The Hopf Conjecture in 4 Dimensions

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1 Introduction

Consider a manifold M of real dimension $2k$. There exists an observed relationship between the vanishing of its higher homotopy groups and the topological Euler characteristic, which has been called the Hopf conjecture. If a topological space has a fundamental group $G = \pi_1(M)$ and vanishing higher homotopy groups, then we say M is a $K(G, 1)$ space. The Hopf conjecture is the following:

Conjecture 1.1. *If a closed, orientable manifold M^{2k} is a $K(G, 1)$ space, then $(-1)^k \chi(M^{2k}) \geq 0$.*

Albeit being called the Hopf conjecture, this was originally conjectured by William Thurston in the 1970s and is a topological version of an earlier conjecture by Heinz Hopf: if a $2k$-dimensional Riemannian manifold has non-negative sectional curvature everywhere, then does the inequality $(-1)^k \chi(M) \geq 0$ hold? Note that, by the Cartan-Hadamard theorem, such a Riemannian manifold has vanishing higher homotopy groups. So Thurston's version of the Hopf conjecture is a stronger statement.

In this thesis, we will look at the 4 dimensional Hopf conjecture; that is, 4-manifolds with vanishing higher homotopy groups have non-negative Euler characteristic. We take a peek at the various techniques that have been used to study $K(G, 1)$ manifolds (also called aspherical manifolds). We will first begin with a discussion of the 2 dimensional case and then discuss various techniques to study the 4 dimensional case. The conjecture is shown to work for complex surfaces, and some partial results which have been obtained by strengthening the hypothesis of the conjecture (placing restrictions on the class of the manifold) will be presented.

2 Definitions

First, excepting section 3 of the thesis which covers the 2 dimensional case, we restrict ourselves to compact, smooth, 4 dimensional manifolds. We will denote a manifold of this type by M and we will denote the universal covering space of M by $\widetilde{M}$. We will also denote the Euler characteristic of M by $\chi(M)$ and the signature of the intersection form on M by $\sigma(M)$.

We also denote a genus g surface by Σ_g. For example, the 2 dimensional torus is Σ_1. We denote spheres by S^2, real projective space by $\mathbb{RP}^2$, and complex projective space by $\mathbb{CP}^2$. If M, N are spaces, the connected sum, wedge product, and disjoint union are denoted $M\#N$, $M \vee N$, and $M \sqcup N$, respectively.

The most important definition in our study is that of an aspherical manifold. We present two definitions.

Definition 2.1. *A topological space X is said to be a $K(G, n)$ space or an* Eilenberg-Maclane *space if $\pi_n(X) = G$ and $\pi_i(X) = 0$ for $i \neq n$.*

Definition 2.2. *A manifold is said to be* aspherical *if it is a $K(G, 1)$ space; that is, all higher homotopy groups vanish.*

Observe that the next definition is identical to definition 2.2

Definition 2.3. *A manifold M is said to be* aspherical *if $\widetilde{M}$ is contractible.*

Since the fundamental group of $\widetilde{M}$ must be trivial and the covering map induces isomorphisms of homotopy groups, $\pi_i(\widetilde{M}) = 0$ for all i. By Hurewicz's theorem, $H_i(\widetilde{M}) = 0$ for all i. Finally, since $\widetilde{M}$ is a weakly contractible manifold, it is contractible. The converse holds for each theorem, showing us the two definitions are identical.

We note that the Hopf conjecture for 4-manifolds boils down to whether or not all aspherical 4-manifolds have non-negative Euler characteristic.

Looking at the Euler characteristic in terms of Betti numbers, we obtain

$$\chi(M) = \sum_i (-1)^i b_i = b_0 - b_1 + b_2 - b_3 + b_4$$

Since M is orientable, $H_0(M) \cong H_4(M) \cong \mathbb{Z}$ and $b_0 = b_4 = 1$. Further, since M is a smooth manifold, we can apply Poincaré duality so that $b_1 = b_3$. We obtain $\chi(M) = 2 - 2b_1 + b_2$. So we can take the liberty of restricting our view to manifolds with $b_1 \geq 2$.

