Turán problems in pseudorandom graphs

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Abstract

Given a graph F, we consider the problem of determining the densest possible pseudorandom graph that contains no copy of F. While this general question is several decades old, recent results have brought a renewed interest in this area due to a connection with classical Ramsey numbers.

We give the first nontrivial upper bound for this problem when F is the Peterson graph. We also give a new construction of the densest known clique-free pseudorandom graphs. Our construction is simpler than similar constructions due to Bishnoi, Ihringer and Pepe; it generalizes an old result of Parsons by considering an appropriate induced subgraph of a construction due to Alon and Krivelevich. Finally, we construct the densest known pseudorandom $K_{2,3}$-free graphs that are also triangle-free.

1 Introduction

Given a family $\mathcal{F}$ of graphs we say a graph G is $\mathcal{F}$-free if it does not contain any member in $\mathcal{F}$ as a subgraph. A fundamental problem in extremal graph theory is to determine the maximum number $\mathrm{ex}(n, \mathcal{F})$ of edges in an $\mathcal{F}$-free graph on n vertices. Here $\mathrm{ex}(n, \mathcal{F})$ is called the *Turán number* of $\mathcal{F}$, and the limit $\pi(F) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{2}$, whose existence was proved by Katona, Nemetz, and Simonovits [17], is called the *Turán density* of $\mathcal{F}$.

For a graph G we use $V(G)$ to denote the vertex set of G, and use $v(G)$ and $e(G)$ to denote the number of vertices and edges in G, respectively. For a set $S \subset V(G)$ we use $e_G(S)$ to denote the number of edges in the induced subgraph $G[S]$. Given two vertex sets $X, Y \subset V(G)$ we use $e_G(X, Y)$ to denote the number of edges in G that have one vertex in X and one vertex in Y (here edges with both vertices in $X \cap Y$ are counted twice, hence $e_G(X, X) = 2e_G(X)$). We will omit the subscript G if it is clear from the context.

Informally, we say that a graph is pseudorandom if its edge distribution behaves like a random graph. In this note we use the following notation, which was firstly introduced by Thomason in his fundamental papers [30, 31], to quantify the randomness of a graph.

For two real numbers $p \in [0, 1]$ and $\alpha \geq 0$, we say a graph G is (p, α)-*jumbled* if it satisfies

$$|e(X, Y) - p|X||Y|| \leq \alpha\sqrt{|X||Y|} \tag{1}$$

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for all $X, Y \subset V(G)$.

A special family of (p, α)-jumbled graphs are the well-known (n, d, λ)-*graphs*. A graph G is an (n, d, λ)-graph if it is a d-regular graph on n vertices and the second largest eigenvalue in absolute value of its adjacency matrix is λ. The well-known Expander mixing lemma (e.g. see [21, Theorem 2.11]) implies that an (n, d, λ)-graph is $(d/n, \lambda)$-jumbled. Conversely, Bilu and Linial [7] proved that an n-vertex d-regular (p, α)-jumbled graph is an (n, d, λ)-graph with $\lambda = O(\alpha \log(d/\alpha))$.

It is known that a random graph $G(n, p)$ is almost surely a (p, α)-jumbled graph with $\alpha = O(\sqrt{np})$ (see e.g. [21, Corollary 2.3]). The proof of Erdős and Spencer in [16] can be extended to show that every (p, α)-jumbled graph on n vertices satisfies that $\alpha = \Omega(\sqrt{np})$ (in particular, $\lambda = \Omega(\sqrt{d})$ for an (n, d, λ)-graph). Therefore, an n-vertex (p, α)-jumbled graph with $\alpha = \Theta(\sqrt{np})$ can be viewed as optimally pseudorandom. The tightness of the bound $\lambda = \Omega(\sqrt{d})$ in general is also witnessed by many well-known explicit constructions. For example, the well-known triangle-free (n, d, λ)-graph constructed by Alon [2] satisfies $d = \Theta(n^{2/3})$ and $\lambda = O(\sqrt{d})$.

Constructions of dense pseudorandom graphs that avoid a certain graph as a subgraph are extremely useful for many problems. In particular, the second author and Verstaëte [25] recently showed that for every fixed integer $t \geq 3$, the existence of K_t-free (n, d, λ)-graphs with $d = \Omega(n^{1-\frac{1}{2t-3}})$ and $\lambda = O(\sqrt{d})$ implies the lower bound $R(t, n) = \Omega^*(n^{t-1})$ for the off-diagonal Ramsey numbers, and this matches the best known upper bound in exponent. More generally, [25] shows that the existence of dense F-free pseudorandom graphs implies a good lower bound for the Ramsey number $R(F, n)$. This motivates us to consider the following pseudorandom version of the Turán problem.

Let $\mathcal{F}$ be a family of graphs and $C > 0$ be a real number. Let $\mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})$ be the maximum number of edges in an n-vertex (p, α)-jumbled $\mathcal{F}$-free graph with $\alpha \leq C\sqrt{np}$. Note that in the definition of $\mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})$ we do not have any restriction on p.

In many applications, it suffices to know the exponent of $\mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})$. So we let

$$\exp(\mathcal{F}) = \lim_{C\to\infty} \limsup_{n\to\infty} \frac{\log \mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})}{\log n}.$$

In other words, $\exp(\mathcal{F})$ is the supremum of β such that there exist a constant C and a sequence $(G_n)_{n=1}^{\infty}$ of $\mathcal{F}$-free (p_n, α_n)-jumbled graphs with

$$\lim_{n\to\infty} v(G_n) = \infty, \quad \lim_{n\to\infty} \frac{\log(p_n v(G_n))}{\log v(G_n)} \geq \beta, \quad \text{and} \quad \alpha_n \leq C\sqrt{p_n v(G_n)}.$$

Using the Expander mixing lemma one can prove that for every integer $t \geq 3$ we have $\exp(K_t) \leq 1 - \frac{1}{2t-3}$. Alon's construction [2] shows that this bound is tight for $t = 3$, that is, $\exp(K_3) = \frac{2}{3}$. It is a major open problem to determine $\exp(K_t)$ in general. Alon and Krivelevich proved in [4] that $\exp(K_t) \geq 1 - \frac{1}{t}$. Recently, Bishnoi, Ihringer, and Pepe [8] improved their bound and proved the following result.

