) - \frac{\ell-3}{2} = h_\ell(m,3),$$

as desired. We construct $X_{\ell,3,n}$ similarly such that

$$|X_{\ell,3,n}| \geq \frac{\ell-1}{2}(n-1) - \frac{\ell-3}{2} = h_\ell(3,n).$$

For the recursive step, assume that we have constructed $X_{\ell,m',n'}$ for all $m' < m$ or $n' < n$. We construct $X_{\ell,m,n}$ as follows. Take a very flat copy of $X_{\ell,m-1,n}$ and a very flat copy of $X_{\ell,m,n-1}$ such that $X_{\ell,m,n-1}$ is very high and far to the right of $X_{\ell,m-1,n}$, the line spanned by any two points in $X_{\ell,m-1,n}$ lies below $X_{\ell,m,n-1}$ and the line spanned by any two points in $X_{\ell,m,n-1}$ lies above $X_{\ell,m-1,n}$. See Figure 1.

Hence, the resulting set does not contain ℓ collinear points and neither an m-cup nor an n-cap. Finally,

$$\begin{aligned}
|X_{\ell,m,n}| &= |X_{\ell,m-1,n}| + |X_{\ell,m,n-1}| \\
&\geq h_\ell(m-1,n) + h_\ell(m,n-1) \\
&= \frac{\ell-1}{2}\binom{m+n-5}{n-2} - \frac{\ell-3}{2}\binom{m+n-7}{n-3} + \frac{\ell-1}{2}\binom{m+n-5}{n-3} - \frac{\ell-3}{2}\binom{m+n-7}{n-4} \\
&= \frac{\ell-1}{2}\binom{m+n-4}{n-2} - \frac{\ell-3}{2}\binom{m+n-6}{n-3},
\end{aligned}$$

as required. □

2.1 Proof of Theorem 1.2

We now use Theorem 2.3 to prove Theorem 1.2, the statement that $ES_\ell(n) \geq (3\ell-1)\cdot 2^{n-5}+1$ for all $\ell, n \geq 3$.

Proof of Theorem 1.2. Let $X_{\ell,m,n}$ be the point set from the proof of Theorem 2.3 with no ℓ collinear points, no m-cup and no n-cap, recalling that

$$|X_{\ell,m,n}| \geq h_\ell(m,n) = \frac{\ell-1}{2}\binom{m+n-4}{n-2} - \frac{\ell-3}{2}\binom{m+n-6}{n-3}.$$

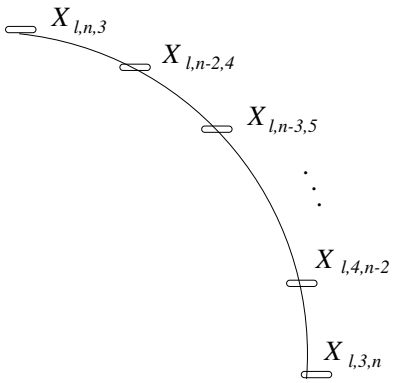

Figure 2: The lower bound construction for $ES_\ell(n)$.

Let S be a unit circle in the plane centered at the origin and consider the arc α along S from $(0,1)$ to $(1,0)$. Place a very small flat copy of $X_{\ell,n,3}$ near $(0,1)$ and a very small flat copy of $X_{\ell,3,n}$ near $(1,0)$. Then evenly spread out very small flat copies of $X_{\ell,n-2,4}, X_{\ell,n-3,5}, \ldots, X_{\ell,n-2-i,4+i}, \ldots, X_{\ell,4,n-2}$ along α from top to bottom, between $X_{\ell,n,3}$ and $X_{\ell,3,n}$. We make each copy flat enough that the line generated by any two points in $X_{\ell,n-2-i,4+i}$ lies below $X_{\ell,n-2-j,4+j}$ for $j < i$ and lies above $X_{\ell,n-2-j,4+j}$ for $j > i$. This implies that the union of such sets contains no ℓ collinear points. See Figure 2. Let P be the resulted point set

$$X_{\ell,n,3} \cup \left(\bigcup_{i=4}^{n-2} X_{\ell,n+2-i,i} \right) \cup X_{\ell,3,n}.$$