3 The Conjecture in 2 Dimensions

This theorem is simple in 2 dimensions due to the classification of compact surfaces. While compact surfaces can be categorized according to genus and orientability, 4-manifolds are in a sense unclassifiable. Any finitely generated group is the fundamental group for some 4-manifold, and so classification reduces to the word problem. Indeed, it is common to restrict oneself to the simply connected case and even then a classification has not yet been found (and may even be impossible!).

The well known classification of compact surfaces is the following statement:

Theorem 3.1. *Let M be a compact surface. Then $M \cong S^2$, $M \cong n\Sigma_1$, or $M \cong n\mathbb{RP}^1$, where $n\Sigma_1$ is a connected sum of tori and $n\mathbb{RP}^1$ is an n-fold connected sum of real projective planes.*

Since we only care about orientable surfaces, we can discard the last case. This simplifies the classification to the following: a compact, orientable surface is either the sphere S^2 or a genus g surface $\Sigma_g \cong g\Sigma_1$. We now show that S^2 is the only non-aspherical manifold (indeed, the term aspherical derives its origins from the fact that all oriented surfaces that are $K(G,1)$ are precisely those which are not a sphere).

Proposition 3.2. *The 2-sphere S^2 has a nontrivial higher homotopy group. In particular, $\pi_2(S^2) = \mathbb{Z}$.*

Proof. We begin by recalling that the singular homology for S^2 is
$H_0(S^2) = H_2(S^2) = \mathbb{Z}$ and $H_k(S^2) = 0$ for $k \neq 0, 2$. Let $h_i : \pi_i(S^2) \to H_k(S^2)$ be the Hurewicz map. Then since S^2 is simply connected, the Hurewicz theorem guarantees that $h_2 : \pi_2(S^2) \to H_2(M) \cong \mathbb{Z}$ is an isomorphism. □

While this proposition is intended to show that S^2 is non-aspherical, the fact that $\pi_2(S^2) = \mathbb{Z}$ will make an appearance later when we prove the non-asphericality of some more interesting manifolds.

To prove that surfaces Σ_g are aspherical, usually a little more is required. For the sake of brevity, only the claim is presented, but the full proof can be found as example 14 in Section 1.B of [**Hat00**].

Proposition 3.3. *Σ_g is an aspherical manifold for $g \geq 1$.*

Now that we have classified all compact closed surfaces into aspherical and non-aspherical surfaces, we can finish the proof by calculating the Euler characteristic of all the aspherical surfaces.

Theorem 3.4. *The Hopf conjecture holds for orientable, closed surfaces. That is, if M is an aspherical surface, then $\chi(M) \leq 0$.*

Proof. If we assume that M is an aspherical surface, then $M \cong \Sigma_g$ for some $g \geq 1$. By the product property of the Euler characteristic, $\chi(\Sigma_1) = \chi(S^1)^2 = 0$. Let D^2 be the closed 2 disk. If we consider $\Sigma_g = S^2 \# g\Sigma_1$, then by the inclusion-exclusion property, we have

$$\begin{aligned}
\chi(\Sigma_g) &= g\chi(\Sigma_1 - D^2) + \chi(S^2 - gD^2) - g\chi(S^1) \\
&= g[\chi(\Sigma_1) - \chi(D^2)] + [\chi(S^2) - g\chi(D^2)] \\
&= -g + 2 - g = 2 - 2g
\end{aligned}$$

Since $g \geq 1$, we have $\chi(M) = \chi(\Sigma_g) = 2 - 2g \leq 0$, finishing the proof. □

4 Results from the Fundamental Group

We present useful results obtained by just looking at the fundamental group of M. The following theorems revolve around the following inequality.