Theorem 1.1 (Bishnoi–Ihringer–Pepe [8]). *Suppose that $t \geq 4$ is an integer. Then $\exp(K_t) \geq 1 - \frac{1}{t-1}$.*

Mattheus and Pavese [24] give a different construction of K_t-free pseudorandom graphs which also matches the bound in Theorem 1.1. In Section 3, we will present yet another construction which also proves Theorem 1.1. Our construction seems simpler as it is just an induced subgraph of an old construction by Alon and Krivelevich in [4].

For bipartite graphs, the pseudorandom version of the Turán problem does not appear to differ much from the ordinary Turán problem since many constructions for the lower bound are pesudorandom. For example, for complete bipartite graphs, the projective norm graphs (see [19, 5]) are optimally pseudorandom (see [28]) and do not contain $K_{s,t}$ with $t \geq (s-1)!+1$. Therefore, together with the well known Kövari–Sós–Turán Theorem [20], we know that $\exp(K_{s,t}) = 2 - \frac{1}{s}$ for all positive integers s, t with $t \geq (s-1)! + 1$. For even cycles, constructions from generalized polygons [22] and an old result of Bondy and Simonovits [10] imply that $\exp(C_6) = \frac{4}{3}$ and $\exp(C_{10}) = \frac{6}{5}$. The value of $\exp(C_{2k})$ for $k \neq 2, 3, 5$ are still unknown due to the lack of constructions. For non-bipartite graphs, $\exp(F)$ is completely different from the ordinary Turán problem (indeed, $\exp(F) < 2$ while $\pi(F) > 0$) and there are very graphs F for which $\exp(F)$ is known. For example, for odd cycles, a construction due to Alon and Kahale [3] together with Proposition 4.12 in [21] implies that $\exp(C_\ell) = \frac{2}{\ell}$ for all odd integers $\ell \geq 3$. More generally, it follows from a result of Kohayakawa Rödl, Schacht, Sissokho, and Skokan [18] that $\exp(F) \leq 1 - \frac{1}{2\nu(F)-2}$ for every triangle-free graph F. Here $\nu(F) = \frac{1}{2}\left(d(F) + D(F) + 1\right)$, where $D(F) = \min\{2d(F), \Delta(F)\}$ and $d(F)$ is the degeneracy of F. A result of Colon, Fox, and Zhao in [12] implies that $\exp(F) \leq 1 - \frac{1}{2d_2(F)+5}$, where $d_2(F)$ is the 2-degeneracy of F whose definition can be found in [12].

We study $\exp(\mathbf{P})$, where $\mathbf{P}$ is the Petersen graph. The Petersen graph was considered by several researchers in related contexts. For example, Tait and Timmons [29] proved that the Erdős–Rényi orthogonal polarity graphs [15] (henceforth the Erdős–Rényi graph), which are optimally pseudorandom C_4-free graphs, contain the Petersen graph as a subgraph. Colon, Fox, Sudakov, and Zhao asked in [11] whether there is a counting lemma for the Petersen graph in an n-vertex C_4-free graph with $\Omega(n^{3/2})$ edges. We know very little about $\exp(\mathbf{P})$, for example, it is not known whether $\exp(\mathbf{P}) \geq \frac{1}{2}$. It seems that the only lower bound we have for $\exp(\mathbf{P})$ is $\exp(\mathbf{P}) \geq \exp(C_5) = \frac{2}{5}$. In the other direction, the best upper bound appears to be $\exp(\mathbf{P}) \leq \frac{4}{5}$ that follows from [18].

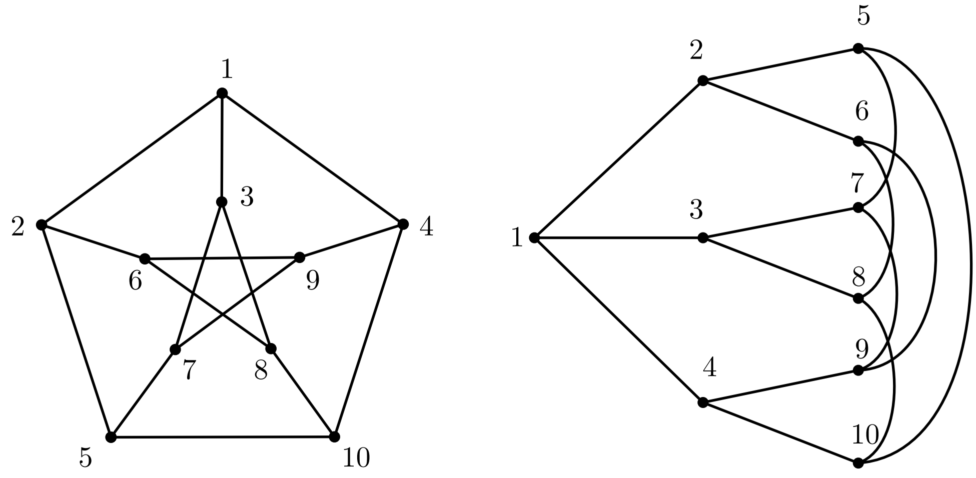

Figure 1: The Petersen graph in two different drawings.

Theorem 1.2. *We have* $\exp(\boldsymbol{P}) \leq \frac{5}{7}$.

We conjecture that $\exp(\mathbf{P}) = \frac{2}{3}$. We think that the construction of K_3-free pseudorandom graphs due to Kopparty does not contain the Petersen graph as a subgraph. If this is true, then it will prove the lower bound $\exp(\mathbf{P}) \geq \frac{2}{3}$. For completeness, we include his construction here.

Let $p \neq 3$ be a prime, and let $\mathbb{F}_q$ be a finite field with where $q = p^h$ for some integer $h \geq 1$. Recall that the *absolute trace function* $\mathrm{Tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ is defined as $\mathrm{Tr}(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{h-1}}$ for every $\alpha \in \mathbb{F}_q$.

Let $V = \mathbb{F}_q^3$, $T = \{x \in \mathbb{F}_q \colon \mathrm{Tr}(x) = 1\}$, and $S \subset \mathbb{F}_q^3$ be a subset defined as

$$S = \left\{(xy, xy^2, xy^3) \colon x \in T,\ y \in \mathbb{F}_q \setminus \{0\}\right\}.$$

Kopparty's construction is the graph G on V in which two vertices $\mathbf{u}, \mathbf{v} \in V$ are adjacent iff $\mathbf{u} - \mathbf{v} \in S$. Using some simple linear algebra one can show that G is triangle-free, and using some results about finite fields and abelian groups one can prove that G is an (n, d, λ)-graph with $n = q^3$, $d = \Theta(\frac{q^2}{p})$, and $\lambda = \Theta(\frac{q}{p})$.