Then

$$\begin{aligned}
|P| &\geq \frac{\ell-1}{2}\left(\binom{n-1}{1} + \left(\sum_{i=0}^{n-6}\binom{n-2}{2+i}\right) + \binom{n-1}{n-2}\right) - \frac{\ell-3}{2}\left(\binom{n-3}{0} + \left(\sum_{i=0}^{n-6}\binom{n-4}{1+i}\right) + \binom{n-3}{n-3}\right) \\
&= \frac{\ell-1}{2}\sum_{i=0}^{n-2}\binom{n-2}{i} - \frac{\ell-3}{2}\sum_{i=0}^{n-4}\binom{n-4}{i} \\
&= \frac{\ell-1}{2}2^{n-2} - \frac{\ell-3}{2}2^{n-4} \\
&= (3\ell-1)2^{n-5}.
\end{aligned}$$

Now suppose that $K \subset P$ is a subset in convex position. If $K \subset X_{\ell,n-2-i,4+i}$ for some $i \geq 0$, then $|K| < n$. If $K \subset X_{\ell,n,3}$, then $|K| < n$ by the structure of $X_{\ell,n,3}$. A similar argument holds if $K \subset X_{\ell,3,n}$.

Suppose then that K has a non-empty intersection with at least two of the parts. Let i be the minimum integer such that $K \cap X_{\ell,n-2-i,4+i} \neq \emptyset$ and j be the maximum integer such that $K \cap X_{\ell,n-2-j,4+j} \neq \emptyset$. Assume that $0 \leq i \leq j \leq n-6$, that is, that K omits both the highest and lowest sets in our construction. By the flatness condition, for all $i < s < j$, we have $|K \cap X_{\ell,n-2-s,4+s}| \leq 1$. Hence,

$$|K| \leq (4+i-1) + (j-i-1) + (n-2-j-1) = n-1.$$

Suppose now that $|K \cap X_{\ell,n,3}| \neq \emptyset$ and the largest j such that $K \cap X_{\ell,n-2-j,4+j} \neq \emptyset$ satisfies $0 \leq j \leq n-6$ (or that no such j exists) and $|K \cap X_{\ell,3,n}| = \emptyset$. If $|K \cap X_{\ell,n,3}| \geq 3$, then $K \cap X_{\ell,n,3}$ is a cup, which means

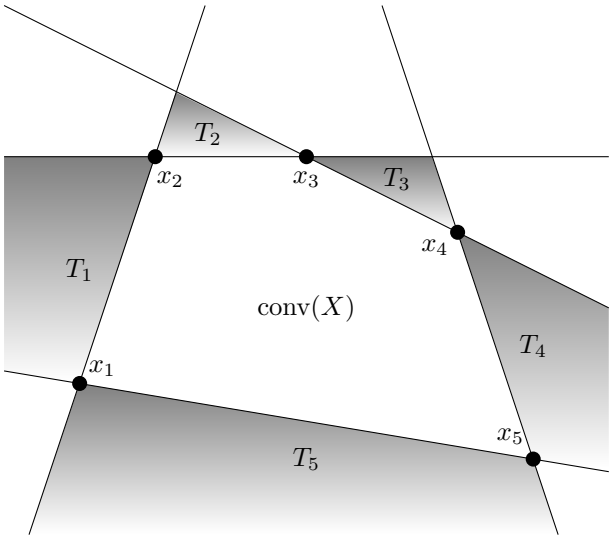

Figure 3: Regions T_i in the support of X.