Definition 4.1 (Winkelnkemper's inequality)**.** *A 4-manifold M is said to satisfy Winkelnkemper's inequality if $|\sigma(M)| \leq \chi(M)$.*

Definition 4.2. *A group G is said to be a* W-group *if it is finitely presented and all 4-manifolds M with $\pi_1(M) \cong G$ satisfy Winkelnkemper's inequality.*

Johnson and Dotschik [**JK93**] give several conditions for G to be a W-group. Three are as follows.

Theorem 4.3. *If G contains a sequence of nested subgroups $K_1 K_2 K_3 \cdots$, each with finite index $[K_n : K_{n+1}]$, so that the Betti number of the group homology $\dim H_i(K_i, \mathbb{R})$ is bounded, then G is a W-group.*

Theorem 4.4. *If G has an extension $1 \to K \to G \to \Delta \to 1$ with K finitely presented and Δ satisfying the hypothesis of the previous theorem, then G is a W-group.*

Theorem 4.5. *If G is a W-group with subgroup K and G' contains a subgroup K' so that $[G : K] = [G' : K']$ and $K \cong K'$ then G' is a W-group.*

From these theorems, we can begin to determine what fundamental groups necessitate an aspherical manifold to have non-negative Euler characteristic.

Proposition 4.6. *$\mathbb{Z}$ is a W-group.*

Proof. We note that the homology of a group G is equivalent to the homology of the classifying space BG. Since the classifying space of $\mathbb{Z}$ is S^1, the homology of $\mathbb{Z}$ is

$$H_k(\mathbb{Z}, \mathbb{R}) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

From this we see that $b_1(\mathbb{Z}) = \dim H_1(S^1;\mathbb{R}) = 1$. Now we choose our infinite sequence of subgroups to be the subgroups $K_i = 2^i\mathbb{Z}$. Each is isomorphic to $\mathbb{Z}$ and each K_i has index 2 in K_{i+1}. This satisfies the conditions of theorem 4.3 and so $\mathbb{Z}$ is a W group. $\square$

Proposition 4.7. *Any finitely generated abelian group is a W-group.*

Proof. Let G be a finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups, $G \cong \mathbb{Z}^n \times \mathbb{Z}_{p_1} \times \cdots \mathbb{Z}_{p_k}$.
Let $K_i = 2^i\mathbb{Z} \times \mathbb{Z}^{n-1} \times \{0\}^k$. Then the index of K_i in G is
$[G : K_i] = |G/K_i| = |\mathbb{Z}/2\mathbb{Z} \times \{0\}^{n-1} \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}| = 2^i p_1 \cdots p_k$ which is finite for all K_i. Since $[G : K_i] = [G : K_{i-1}] \cdot [G : K_i]$, the index of each subgroup in the subgroup above it is finite. Further, $K_i \cong \mathbb{Z}^n$ and so
$\dim H_1(K_i;\mathbb{R}) = \dim H_1(\mathbb{Z}^n;\mathbb{R}) = \dim H_1((S^1)^n;\mathbb{R}) = 1$. So G satisfies the conditions of theorem 4.3 and is a W-group. $\square$

We also have the following general fact about manifolds, which is proposition 2.45 in Hatcher's *Algebraic Topology* [**Hat00**].

Theorem 4.8. *If a finite-dimensional CW-complex X is a $K(G,1)$, then $\pi_1(X)$ is torsion-free.*

Thus, we can restrict ourselves to studying nonabelian fundamental groups with no torsion (in particular, $\pi_1(X)$ is finite excepting the trivial case).

5 The Hopf Conjecture for Complex Surfaces

We now take a look at compact complex surfaces, which make up a large class of examples of 4-manifolds. Furthermore, this allows us to use some tools of algebraic geometry to get a better look at these manifolds. In particular, we will be using the Enriques-Kodaira classification for compact complex surfaces.