Our next result about $K_{2,3}$ was motivated by an old problem of Erdős [14], which asks if

$$\mathrm{ex}(n, \{K_3, C_4\}) = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2}$$

is true. A construction due to Parsons [26] for the lower bound comes from the Erdős–Rényi graph by removing half of its vertices. Since the Erdős–Rényi graph is optimally pseudorandom, Parsons' construction also implies that $\mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, C_4\}) \geq \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2}$ for some absolute constant C. In [1], Allen, Keevash, Sudakov, and Verstraëte proved that the extremal constructions for $\mathrm{ex}(n, \{K_3, K_{2,t}\})$ cannot be bipartite for every $t \geq 3$ by constructing a $\{K_3, K_{2,t}\}$-free graph whose number of edges is greater than the maximum number of edges in a $\{K_3, K_{2,t}\}$-free bipartite graph. However, their construction is $(t-1)$-partite, and therefore it does not give a lower bound for $\mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, K_{2,3}\})$. The previous best lower bound is

$$\mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, K_{2,3}\}) \geq \mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, C_4\}) \geq \left(\frac{1}{2\sqrt{2}} - o(1)\right) n^{3/2}$$

that follows from Parsons' construction. We improve this and present a construction of the densest known $\{K_3, K_{2,3}\}$-free pseudorandom graphs.

Theorem 1.3. *We have* $\mathrm{ex}_{\mathrm{rand}}(n, 2, \{K_3, K_{2,3}\}) \geq \left(\frac{1}{2} - o(1)\right) n^{3/2}$.

In Section 2 we prove Theorem 1.2. In Section 3 we present our construction which proves Theorem 1.1. In Section 4, we prove Theorem 1.3.

2 Proofs of Theorem 1.2

We prove Theorem 1.2 in this section. Let us present some simple lemmas first.

Lemma 2.1. *Suppose that G is a (p, α)-jumbled graph on n vertices. Then G contains an induced subgraph on at least $\frac{n}{2} - \frac{\alpha}{p}$ vertices with minimum degree at least $\frac{np}{4}$.*

Proof. Let $V = V(G)$. We start with $B = \emptyset$. In each step, if there exists a vertex $v \in V \setminus B$ with degree less than $\frac{np}{4}$ in the induced subgraph $G[V \setminus B]$, then we add v into B. We terminate this process if the induced subgraph $G[V \setminus B]$ has minimum degree at least $\frac{np}{4}$ or if $|B| \geq \frac{n}{2} + \frac{\alpha}{p}$.

Suppose that we stopped the process because of $|B| \geq \frac{n}{2} + \frac{\alpha}{p}$. Then it follows from the definition of B that $e(B) < |B|\frac{np}{4}$. Indeed, if the vertices were added to B in the order $v_1, v_2, \ldots$, then the number of neighbors of v_i within $B \setminus \{v_1, \ldots, v_{i-1}\}$ is at most $\frac{pn}{4}$. On the other hand, it follows from (1) that $e(B) = \frac{1}{2}e(B,B) \geq \frac{1}{2}\left(p|B|^2 - \alpha|B|\right)$. Therefore,

$$|B|\frac{np}{4} > \frac{1}{2}\left(p|B|^2 - \alpha|B|\right),$$

which implies that $|B| < \frac{n}{2} + \frac{\alpha}{p}$, a contradiction. Therefore $|V \setminus B| \geq \frac{n}{2} - \frac{\alpha}{p}$, and the induced subgraph $G[V \setminus B]$ has minimum degree at least $\frac{np}{4}$. ∎

Let us introduce two more definitions before stating the next lemma. We say a bipartite graph $G = G[A,B]$ with parts A, B is *(p,α)-bi-jumbled* if

$$|e(X,Y) - p|X||Y|| \leq \alpha\sqrt{|X||Y|}. \tag{2}$$

holds for all $X \subset A$ and for all $Y \subset B$.

For a graph G and a subset $S \subset V(G)$ the *neighborhood* $N_G(S)$ of S is defined as

$$N_G(S) = \{u \in V(G) \setminus S \colon \exists v \in S \text{ such that } \{u,v\} \in G\}.$$

If it is clear from the context, we will omit the subscript G.

The next lemma is about the expansion property of a bi-jumbled graph.

Lemma 2.2. *Suppose that $G = G[A,B]$ is a (p,α)-bi-jumbled graph with minimum degree δ. Then for every set $S \subset A$, we have*

$$|N(S)| \geq \min\left\{\frac{\delta}{2p}, \frac{\delta^2}{4\alpha^2}|S|\right\}$$

Proof. Given $S \subset A$, let $N = N(S)$ denote the neighborhood of S in G. Since the minimum degree of G is δ, we have $e(S,N) \geq |S|\delta$. On the other hand, it follows from (2) that $e(S,N) \leq p|S||N| + \alpha\sqrt{|S||N|}$. Therefore, we have $p|S||N| + \alpha\sqrt{|S||N|} \geq |S|\delta$. By the pigeonhole principle, we have either $p|S||N| \geq \frac{|S|\delta}{2}$ or $\alpha\sqrt{|S||N|} \geq \frac{|S|\delta}{2}$. In the former case, we obtain $|N| \geq \frac{\delta}{2p}$, and in the latter case, we obtain $|N| \geq \frac{\delta^2}{4\alpha^2}|S|$. ∎

We will prove the following theorem will implies Theorem 1.2.

Theorem 2.3. *There exists a constant C such that the following holds for all sufficiently large n. Suppose that G is a (p,α)-jumbled graph on n vertices with $\alpha \leq \frac{C}{2}\sqrt{np}$ and $np \geq Cn^{5/7}$. Then G contains a copy of the Petersen graph.*

Proof. Let C be a sufficiently large constant. Let $n \gg C$ be a sufficiently large integer. Suppose to the contrary that there exist a $\mathbf{P}$-free (p,α)-jumbled graph G on n vertices with $\alpha \leq \frac{C}{2}\sqrt{np}$ and $np \geq Cn^{5/7}$ (note that $p \geq Cn^{-2/7}$). Let $d = np$.