that $K \subset X_{\ell,n,3}$ and hence $|K| \leq n-1$. Otherwise, $|K \cap X_{\ell,n,3}| \leq 2$ and $|K| \leq 2 + (n-3-j) + j = n-1$. A similar argument applies if $|K \cap X_{\ell,3,n}| \neq \emptyset$. Finally, if $|K \cap X_{\ell,n,3}| \neq \emptyset$ and $|K \cap X_{\ell,3,n}| \neq \emptyset$, then $|K| \leq 2 + (n-5) + 2 = n-1$. Hence, $ES_\ell(n) \geq (3\ell-1)2^{n-5} + 1$, as required. □

3 A positive fraction cups-caps theorem for arbitrary point sets

In this subsection, we establish a positive fraction cups-caps theorem for arbitrary point sets. Given a k-cap (k-cup) $X = \{x_1, \ldots, x_k\}$, where the points appear in order from left to right, we define the *support of* X to be the collection of open regions $\mathcal{C} = \{T_1, \ldots, T_k\}$, where T_i is the region outside of conv(X) bounded by the segment $\overline{x_i x_{i+1}}$ and by the lines $x_{i-1}x_i$, $x_{i+1}x_{i+2}$ (where $x_{k+1} = x_1$, $x_{k+2} = x_2$, etc.). See Figure 3.

Theorem 3.1. *There is a constant c_1 such that the following holds. Let P be an N-point planar set with no ℓ points on a line and $N > c_1 \ell \cdot 2^{32k}$. Then there is a k-element subset $X \subset P$ that is either a k-cup or a k-cap such that, for the regions $T_1, \ldots, T_{k-1}$ from the support of X, the point sets $P_i = P \cap T_i$ satisfy $|P_i| \geq N/2^{32k}$. In particular, every $(k-1)$-tuple obtained by selecting one point from each P_i, $i = 1, \ldots, k-1$, is in convex position.*

Let us remark that a positive-fraction cups-caps theorem for point sets in general position was first proved by Pach and Solymosi [10] and can be found more explicitly in [12]. Its proof is a simple supersaturation argument using (1). Unfortunately, this approach for point sets with no ℓ collinear members gives a rather poor dependency on ℓ. Instead, we will make use of simplicial partitions together with the probabilistic method. First, we need some simple definitions. A *cell* $\Delta \subset \mathbb{R}^2$ is a 1 or 2-dimensional simplex, i.e., closed segments or triangles. Given a cell $\Delta \subset \mathbb{R}^2$, we say that a line L *crosses* Δ if L intersects, but does not contain, Δ.

Lemma 3.2 ([2], Section 3)**.** *Let P be a set of N points in the plane. Then, for any integer $1 \leq r \leq N$, there are pairwise disjoint subsets $P_1, \ldots, P_r$ of P and disjoint cells $\Delta_1, \ldots, \Delta_r$ in $\mathbb{R}^2$, with $P_i \subset \Delta_i$, such that $|P_i| \geq N/(8r)$ and every line in the plane crosses at most $O(\sqrt{r})$ cells Δ_i.*

Let us remark that in the original version of simplicial partitions due to Matoušek [7], the cells Δ_i may not necessarily be disjoint. However, in a newer version due to Chan [2], disjointness can also be guaranteed.

Proof of Theorem 3.1. Let $c_1 > c_2$ be large constants that will be determined later. Set $r = c_2 2^{16k}$. Then we apply Lemma 3.2 with parameter r to obtain pairwise disjoint subsets $P_1, \ldots, P_r \subset P$ and pairwise disjoint cells $\Delta_1, \ldots, \Delta_r \subset \mathbb{R}^2$ such that $|P_i| \geq N/(8r)$ and $P_i \subset \Delta_i$. Moreover, every line in the plane crosses at most $O(\sqrt{r})$ cells Δ_i. Since

$$|P_i| \geq \frac{N}{8r} \geq \frac{c_1 \ell 2^{32k}}{8 c_2 2^{16k}} > \ell,$$

no line contains a cell Δ_i. We call a triple (P_i, P_j, P_s) of parts *bad* if there is a line intersecting all three cells Δ_i, Δ_j and Δ_s. Otherwise, we call the triple (P_i, P_j, P_s) *good*.