We first begin with a quick look at blow ups, which are central to the classification. We first motivate the definition by a simple example.
Consider the complex plane $\mathbb{C}^n$ and let 0 denote the origin. The blow-up "zooms in" on the origin by replacing it with the space of all possible directions from the origin. To make this more concrete, we let $\mathbb{CP}^n$ be the space of lines coming from the origin and construct the space

$$K = \{(p, \ell) \mid p \text{ and } 0 \text{ lie in } \ell\} \subset \mathbb{C} \times \mathbb{CP}^n.$$

Then for each $p \neq 0$, there is precisely one element of K of the form (p, ℓ) since two points in the plane exactly determine a line. However, there are infinitely many points of the form $(0, \ell)$, and those points form a copy of $\mathbb{CP}^n$ in K. So the blow-up is the same space but with a copy of $\mathbb{CP}^n$ attached to the blow-up point. This is made more general in the following definition.

Definition 5.1 (Blow-Up). *Let X be a complex surface with complex dimension n. The blow-up of X at a point x, which we will denote $\widehat{X}$, is a space with a map $p : \widehat{X} \to X$ such that p induces a homeomorphism outside of x and $p^{-1}(x)$ is a copy of $\mathbb{CP}^{n-1}$, called the exceptional divisor. In particular, $\widehat{X} \cong X \# \overline{\mathbb{CP}^2}$. If such a map exists from X to another space Y, we call Y the blow-down of X. If no such Y exists that is a blow down of X, we call X a minimal surface. Lastly, if there exists a series of blow-downs $p_1, p_2, \cdots, p_k$ from X to Y and Y is minimal, we call Y a minimal model of X.*

The Enriques-Kodaira classification works by assigning a surface a Kodaira dimension, denoted $\kappa(M)$. This number gives the size of the canonical model, a ring which encodes information about the cotangent bundle. In particular, it assigns a surface the Kodaira dimension of its minimal model (this process is well-defined). The precise definition of the Kodaira dimension is unnecessary here, but a full treatment is given in Chapter I Section VII of Barth's *Compact Complex Surfaces* [**Bar04**].

Now, we take a look at the table that classifies minimal compact complex surfaces. This comes from Chapter VI, Section I of [**Bar04**].

Class of M	$\kappa(M)$	$b_1(M)$
1) minimal rational surfaces		0
2) minimal surfaces of class VII	$-\infty$	1
3) ruled surfaces of genus ≥ 1		$2g$
4) Enriques surfaces		0
5) Bi-elliptic surfaces		2
6) Kodaira surfaces	0	
a) primary		3
b) secondary		1
7) K3 surfaces		0
8) Tori		4
9) Minimal properly elliptic surfaces	1	
10) minimal surfaces of general type	2	0 mod 2

Definitions for the surfaces featured in the classification are as follows. Note that a fiber bundle is holomorphic if the projection map $p : M \to B$ can be written as a complex differentiable function from an open subset of $\mathbb{C}^2$ to $\mathbb{C}$ in a neighborhood of any point $b \in B$.

Definition 5.2 (Surfaces in the Enriques-Kodaira classification)**.**

- *Rational Surfaces are surfaces that are birationally equivalent to* $\mathbb{CP}^2$*.*
- *Surfaces of Class VII are surfaces with* $\kappa(M) = -\infty$ *and* $b_1 = 1$*. There are several examples of Class VII surfaces, but there does not exist a classification.*
- *Ruled surfaces are surfaces which admit a fiber bundle by* $\mathbb{CP}^1$ *over a surface of genus* g*.*
- *Bi-elliptic surfaces are surfaces with* $b_1 = 2$ *which admit a holomorphic fiber bundle of an elliptic curve over an elliptic curve.*
- *Primary Kodaira surfaces are surfaces with* $b_1 = 3$ *and which admit a holomorphic fiber bundle of an elliptic curve over an elliptic curve.*
- *A torus refers to the 4-torus, i.e.* $S^1 \times S^1 \times S^1 \times S^1$*.*
- *Properly elliptic surfaces are fibrations of elliptic curves over a smooth complex curve.*
- *Surfaces of general type are simply surfaces with* $\kappa = 2$*.*

We can see by this chart and the definitions a general rule associated with Kodaira dimensions: the higher the dimension, the more general the surface and the harder it is to say anything concrete about it. One can work case by case and use specific properties of the surfaces in each case, but the more general surfaces have more freedom and thus less shared properties.