By Lemma 2.1, G contains an induced subgraph G' on at least $\frac{n}{2} - \frac{\alpha}{p} \geq \frac{n}{2} - \frac{C\sqrt{np}}{p} = \frac{n}{2} - o(n)$ vertices with minimum degree at least $\frac{d}{4}$. Let $V = V(G')$. From now on, we consider G' only. Fix a vertex $v_1 \in V$ and three distinct vertices $v_2, v_3, v_4 \in N(v)$. Let $V_1 \subset N(v_2) \setminus \{v_1, v_2, v_3, v_4\}$, $V_2 \subset N(v_3) \setminus \{v_1, v_2, v_3, v_4\}$, and $V_3 \subset N(v_4) \setminus \{v_1, v_2, v_3, v_4\}$ be three disjoint sets of size $d/15$. Since $\min\{|N(v_1)|, |N(v_2)|, |N(v_3)|\} \geq d/4 - 4 \geq d/5$, such three sets V_1, V_2, V_3 exist. Let $H = G[V_1, V_2, V_3]$ be the induced 3-partite subgraph of G' on $V_1 \cup V_2 \cup V_3$.

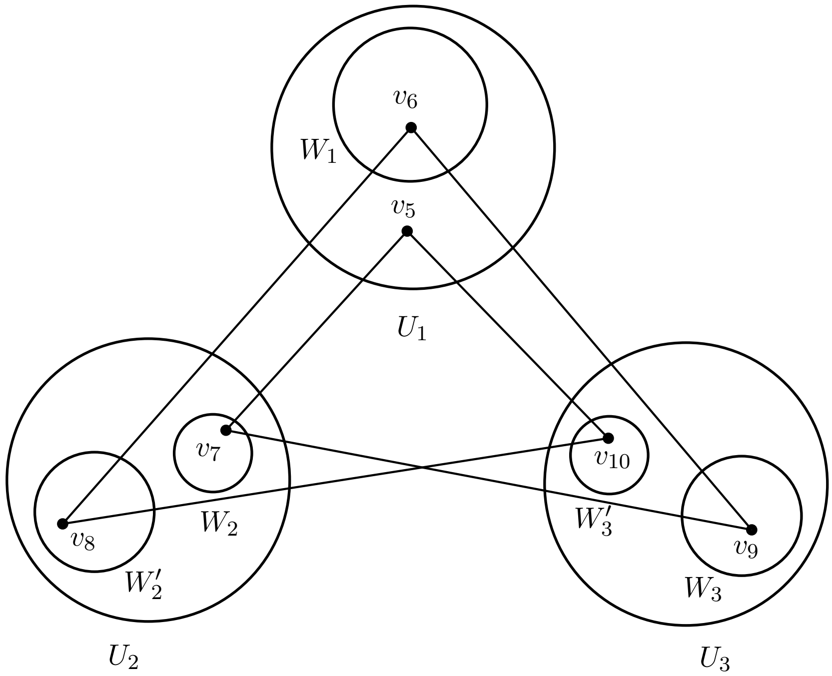

Figure 2: $v_5v_7v_9v_6v_8v_{10}v_5$ is a copy of C_6.

Claim 2.4. *For every $i \in [3]$ there exists a set $U_i \subset V_i$ of size $d/30$ such that every vertex in U_i has at least $\frac{d^2}{60n}$ neighbors in both U_{i+1} and U_{i+2} (indices $\{1,2,3\}$ taken modulo 3).*

Proof. We start with $B = \emptyset$. If there exists $i \in [3]$ and a vertex $v \in V_i \setminus B$ such that v has less than $\frac{d^2}{60n}$ neighbors in $V_{i+1} \setminus B$ or $V_{i+2} \setminus B$, then we add v into B. We stop this process if either there is no such i or $|B| \geq d/30$.

Suppose that we stopped the process because of $|B| \geq d/30$. By the Pigeonhole principle, there exists $i \in [3]$ such that the size of $B_i = B \cap V_i$ has size at least $|B|/3 = d/90$. Without lose of generality we may assume that $i = 1$. It follows from the definition of B that every vertex in B_1 has less than $\frac{d^2}{60n}$ neighbors in either $V_2 \setminus B$ or $V_3 \setminus B$. By symmetry, we may assume that the former case holds, that is, there exists a set $B_1' \subset B_1$ of size $\frac{1}{2} \cdot \frac{d}{90} = \frac{d}{180}$ such that every vertex in B_1' has less than $\frac{d^2}{60n}$ neighbors in $V_2 \setminus B$. Then we have $e(B_1', V_2 \setminus B) \leq |B_1'|\frac{d^2}{60n}$. On the other hand, it follows from (1) that $e(B_1', V_2 \setminus B) \geq \frac{d}{n}|B_1'||V_2 \setminus B| - \frac{C}{2}d^{1/2}\sqrt{|B_1'||V_2 \setminus B|}$. Therefore, we have

$$\begin{aligned}\frac{d^3}{180 \cdot 60n} = |B_1'|\frac{d^2}{60n} &\geq \frac{d}{n}|B_1'||V_2 \setminus B| - \frac{C}{2}d^{1/2}\sqrt{|B_1'||V_2 \setminus B|} \\ &\geq \frac{d}{n}\frac{d}{180}\left(\frac{d}{15} - \frac{d}{30}\right) - \frac{C}{2}d^{1/2}\sqrt{\frac{d}{180}\left(\frac{d}{15} - \frac{d}{30}\right)} \\ &\geq \frac{d^3}{180 \cdot 30n} - Cd^{3/2}\end{aligned}$$

Since $d \geq Cn^{5/7} \gg n^{2/3}$, we have $\frac{d^3}{180 \cdot 30n} - Cd^{3/2} \geq \frac{d^3}{180 \cdot 60n}$, a contradiction. Therefore, the process stopped before $|B| \geq \frac{d}{3}$. Now for $i \in [3]$ let $U_i = V_i \setminus B$. Then $|U_i| \geq \frac{d}{30}$ and every vertex in U_i has at least $\frac{d^2}{60n}$ neighbors in both U_{i+1} and U_{i+2}. ∎

Observe that to prove G contains a copy of the Petersen graph it suffices to find distinct vertices $v_5, v_6 \in U_1$, $v_7, v_8 \in U_2$, and $v_9, v_{10} \in U_3$ such that $v_5v_7v_9v_6v_8v_{10}v_5$ forms a copy of C_6 in G. Let us start by fixing a vertex $v_5 \in U_1$ (see Figure 2). It follows from Claim 2.4 that

$$\min\{|N_G(v_5) \cap U_2|, |N_G(v_5) \cap U_3|\} \geq \frac{d^2}{60n} \geq n^{3/7}.$$

So we can choose sets $W_2 \subset N_G(v_5) \cap U_2$ and $W_3' \subset N_G(v_5) \cap U_3$ both of size $n^{3/7}$.