If there are three disjoint cells $\Delta_i, \Delta_j, \Delta_s$ and a line L that intersects all three, then we can translate and rotate L so that L is tangent to two of the cells and intersects the third. Hence, for every bad triple (P_i, P_j, P_s), there is a line L tangent to two of the cells, say Δ_i and Δ_j, such that L intersects Δ_s. For every pair $\{i, j\}$, there are at most 4 tangent lines for Δ_i and Δ_j and, by our application of Lemma 3.2, there are at most $O(\sqrt{r})$ parts Δ_s that intersect any of these 4 lines. Hence, the number of bad triples (P_i, P_j, P_s) is at most $O(r^2\sqrt{r}) = c' r^{5/2}$, where c' is an absolute constant.

We pick each part P_i with probability $p = 1/(\sqrt{4c'} r^{3/4})$. Then the expected number of parts chosen is pr and the expected number of bad triples among them is at most

$$p^3 c' r^{5/2} \leq p^2 c' r^{5/2} \leq pr/4.$$

Hence, by the Chernoff bound, we can select at least $3pr/4$ parts P_i such that the number of bad triples among them is at most $pr/2$. By deleting one part from each bad triple, we obtain $pr/4$ parts P_i such that every triple among them is good.

For simplicity, let $P_1, \ldots, P_t$ be the remaining parts, where $t = pr/4 = \Theta(r^{1/4})$. By sweeping a vertical line from left to right, we can greedily pick subsets $P_i' \subset P_i$, $1 \leq i \leq t$, such that no vertical line intersects any two of the convex sets $C_i = \mathrm{conv}(P_i')$ and

$$|P_i'| \geq |P_i|/t > \Omega(N/r^{5/4}).$$

Without loss of generality, we can assume that the subsets $P_1', \ldots, P_t'$ appear from left to right. That is, the x-coordinate of each point in P_i' is less than the x-coordinate of each point in P_j' for $i < j$. Let Q be the t-element point set obtained by selecting one point from each of the remaining P_i'. Then Q is in general position. By setting c_2 sufficiently large, we have $|Q| = t = pr/4 \geq 4^{2k}$. By the Erdős–Szekeres cups-caps theorem (1), there is either a $(2k)$-cup or a $(2k)$-cap $X \subset Q$. We will assume that X is a $(2k)$-cap, since a symmetric argument works otherwise. Let $X = \{x_1, \ldots, x_{2k}\}$ be the points of X ordered from left to right and let us now assume that P_i' is the part that corresponds to the point $x_i \in X$.

Observation 3.3. *If $q_1 \in P_1', \ldots, q_{2k} \in P_{2k}'$, then $q_1, \ldots, q_{2k}$ forms a $(2k)$-cap.*

Proof. It suffices to show that every triple in $\{q_1, \ldots, q_{2k}\}$ forms a cap. For the sake of contradiction, suppose (q_i, q_j, q_s) is a cup. Since (x_i, x_j, x_s) is a cap, this implies that the convex sets $\mathrm{conv}(P_i'), \mathrm{conv}(P_j'), \mathrm{conv}(P_s')$ can be pierced by a line, a contradiction. □