The reader may note that several definitions were omitted from the definition list (the Enriques surface, K3 surface, and the Kodaira surface of secondary type). This is because they are not particularly interesting; since the first Betti number satisfies $b_1 \leq 1$, they trivially satisfy the antecedent of the Hopf conjecture. Rational surfaces were included, however, because while they have non-negative Euler characteristic, they become objects of consideration when consider non-minimal complex surfaces. It is also of note that in fact there exist no aspherical ruled surfaces.

Theorem 5.3. *There exist no aspherical ruled surfaces.*

Proof. Let M be a ruled 4-manifold. Firstly, we note that $\mathbb{CP}^1 \cong S^2$ as manifolds and so we can treat M as an S^2 fibration over the Riemannian manifold B. Then we have a fibration $p : M \to B$, where $p^{-1}(x) = S^2$ for $x \in B$. Since M is compact, B is as well, and so $B \cong \Sigma_g$ for $g > 0$ or $B \cong S^2$. In the first case, the long exact sequence

$$\dots \to \pi_3(B) \to \pi_2(S^2) \to \pi_2(M) \to \pi_2(B) \to \dots$$

yields the short exact sequence

$$0 \to \mathbb{Z} \to \pi_2(M) \to 0$$

since Σ_g is itself aspherical. Then $\pi_2(M) \cong \mathbb{Z}$ and M is not aspherical.
In the second case, we can consider an open circular cap $\tilde{V}$ in $B \cong S^2$, then remove an open circular cap that is a subset of $\tilde{V}$ to obtain a punctured sphere, $\tilde{U}$. Note that $\tilde{U} \cap \tilde{V}$ is an open annulus. Then let U and V be the fibration restricted to $\tilde{U}$ and $\tilde{V}$. We can then apply the Seifert-van Kampen theorem on U and V. Since both have contractible base spaces, they are homeomorphic to the trivial bundle $D^2 \times S^2$. So

$$\pi_1(M) \cong \pi_1(D^2 \times S^2) \underset{\pi_1(U \cap V)}{*} \pi_1(S^2 \times S^2) \cong \{0\} \underset{\pi_1(U \cap V)}{*} \{0\} = \{0\}.$$

Looking at the sequence again, we have $\pi_2(M) \xrightarrow{p_*} \pi_2(S^2) \to 0$. Since p_* must be surjective and $\pi_2(S^2) \cong \mathbb{Z}$, $\pi_2(M)$ is non-empty and so M is not aspherical. $\square$

We now endeavor to prove that all compact complex surfaces of real dimension 4 satisfy the Hopf conjecture.

Theorem 5.4. *Let M be a closed aspherical minimal complex surface of real dimension 4. Then $\chi(M) \geq 0$.*

Proof. We naturally work case-by-case through the possible Kodaira dimensions of M and check that each subcategory satisfies $\chi(M) \geq 0$.

Case 1: M has a Kodaira dimension of $-\infty$:

This case is relatively simple. We begin with the case of rational surfaces. It is known that the fundamental group is a birational invariant; so $\pi_1(M) \cong \pi_1(\mathbb{CP}^2)$. But $\mathbb{CP}^n$ is simply connected for $n \geq 1$ and so $\pi_1(M) = 0$. This case then degenerates to the contractible case.

As mentioned earlier, the minimal surfaces of class VII have no general classification, but this is no issue since it is known that $b_1(M) = 1$. So $\chi(M) \geq 0$ in this case.

From theorem 5.3 we know there exist no aspherical ruled surfaces for us to consider, and we are done with the $\kappa(M) = -\infty$.