Applying Lemma 2.2 to the induced bipartite graph $G'[U_2, U_3]$ and W_2 we obtain that

$$|N(W_2) \cap U_3| \geq \min\left\{\frac{\frac{d^2}{60n}}{2p}, \frac{\left(\frac{d^2}{60n}\right)^2}{4\alpha^2}|W_2|\right\} \geq \min\left\{\frac{d}{120}, \frac{d^3}{4 \cdot 60^2 C^2 n^2}|W_2|\right\} \geq 2n^{4/7},$$

where the last inequality follows from our assumption that $d \geq Cn^{5/7}$ and C is sufficiently large. Therefore, we can choose a set $W_3 \subset (N(W_2) \cap U_3) \setminus W_3'$ of size $n^{4/7}$. Similarly, we can choose a set $W_2' \subset (N(W_3') \cap U_2) \setminus W_2$ of size $n^{4/7}$.

Now applying Lemma 2.2 to the bipartite graph $G'[U_3, U_1]$ and W_3, we obtain

$$|N(W_3) \cap V_1| \geq \min\left\{\frac{d}{120}, \frac{d^3}{4 \cdot 60^2 C^2 n^2}|W_3|\right\} \geq 2n^{5/7}.$$

Again, here we used our assumption that $d \geq Cn^{5/7}$ and C is sufficiently large. Therefore, we can choose a set $W_1 \subset (N(W_3) \cap U_1) \setminus \{v_5\}$ of size $n^{5/7}$.

Applying (1) to sets W_1, W_2', we obtain

$$e(W_1, W_2') \geq p|W_1||W_2'| - \alpha\sqrt{|W_1||W_2'|} \geq Cn^{-\frac{2}{7}+\frac{4}{7}+\frac{5}{7}} - \frac{C}{2}n^{\frac{5}{14}+\frac{4}{14}+\frac{5}{14}} = \frac{C}{2}n.$$

Therefore, there exists $v_6 \in W_1$ and $v_8 \in W_2'$ such that $\{v_6, v_8\}$ is an edge in G'. Since $W_2' \subset N(W_3')$, there exists $v_{10} \in W_3'$ such that $\{v_8, v_{10}\}$ is an edge in G'. Similarly, there exist $v_9 \in W_3$ and $v_7 \in W_2$ such that $\{v_6, v_9\}$ and $\{v_9, v_7\}$ are edges in G' (see Figure 2). Note that $v_5v_7v_9v_6v_8v_{10}v_5$ forms a copy of C_6 in G', so this completes the proof of Theorem 2.3. ∎

3 K_t-free pseudorandom graphs

In this section we present a different construction of pseudorandom K_t-free graphs with the same edge density as constructions in [8] and [24].

Denote by $\mathrm{PG}(t-1, q)$ the $(t-1)$-dimensional projective space over $\mathbb{F}_q$, i.e. $\mathrm{PG}(t-1, q) = \mathbb{F}_q^t/\sim$, where two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^t$ are equivalent under $\sim$ if there exists a non-zero element $a \in \mathbb{F}_q$ such that $\mathbf{x} = a\mathbf{y}$. For a vector $\mathbf{x} \in \mathbb{F}_q^t$ we use $[\![\mathbf{x}]\!]$ to denote its equivalence class in $\mathbb{F}_q^t/\sim$. It is easy to see that the number of points in $\mathrm{PG}(t-1, q)$ is $\frac{q^t-1}{q-1} = (1+o(1))q^{t-1}$. Recall that the dot-product $\mathbf{x} \cdot \mathbf{y}$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^t$ is defined as $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^t x_i y_i$. A point $\mathbf{x} = (x_1, \ldots, x_t) \in \mathbb{F}_q^t$ is called

- *absolute* if $\mathbf{x} \cdot \mathbf{x} = 0$,
- *square* if $\mathbf{x} \cdot \mathbf{x} = a^2$ for some $a \in \mathbb{F}_q$,
- *non-square* if $\mathbf{x} \cdot \mathbf{x} \neq a^2$ for all $a \in \mathbb{F}_q$,

We use $X_0(t,q), X_\square(t,q), X_\boxtimes(t,q)$ to denote the collection of all absolute points, square points, and non-square points in $\mathbb{F}_q^t$, respectively. If t and q are clear from the context, we will omit them and use $X_0, X_\square, X_\boxtimes$ for simplicity. It is easy to see from the definition that if $\mathbf{x} \in X_0$, $\mathbf{x} \in X_\square$, or $\mathbf{x} \in X_\boxtimes$, then $[\![\mathbf{x}]\!] \subset X_0$, $[\![\mathbf{x}]\!] \subset X_\square$, or $[\![\mathbf{x}]\!] \subset X_\boxtimes$, respectively.

Recall that a *character* of a group H is a homomorphism $\psi\colon H \to \mathbb{C}^\times$, and the *quadratic character* $\chi(\cdot)$ of $\mathbb{F}_q$ is defined as

$$\chi(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \text{ is a square}, \\ -1, & \text{if } x \text{ is a non-square}. \end{cases}$$

Let $\mathrm{AK}(t-1,q)$ be the graph whose vertices are non-absolute points of $\mathrm{PG}(t-1,q)$ and two vertices $[\![\mathbf{x}]\!]$ and $[\![\mathbf{y}]\!]$ are adjacent iff $\mathbf{x}\cdot\mathbf{y} = 0$. Note that $\mathrm{AK}(2,q)$ is just the Erdős–Renyi graph. In [4], Alon and Krivelevich proved that $\mathrm{AK}(t-1,q)$ is a K_{t+1}-free (n,d,λ)-graph with $n = (1+o(1))q^{t-1}$, $d = \Theta(n^{1-\frac{1}{t}})$, and $\lambda = \Theta(\sqrt{d})$.