Set $X' = \{x_1, x_3, \ldots, x_{2k-1}\}$. Let $T_1, \ldots, T_k$ be the support of X'. Then the k parts $P_2', P_4', \ldots, P_{2k}'$ must lie in $T_1, \ldots, T_k$, respectively. Moreover, by setting c_1 sufficiently large, each such part P_{2i}' satisfies

$$|P_{2i}'| \geq \Omega\left(\frac{N}{r^{5/4}}\right) \geq \frac{N}{2^{32k}},$$

as required. □

4 Big line or big convex polygon – Proof of Theorem 1.1

For the proof of Theorem 1.1, we will need the following more general version of Theorem 2.1. Let K be a convex set in the plane. Then we say that the point set P *avoids* K if the line generated by any two points in P is disjoint from K. We say that K and P are *separated* if there is a line that separates K and $\mathrm{conv}(P)$. Suppose now that K is a convex set in the plane, P is a finite point set that avoids K and K and P are separated. Then, given a subset $X \subset P$, we say that X is an *inner-cap* with respect to K if, for each point $x \in X$, there is a line that separates x from $(X \setminus \{x\}) \cup K$. Similarly, we say that $X \subset P$ is an *outer-cup* with respect to K if, for each point $x \in X$, there is a line that separates $\{x\} \cup K$ from $X \setminus \{x\}$.

Theorem 4.1. *There is an absolute constant $c > 0$ such that the following holds. Let K be a convex set in the plane and let P be a finite point set in the plane that avoids K. If K and P are separated and*

$$|P| \geq c(\min\{m-1, n-1\} + \ell) \cdot \binom{m+n-4}{n-2},$$

then P contains either ℓ collinear points, an outer-cup with respect to K of size m or an inner-cap with respect to K of size n.

Proof. Let $|P| = N$. Without loss of generality, we can assume that the line L which separates K and P is horizontal, that K lies below L and that P lies above L. By considering $\mathrm{conv}(K \cup \{p\})$ for each $p \in P$, we can radially order the elements in $P = \{p_1, \ldots, p_N\}$ with respect to K in clockwise order, from left to right.

Notice that every non-collinear triple in P is either an inner-cap with respect to K or an outer-cup with respect to K. Moreover, for $i < j < s < t$, if $\{p_i, p_j, p_s\}$ and $\{p_j, p_s, p_t\}$ are both inner-caps with respect to K (outer-cups with respect to K), then every triple in $\{p_i, p_j, p_s, p_t\}$ is an inner-cap with respect to K (outer-cup with respect to K). Thus, by following the proof of Theorem 2.1 almost verbatim, the statement follows. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let P be an N-element point set in the plane, where $N = \ell^2 \cdot 2^{n+C\sqrt{n \log n}}$ with C a sufficiently large absolute constant. We can assume that no two points in P have the same x-coordinate. Moreover, we can assume that there are no ℓ collinear members in P, since otherwise we would be done.

For the sake of contradiction, suppose P does not contain n points in convex position. Set $k = 2\lceil\sqrt{n \log n}\rceil$. We apply Theorem 3.1 to P with parameter $k+3$, obtaining a subset $X = \{x_1, \ldots, x_{k+3}\} \subset P$ such that X is either a cup or a cap, where we assume that the points of X appear in order from left to right. Moreover, the regions $T_1, \ldots, T_{k+2}$ in the support of X satisfy

$$|T_i \cap P| \geq \frac{N}{2^{32(k+3)}}.$$

Set $P_i \subset T_i \cap P$ to be the set of points of P in the interior of T_i, for $i = 1, \ldots, k+2$. Hence,

$$|P_i| \geq \frac{N}{2^{32(k+3)}} - 3\ell \geq \frac{N}{2^{40k}}.$$

We will now assume that X is a cap, since a symmetric argument works in the other case.

Consider the subset $P_i \subset P$ and the region T_i for some fixed $i \in \{2, \ldots, k+1\}$. Let B_i be the closed segment $\overline{x_{i-1}x_{i+2}}$. The point set P_i naturally comes with a partial order $\prec_i$, where $p \prec_i q$ if $p \neq q$ and $p \in \text{conv}(B_i \cup q)$. Note that $p \prec_i q$ if p lies on the boundary of $\text{conv}(B_i \cup q)$. Following Holmsen et al. [6], for each P_i, let

1. h_i be the size of the longest antichain with respect to $\prec_i$,
2. v_i be the size of the longest chain with respect to $\prec_i$,
3. a_i be the size of the largest inner-cap with respect to x_{i+1} that is also a chain with respect to $\prec_i$,
4. b_i be the size of the largest inner-cap with respect to x_i that is also a chain with respect to $\prec_i$,
5. w_i be the size of the largest inner-cap with respect to B_i that is also an antichain with respect to $\prec_i$ and
6. z_i be the size of the largest outer-cup with respect to B_i that is also an antichain with respect to $\prec_i$.