Case 2: M has a Kodaira dimension of 0:

We begin by recalling again that $b_1(M) \leq 1$ implies that $\chi(M) \geq 0$. This allows us to immediately rule out the cases of the Enriques surface, the secondary Kodaira surfaces, and the K3 surface. We now tackle the rest of the cases.

The case of bi-elliptic surfaces and primary Kodaira surfaces can be considered together because they are both fibrations of an elliptic curve over an elliptic curve. However, this is simply a fibration of a torus over another torus. Not only are both of these surfaces aspherical, but they indeed satisfy the Hopf conjecture, since $\chi(M) = \chi(\Sigma_1)^2 = 0$.

Lastly, we cover the 4-torus, T^4. The 4-torus is aspherical since $\pi_n(T^4) = \pi_n(S^1) \times \pi_n(S^1) \times \pi_n(S^1) \times \pi_n(S^1)$, and $\pi_n(S^1) = 0$ for $n \geq 2$. Then $\chi(T^4) = \chi(S^1)^4 = 0$. This finishes the case $\kappa(M) = 0$.

Case 3: M has a Kodaira dimension of 1:

The only case we have to consider is that of the properly elliptic surfaces; that is, elliptic fibrations over smooth curves. Let the fibration be denoted $p : M \to C$ and let $S \subset C$ be the set of points whose inverse images under p give the singular fibers of the fibration. Then, according to Theorem 6.10 of [**SS09**], S is finite and the Euler characteristic of an elliptic fibration over $\mathbb{C}$ is of the form $\chi(M) = \sum_{x \in S} \chi(p^{-1}(x))$. Further, according to the classification of singular fibers given in [**SS09**], $\chi(F_x) = 0, b_0(F_x)$, or $b_0(F_x) + 1$ (where here b_0 is the zeroth Betti number, representing path connected components). Thus, $\chi(M)$ is a sum of non-negative integers and $\chi(M) \geq 0$.

Case 4: M has a Kodaira dimension of 2:

Here we look at the surfaces of general type. While such a task seems daunting, it is actually trivial. Theorem 1.1 in Chapter VII of [**Bar04**] gives a number of conditions the Chern numbers c_1^2 and c_2 of a surface with $\kappa(M) = 2$ must conform to. Luckily, one of these conditions is $c_2 > 0$. Because we are working in dimension 4 and c_2 is the top Chern class and is equal to the Euler class (and hence, the Euler characteristic) of M (chapter 1 in [**Bar04**]). However, it is known that the top Chern class of a manifold is precisely its Euler characteristic. So $\chi(M) = c_2 > 0$, finishing this case and the proof in general. $\square$

We have seen that every minimal model satisfies the Hopf conjecture. To prove that every compact complex surface satisfies the conjecture, we take a look at the effect that blowing up has on both the Euler characteristic and on the second homotopy group. We begin with the following useful proposition.

Proposition 5.5. *Let M be a compact complex surface of complex dimension 2 and $\widehat{M}$ a blow-up of M at a point p. Then $\chi(\widehat{M}) = \chi(M) + 1$.*

Proof. We use the inclusion-exclusion formula for the Euler characteristic. The blowup process M has the topological effect of adding a copy of $\overline{\mathbb{CP}^2}$, $\mathbb{CP}^2$ with the opposite orientation, via a connected sum. So $\overline{M} = M\#\overline{\mathbb{CP}^2}$ and we obtain

$$\chi(\overline{M}) = \chi(M\#\overline{\mathbb{CP}^1}) = \chi(M) + \chi(\mathbb{CP}^2) - \chi(S^4) = \chi(M) + 3 - 2 = \chi(M) + 1.$$

So blowing up increments the Euler characteristic by 1. □

We lastly focus precisely on the effect that a blow-up has on a ruled manifold; specifically, we show that a n-fold wedge connected sum of a genus ≥ 1 ruled manifold with $\overline{\mathbb{CP}^2}$ forces the second homotopy group to be non-trivial. This is the final tool that allows us to prove the Hopf conjecture for all compact complex surfaces.