Parsons [26] proved that for q odd, the induced subgraph of $\mathrm{AK}(2,q)$ on $X_\boxtimes/\!\sim$ is K_3-free. Indeed, suppose to the contrary that there exist three distinct points $[\![\mathbf{x}_1]\!], [\![\mathbf{x}_2]\!], [\![\mathbf{x}_3]\!] \in X_\boxtimes/\!\sim$ that induce a copy of K_3 in $\mathrm{AK}(2,q)$. Then $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are pairwise orthogonal, which means that there exists a non-zero element $a \in \mathbb{F}_q$ such that $\mathbf{x}_3 = a\mathbf{x}_1 \times \mathbf{x}_2$, where $\mathbf{x}_1 \times \mathbf{x}_2$ is the *cross-product* of $\mathbf{x}_1$ and $\mathbf{x}_2$. Therefore, we have

$$\mathbf{x}_3 \cdot \mathbf{x}_3 = (a\mathbf{x}_1 \times \mathbf{x}_2) \cdot (a\mathbf{x}_1 \times \mathbf{x}_2) = a^2 \cdot (\mathbf{x}_1 \cdot \mathbf{x}_1) \cdot (\mathbf{x}_2 \cdot \mathbf{x}_2),$$

where in the last equality we used the fact that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. Applying the quadratic character $\chi(\cdot)$ to both sides of the equation above we obtain

$$-1 = \chi(\mathbf{x}_3 \cdot \mathbf{x}_3) = \chi(a^2) \cdot \chi(\mathbf{x}_1 \cdot \mathbf{x}_1) \cdot \chi(\mathbf{x}_2 \cdot \mathbf{x}_2) = 1 \cdot (-1) \cdot (-1) = 1,$$

a contradiction. Therefore, the induced subgraph of $\mathrm{AK}(2,q)$ on $X_\boxtimes/\!\sim$ is K_3-free.

Our aim in this section is to extend Parsons' result to all $t \geq 4$.

First, the cross-product can be extended from 3-dimensional space to r-dimensional space for every $r \geq 4$. We refer the reader to [27] and [13] for the formal definition. Here we only recall some basic properties of the cross-product in r-dimensional space.

Fact 3.1 (see e.g. [13])**.** *Suppose that $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{t-1} \in \mathbb{F}_q^t$ are $t-1$ vectors. Then*

1. *$\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1}$ is skew-symmetric and linear in each $\boldsymbol{x}_i$,*
2. *$\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1}$ is a vector that is orthogonal to each of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{t-1}$,*
3. *$\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1} = 0$ iff $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{t-1}$ are linearly dependent.*

A proof of the following theorem can be found in [13].

Theorem 3.2 (see e.g. [13])**.** *Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{t-1}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{t-1} \in \mathbb{F}_q^t$. Then*

$$(\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1}) \cdot (\boldsymbol{y}_1 \times \cdots \times \boldsymbol{y}_{t-1}) = \det \begin{bmatrix} \boldsymbol{x}_1 \cdot \boldsymbol{y}_1 & \dots & \boldsymbol{x}_1 \cdot \boldsymbol{y}_{t-1} \\ \vdots & \ddots & \vdots \\ \boldsymbol{x}_{t-1} \cdot \boldsymbol{y}_1 & \dots & \boldsymbol{x}_{t-1} \cdot \boldsymbol{y}_{t-1} \end{bmatrix}. \tag{3}$$

Our main result in this section is as follows.

Theorem 3.3. *Suppose that q is an odd prime power and $t \geq 3$ is an odd integer. Then the induced subgraph of* $\mathrm{AK}(t-1,q)$ *on* $X_{\boxtimes}/\sim$ *is a* K_t*-free* (p,α)*-jumbled graph with* $p = \Theta\left(n^{-\frac{1}{t-1}}\right)$ *and* $\alpha = \Theta(\sqrt{np})$*, where* $n = |X_{\boxtimes}/\sim|$*.*

Proof. Suppose to the contrary that there exists a set S of t distinct points $[\![\mathbf{x}_1]\!], \ldots, [\![\mathbf{x}_t]\!] \in X_{\boxtimes}/\sim$ such that the induced subgraph of $\mathrm{AK}(t-1,q)$ on S is complete. Then it follows from Fact 3.1 that there exists a nonzero element $a \in \mathbb{F}_q$ such that $\mathbf{x}_t = a\mathbf{x}_1 \times \cdots \times \mathbf{x}_{t-1}$. Therefore, by (3), we have

$$\begin{aligned}\mathbf{x}_t \cdot \mathbf{x}_t &= (a\mathbf{x}_1 \times \cdots \times \mathbf{x}_{t-1}) \cdot (a\mathbf{x}_1 \times \cdots \times \mathbf{x}_{t-1}) \\ &= a^2 \cdot \det \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{x}_1 & \ldots & \mathbf{x}_1 \cdot \mathbf{x}_{t-1} \\ \vdots & \ddots & \vdots \\ \mathbf{x}_{t-1} \cdot \mathbf{x}_1 & \ldots & \mathbf{x}_{t-1} \cdot \mathbf{x}_{t-1} \end{bmatrix} = a^2 \cdot \prod_{i=1}^{t-1} (\mathbf{x}_i \cdot \mathbf{x}_i).\end{aligned}$$

In the last equality we used the fact that $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ for all $i \neq j$. Applying the quadratic character $\chi(\cdot)$ to both sides of the equation above, we obtain

$$-1 = \chi(\mathbf{x}_t \cdot \mathbf{x}_t) = \chi\left(a^2 \cdot \prod_{i=1}^{t-1} (\mathbf{x}_i \cdot \mathbf{x}_i)\right) = \chi(a^2) \cdot \prod_{i=1}^{t-1} \chi(\mathbf{x}_i \cdot \mathbf{x}_i) = 1 \cdot (-1)^{t-1} = 1,$$

a contradiction. ∎

For the case that $t \in \mathbb{N}$ is even we use a different argument.