By Dilworth's theorem [3], we have $v_i h_i \geq |P_i|$. We also clearly have $z_i, a_i, b_i, w_i < n$. We now make the following observations.

Observation 4.2.

$$w_2 + w_4 + \cdots + w_k < n$$

and

$$w_3 + w_5 + \cdots + w_{k+1} < n.$$

Proof. Recall that $k = 2\lceil \sqrt{n \log n} \rceil$ is even. Let us consider the sets $P_2, P_4, \ldots, P_k$. Suppose we have subsets $S_2 \subset P_2, S_4 \subset P_4, \ldots, S_k \subset P_k$ such that S_i is an antichain with respect to $\prec_i$, an inner-cap with respect to B_i and satisfies $|S_i| = w_i$. Then we claim that $S = S_2 \cup S_4 \cup \cdots \cup S_k$ is a cap and, therefore, in convex position. Let $p \in S_i$. Then there is a line L through p that has the property that all of the other points in S_i lie below L and L does not intersect B_i. Since L does not intersect B_i, all of the points in $S \setminus \{p\}$ must lie below L. But then, we must have that

$$w_2 + w_4 + \cdots + w_k = |S| < n,$$

as required. A similar argument works for the parts $P_3, P_5, \ldots, P_{k+1}$ to prove the second inequality. □

By Observation 4.2, we have

$$w_2 + w_3 + \cdots + w_{k+1} < 2n.$$

Let $P_i' \subset P_i$ be a chain with respect to $\prec_i$. Clearly P_i' avoids x_i and x_{i+1}. Moreover, if P_i' contains an outer-cup with respect to x_i, then it must be an inner-cap with respect to x_{i+1}. Therefore, if

$$|P_i'| > c(\min\{m-1, n-1\} + \ell) \cdot \binom{m+n-4}{n-2},$$

then, by Theorem 4.1 applied to the convex set $K = \{x_i\}$, the set P_i' contains either an outer-cup with respect to x_i of size m, which is an inner-cap with respect to x_{i+1} of size m, or an inner-cap with respect to x_i of size n. See Figure 4.

Observation 4.3. *If there are subsets $Y_{i-1} \subset P_{i-1}$ and $Y_i \subset P_i$ such that Y_{i-1} is a chain with respect to $\prec_{i-1}$ and an inner-cap with respect to x_i and Y_i is a chain with respect to $\prec_i$ and an inner-cap with respect to x_i, then $Y_{i-1} \cup Y_i$ is in convex position.*

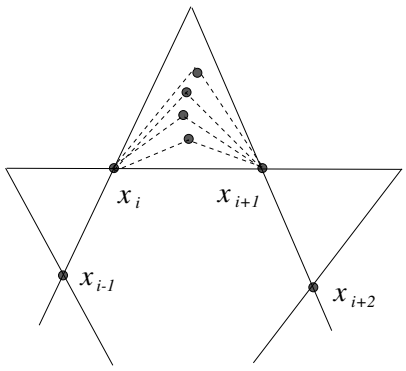

Figure 4: Four points in P_i' that form an outer-cup with respect to x_i, which is an inner-cap with respect to x_{i+1}.