Theorem 5.6. *Let M be a ruled manifold with genus greater than 0; that is, M is a $\mathbb{CP}^1$ bundle over Σ_g. Then $M\#n\overline{\mathbb{CP}^2}$ is not aspherical.*

Proof. We show that the second homotopy group of $M\#n\overline{\mathbb{CP}^2}$ is non-trivial using the Hurewicz theorem.

We first note that the universal covering space of Σ_g is $\mathbb{R}^2$. Let $p : \mathbb{R}^2 \to \Sigma_g$ be the covering map. Given a point $x \in \Sigma_g$, we obtain a discrete subset $p^{-1}(x) = L \subset \mathbb{R}^2$. Since $\pi_1(\Sigma_g)$ acts on $p^{-1}(x)$ (via monodromy) both transitively and freely, there is a bijection between L and $\pi_1(M)$.

Now let $f : M \to \Sigma_g$ be the projection for the fiber bundle M. Then we can pull back the projection of the fiber bundle under the projection for the universal covering space: we obtain a function $p^*(f) : N \to \mathbb{R}^2$, with $\left(p^*(f)\right)^{-1}(y) \cong \mathbb{CP}^1$. Then this is a new fiber bundle; in particular, it is a covering space of M. However, since $\mathbb{R}^2$ is a contractible space the new fiber bundle must be a product; since the fiber is $\mathbb{CP}^1$, the bundle is simply $\mathbb{CP}^1 \times \mathbb{R}^2$. Thus, $\widetilde{M} \cong S^2 \times \mathbb{R}^2$, since this product is clearly simply connected.

Let $p' : \widetilde{M} \to M$. Now consider the connected sum $M\#n\overline{\mathbb{CP}^2}$. Then each $\mathbb{CP}^2$ copy is summed at a point where a copy of B^4 has been removed; label these points x_i. Then let $L_i = p^{-1}(x_i) \subset \mathbb{R}^2$. We can choose open balls B_i^4 small enough so that $p^{-1}(B_i^4) \cap p^{-1}(B_j^4) = \emptyset$ for $i \neq j$. Now, we can pull back once again to the universal covering space. Since $\mathbb{CP}^2$ is simply connected, we find that the universal covering space of $M\#n\overline{\mathbb{CP}^2}$ is simply a connected sum of copies of $\mathbb{CP}^2$ near each point $p'^{-1}(x_i)$. Since S^2 is simply connected, each $p'^{-1}(x_i)$ gives a point (a, b)

with $a \in S^2$ and $b \in L_i$.

Now we denote $\widetilde{M}\#_L\overline{\mathbb{CP}^2}$ as the connected sum of a copy of $\mathbb{CP}^2$ at $p'^{-1}\left(\bigcup L_i\right)$. We also define the wedge product $\widetilde{M} \vee_L \overline{\mathbb{CP}^2}$ as a wedge sum at each point $x \in L \subset \widetilde{M}$ and the disjoint union $\sqcup_L S^3$ as a disjoint collection of S^3's for each point in L (and we consider $\sqcup_L S^3$ as a subset of $\widetilde{M}$). Now, we note that every embedded ball B^4 in $S^2 \times \mathbb{R}^2$ has a neighborhood that retracts onto its boundary S^3; thus $(\widetilde{M}, \sqcup_L S^3)$ is a good pair and we can use the long exact reduced homology sequence

$$\cdots \to \tilde{H}_i\left(\sqcup_L S^3\right) \to \tilde{H}_i\left((S^2 \times \mathbb{R}^2)\#_L\overline{\mathbb{CP}^2}\right) \to \tilde{H}_i\left((S^2 \times \mathbb{R}^2) \vee_L \overline{\mathbb{CP}^2}\right) \to \cdots$$