Proposition 3.4. *For every vertex $v \in \mathrm{PG}(t,q)$ the induced subgraph of* $\mathrm{AK}(t,q)$ *on* $N(v)$ *is isomorphic to* $\mathrm{AK}(t-1,q)$*.*

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_t$ be the standard orthonormal basis of the t-dimensional space $\mathbb{F}_q^t$. Fix a vector $\mathbf{v} \in \mathbb{F}_q^{t+1} \setminus X_0$ and let $\mathbf{e}'_1, \ldots, \mathbf{e}'_t$ be an orthonormal basis of the t-dimensional space $\mathbf{v}^{\perp}$, where $\mathbf{v}^{\perp} = \{\mathbf{w} \in \mathbb{F}_q^{t+1} \in : \mathbf{w} \cdot \mathbf{v} = 0\}$. Define the map $\phi \colon \mathbb{F}_q^t \to \mathbf{v}^{\perp}$ by sending $\sum_{i=1}^t a_i \mathbf{e}_i$ to $\sum_{i=1}^t a_i \mathbf{e}'_i$. Clearly, the map ϕ is linear and induces a bijection between $\mathbb{F}_q^t/\sim$ and $\mathbf{v}^{\perp}/\sim$. Moreover, ϕ sends absolute points to absolute points. Now suppose that $\mathbf{u}_1 = \sum_{i=1}^t a_i \mathbf{e}_i$ and $\mathbf{u}_2 = \sum_{i=1}^t b_i \mathbf{e}_i$ are two distinct points in $\mathbb{F}_q^t$. Then

$$\psi(\mathbf{u}_1) \cdot \psi(\mathbf{u}_2) = \left(\sum_{i=1}^t a_i \mathbf{e}'_i\right) \cdot \left(\sum_{i=1}^t b_i \mathbf{e}'_i\right) = \sum_{i=1}^t a_i b_i = \left(\sum_{i=1}^t a_i \mathbf{e}_i\right) \cdot \left(\sum_{i=1}^t b_i \mathbf{e}_i\right) = \mathbf{u}_1 \cdot \mathbf{u}_2.$$

This implies that the map ϕ preserves the orthogonality of two vectors, and hence, it sends edges (resp. non-edges) in $\mathrm{AK}(t,q)$ to an edge (resp. non-edge) in the induced subgraph of $\mathrm{AK}(t,q)$ on $\mathbf{v}^{\perp}$. Therefore, ϕ induces an isomorphism between $\mathrm{AK}(t-1,q)$ and the induced subgraph of $\mathrm{AK}(t,q)$ on $N(\mathbf{v})$. ∎

Lemma 3.5. *Suppose that $\alpha \in (0,1)$ is a constant and $V_1 \subset \mathrm{PG}(t,q)$ is a subset of size $\alpha \cdot |\mathrm{PG}(t,q)|$ in the graph* $\mathrm{AK}(t,q)$*. Then there exists a vertex $v \in V_1$ such that*

$$\frac{|N(v) \cap V_1|}{|N(v)|} \geq (1 - o_q(1))\alpha.$$

Proof. Suppose to the contrary that there exists an absolute constant $\epsilon > 0$ such that $\frac{|N(v)\cap V_1|}{|N(v)|} \le (1-\epsilon)\alpha$ for all $v \in V_1$ and for all q. Choose q to be sufficiently large. Let $n = |\mathrm{PG}(t,q)|$ be the number of vertices in $\mathrm{AK}(t,q)$, and let d be the degree of $\mathrm{AK}(t,q)$. Then, it follows from our assumption that

$$\begin{aligned} e(V_1) = \frac{1}{2}\sum_{v\in V_1} |N(v)\cap V_1| &\le \frac{1}{2}(1-\epsilon)\alpha|N(v)||V_1| \\ &= \frac{1}{2}(1-\epsilon)\frac{d}{n}|V_1|^2 \\ &= \frac{1}{2}\frac{d}{n}|V_1|^2 - \frac{\epsilon}{2}\frac{d}{n}|V_1|^2 = \frac{1}{2}\frac{d}{n}|V_1|^2 - \frac{\epsilon\alpha}{2}d|V_1|. \end{aligned}$$

Since $\frac{\epsilon\alpha}{2}d \gg \sqrt{d}$, this contradicts the fact that $\mathrm{AK}(t,q)$ is $(\frac{d}{n}, \Theta(\sqrt{d}))$-jumbled. ∎

Now we are ready to prove Theorem 1.1 for even t. Our construction will be an induced subgraph of $\mathrm{AK}(t,q)$ on a subset of the neighborhood of a vertex.

Proof of Theorem 1.1 for even t. Let $t \in \mathbb{N}$ be an even number. Let V denote the vertex set of $\mathrm{AK}(t,q)$. Let $V_1 = X_{\boxtimes}/\!\sim$. Since $|V_1| = (1/2+o(1))|\mathrm{PG}(t,q)|$, by Lemma 3.5, there exists a vertex $[\![\mathbf{v}]\!] \in V_1$ such that $|N([\![\mathbf{v}]\!])\cap V_1| \ge \left(\frac{1}{2}-o(1)\right)|N([\![\mathbf{v}]\!])|$. Let $U = N([\![\mathbf{v}]\!])\cap V_1$. By Proposition 3.4, the induced subgraph of $\mathrm{AK}(t,q)$ on the set $N([\![\mathbf{v}]\!])$ is isomorphic to $\mathrm{AK}(t-1,q)$, which is (p,α)-jumbled with $p = \Theta(m^{-\frac{1}{t-1}})$ and $\alpha = \Theta(\sqrt{mp})$, where $m = |\mathrm{PG}(t-1,q)|$. On the other hand, by Theorem 3.3, the induced subgraph of $\mathrm{AK}(t,q)$ on the set $X_{\boxtimes}/\!\sim$ is K_{t+1}-free. Therefore, the induced subgraph of $\mathrm{AK}(t,q)$ on the set U is K_t-free. This proves Theorem 1.1 for even t. ∎

4 $\{K_{2,3}, K_3\}$-free pseudorandom graphs

In this section we present a construction of $\{K_{2,3}, K_3\}$-free pseudorandom graphs.

Suppose that H is a finite group and $S \subset H$ is a symmetric subset, i.e. $S = S^{-1}$. Then the *Cayley graph* $\mathrm{Cay}(H,S)$ is a graph on H with edge set

$$\{\{v, vs\}\colon v \in H \text{ and } s \in S\}.$$

The spectrum, i.e. the eigenvalues of the adjacency matrix, of a Cayley graph can be represented by the characters of of H (see e.g. [23, 6]). For our purpose, we only need the following result for the case that H is an abelian group.