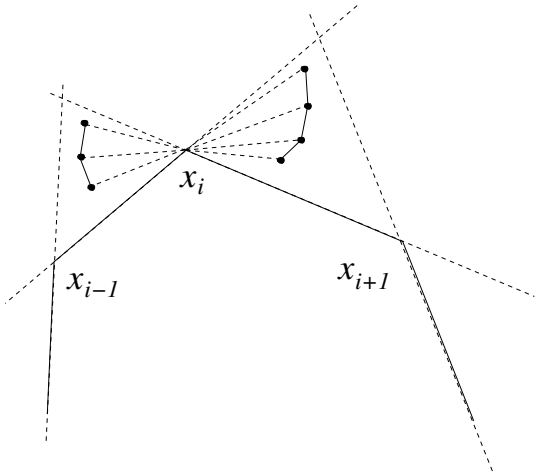

Figure 5: An inner-cap of size 3 with respect to x_i in P_{i-1} and an inner-cap of size 4 in P_i with respect to x_i, which gives 7 points in convex position.

Proof. It suffices to show that every four points in $Y_{i-1} \cup Y_i$ are in convex position. If all four points lie in Y_i, then they are in convex position. Likewise, if they all lie in Y_{i-1}, they are again in convex position. Suppose we take two points $p_1, p_2 \in Y_{i-1}$ and two points $p_3, p_4 \in Y_i$. Since Y_{i-1} and Y_i are both chains with respect to $\prec_{i-1}$ and $\prec_i$ respectively, the line spanned by p_1, p_2 does not intersect the region T_i and the line spanned by p_3, p_4 does not intersect the region T_{i-1}. Hence, p_1, p_2, p_3, p_4 are in convex position. Now suppose we have $p_1, p_2, p_3 \in Y_{i-1}$ and $p_4 \in Y_i$. Since the three lines L_1, L_2, L_3 spanned by p_1, p_2, p_3 all intersect the segment B_{i-1}, both x_i and p_4 lie in the same region in the arrangement of $L_1 \cup L_2 \cup L_3$. Therefore, p_1, p_2, p_3, p_4 are in convex position. The same argument works for the case where $p_1 \in Y_{i-1}$ and $p_2, p_3, p_4 \in Y_i$. See Figure 5. □

By Observation 4.3, we have $a_i + b_{i+1} < n$ for all i. By applying Theorem 4.1 with $K = \{x_i\}$, we have

$$v_i < c(\ell + n)\binom{(a_i + 1) + (b_i + 1) - 4}{(a_i + 1) - 2} = c(\ell + n)\binom{a_i + b_i - 2}{a_i - 1}.$$

Likewise, by applying Theorem 4.1 with $K = B_i$, we have

$$h_i < c(\ell+n)\binom{(w_i+1)+(z_i+1)-4}{(w_i+1)-2} = c(\ell+n)\binom{w_i+z_i-2}{w_i-1}.$$

Putting everything together, we obtain

$$\begin{aligned}
\frac{N^k}{2^{40k^2}} &\leq \prod_{i=2}^{k+1} |P_i| \\
&\leq \prod_{i=2}^{k+1} v_i h_i \\
&\leq \prod_{i=2}^{k+1} c^2(\ell+n)^2 \binom{a_i+b_i-2}{a_i-1}\binom{w_i+z_i-2}{w_i-1} \\
&< \prod_{i=2}^{k+1} c^2(\ell+n)^2 2^{a_i+b_i}(2n)^{w_i} \\
&< (c(\ell+n))^{2k} 2^{(k+1)n+2n\log(2n)},
\end{aligned}$$

where c is the absolute constant from Theorem 4.1. Therefore, we have

$$N < c^2(\ell+n)^2 2^{n+3(n/k)\log(2n)+40k}.$$

Since $k = 2\lceil\sqrt{n\log n}\rceil$, this gives us

$$N < \ell^2 \cdot 2^{n+O(\sqrt{n\log n})}.$$

Since $|P| = N = \ell^2 \cdot 2^{n+C\sqrt{n\log n}}$, by setting C sufficiently large, we have a contradiction. □

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