Recall that an abelian group H can be represented as $H = \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$ for some integers k and $n_1,\dots,n_k$. For abelian groups we have a simple description of all the characters. For each $\mathbf{a} = (a_1,\dots,a_k) \in \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$ we have a character $\psi_{\mathbf{a}}\colon H \to \mathbb{C}$ defined by

$$\psi_{\mathbf{a}}(h_1,\dots,h_k) = \prod_{i=1}^k \omega_{n_i}^{a_i h_i},$$

where $\omega_t = e^{2\pi i/t}$.

Lemma 4.1 (see e.g. [23, 6])**.** *Suppose that $H = \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$ is an abelian group. Then the spectrum of the Cayley graph* $\mathrm{Cay}(H,S)$ *is*

$$\left\{\sum_{s\in S}\psi_{\boldsymbol{a}}(\boldsymbol{s})\colon \boldsymbol{a}\in H\right\}.$$

Our main result is as follows.

Theorem 4.2. *Suppose that $p \neq 3$ is a prime number, $H = \mathbb{Z}_p^2$, and*

$$S = \left\{(x, x^3) \colon x \in \mathbb{Z}_p \setminus \{0\}\right\}.$$

Then $\mathrm{Cay}(H, S)$ *is a* $\{K_3, K_{2,3}\}$*-free* (n, d, λ)*-graph with* $n = p^2$, $d = p - 1$, *and* $\lambda \leq 2\sqrt{p} + 1$.

We will use the following well known estimate of Weil in the proof of Theorem 4.2.

Recall that the *order* of a character χ is the smallest positive integer d such that $\chi^d = \chi_0$, where χ_0 is the trivial character.

Theorem 4.3 (see e.g. [9, Theorem 13.3])**.** *Let χ be a character of order $d > 1$. Suppose that $f(X) \in \mathbb{F}[X]$ has precisely m distinct zeros and it is not a dth power, that is $f(X)$ is not the form $c\,(g(X))^d$, where $c \in \mathbb{F}$ and $g(X) \in \mathbb{F}[X]$. Then*

$$\left|\sum_{x \in \mathbb{F}} \chi\left(f(x)\right)\right| \leq (m-1)\sqrt{p}.$$

Proof of Theorem 4.2. Let $G = \mathrm{Cay}(H, S)$. Let $n = p^2$. It is clear that the number of vertices in G is n, and it follows from the definition of Cayley graphs that G is $|S|$-regular. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A_G of G. Since G is regular, we have $\lambda_1 = |S| = p - 1$.

First we prove that G is K_3-free. Suppose to the contrary that there exist three vertices $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{Z}_p^2$ that form a copy of K_3 in G. Assume that $\mathbf{v} - \mathbf{u} = (a, a^3)$, $\mathbf{w} - \mathbf{v} = (b, b^3)$, and $\mathbf{u} - \mathbf{w} = (c, c^3)$. Then

$$\begin{aligned} a + b + c &= 0, \\ a^3 + b^3 + c^3 &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 = (a+b+c)(a^2+b^2+c^2) - (a^3+b^3+c^3) &= ab(a+b) + ac(a+c) + bc(b+c) \\ &= -3abc. \end{aligned}$$

Since $p \neq 3$, we must have $0 \in \{a, b, c\}$, a contradiction.

Next we prove that G is $K_{2,3}$-free. It is equivalent to show that every pair of vertices $\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{Z}_p^2$ has at most two common neighbors. Let $a = \mathbf{u}_1 - \mathbf{v}_1$ and $b = \mathbf{u}_2 - \mathbf{v}_2$. A common neighbor of $\mathbf{u}$ and $\mathbf{v}$ implies that there exist $x, y \in \mathbb{Z}_p \setminus \{0\}$ such that

$$\begin{aligned} y - x &= a, \text{ and} \\ y^3 - x^3 &= b. \end{aligned}$$

These two equations imply that $(x+a)^3 = x^3 + b$, which simplifies to $3ax^2 + 3a^2x + a^3 - b = 0$. Since $(a, b) \neq (0, 0)$ and $p \neq 3$, this quadratic equation in x has at most two solutions in $\mathbb{Z}_p \setminus \{0\}$. Therefore, $\mathbf{u}$ and $\mathbf{v}$ have at most two common neighbors.

Finally, we prove that $|\lambda_i| \leq 2\sqrt{p}$ for all $i \in [2, n]$. By Lemma 4.1, for every $i \in [n]$ there exists $(a_1, a_2) \in \mathbb{Z}_p^2$ such that

$$\lambda_i = \lambda_{(a_1, a_2)} = \sum_{\mathbf{s} \in S} \varphi_{(a_1, a_2)}(\mathbf{s}) = \sum_{x=1}^{p-1} \omega_p^{a_1 x + a_2 x^3}$$

If $(a_1, a_2) = (0, 0)$, then $\lambda_{(a_1,a_2)} = |S| = p - 1$, and this corresponds to λ_1. So we may assume that $(a_1, a_2) \neq (0, 0)$.

First, it is easy to see that the character $\chi \colon \mathbb{Z}_p \to \mathbb{C}^\times$ defined by $\chi(\alpha) = \omega_p^\alpha$ for all $\alpha \in \mathbb{Z}_p$ has order p. On the other hand, since $(a_1, a_2) \neq (0, 0)$ and $p \neq 3$, the polynomial $f(X) = a_1 X + a_2 X^3$ is not of the form $c\,(g(X))^p$ for any $c \in \mathbb{Z}_p$ and for any polynomial $g(X)$. Therefore, it follows from Theorem 4.3 that

$$\left|\sum_{x=1}^{p-1} \omega_p^{a_1 x + a_2 x^3}\right| = \left|\sum_{x=0}^{p-1} \omega_p^{a_1 x + a_2 x^3} - 1\right| \leq \left|\sum_{x=0}^{p-1} \omega_p^{a_1 x + a_2 x^3}\right| + 1 \leq 2\sqrt{p} + 1.$$

This implies that $|\lambda_i| \leq 2\sqrt{p} + 1$ for all $i \in [n] \setminus \{1\}$, and hence completes the proof of Theorem 4.2. ∎

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