Let $X = \widetilde{M}\#_L\overline{\mathbb{CP}^2}$. We look at $i = 2$ to get

$$\cdots \to \tilde{H}_2\left(\sqcup_L S^3\right) \to \tilde{H}_2\left(X\right) \to \tilde{H}_2\left((S^2 \times \mathbb{R}^2) \vee_L \overline{\mathbb{CP}^2}\right) \to \tilde{H}_1\left(\sqcup_L S^3\right) \to \cdots$$

Then because $\tilde{H}_i(S^3) = \mathbb{Z}$ only if $i = 3$, we have that $\tilde{H}_1(\sqcup_L S^3)$ and $\tilde{H}_2(\sqcup_L S^3)$ are both trivial. Plugging into the long exact sequence gives a short exact sequence from which we obtain $\tilde{H}_2(X) \cong \tilde{H}_2\left((S^2 \times \mathbb{R}^2) \vee_L \overline{\mathbb{CP}^2}\right)$.

Now given that $(S^2 \times \mathbb{R}^2)$ and the collection $\sqcup_L\overline{\mathbb{CP}^2}$ have interiors that cover their wedge product and that their intersection is a collection of points, the Mayer-Vietoris sequence gives us the isomorphism

$$\tilde{H}_2\left((S^2 \times \mathbb{R}^2) \vee_L \overline{\mathbb{CP}^2}\right) \cong \tilde{H}_2\left(S^2 \times \mathbb{R}^2\right) \oplus \tilde{H}_2(\sqcup_L\overline{\mathbb{CP}^2}).$$

Now the homology of disjoint sum breaks up into direct sums of homology groups so that

$$\tilde{H}_2\left((S^2 \times \mathbb{R}^2) \vee_L \overline{\mathbb{CP}^2}\right) \cong \tilde{H}_2\left(S^2 \times \mathbb{R}^2\right) \oplus \bigoplus_L \tilde{H}_2(\overline{\mathbb{CP}^2}).$$

Lastly, we finally precisely calculate the homology of $\widetilde{M}$. We know that $\tilde{H}_2(\overline{\mathbb{CP}^2}) = \mathbb{Z}$. Using the Künneth formula along with the fact that $\tilde{H}_k(\mathbb{R}^2) = 0$, we get that $\tilde{H}_2(S^2 \times \mathbb{R}^2) = 0$. Thus, our final answer is

$$\tilde{H}_2(X) = \bigoplus_{\pi_1(M)} \mathbb{Z}^n.$$

And so the second homology group $H_2(X) \neq 0$. By Hurewicz' theorem, $\pi_2(X) = H_2(X) \neq 0$. Then since X is the universal covering of $M\#n\overline{\mathbb{CP}^2}$, $\pi_2(M\#n\mathbb{CP}^2) \cong \pi_2(X) \neq 0$ and $M\#n\overline{\mathbb{CP}^2}$ fails to be aspherical. Therefore, no blow-up of a ruled surface is aspherical. $\square$

We are now in a position to prove the Hopf conjecture for all compact complex manifolds.

Theorem 5.7. *Let M be a closed aspherical complex surface. Then $\chi(M) \geq 0$.*

Proof. The final proof is very simple. We first note that M has a minimal model from which M is obtained by a finite sequence of blow-ups. Theorem 5.4 tells us that every class of minimal surface has non-negative Euler characteristic. However, there is an exception which is the class of ruled manifolds which all have negative Euler characteristic.

So in the first case, if M has a minimal model that is a ruled manifold, then we know by Theorem 5.6 that M is in fact not aspherical and this case can be ruled out completely.

Now let M have a minimal model that is not a ruled manifold. Denote this model N. Then, topologically, $M \cong M\#n\overline{\mathbb{CP}^2}$ for some $n \geq 0$. Then, by Proposition 5.5, we have that $\chi(M) = \chi(N)+n$. However, since N is a minimal model then from the argument of the first paragraph $\chi(N) \geq 0$ and $\chi(M) \geq 0$. So if M were aspherical then its Euler characteristic would be non-negative, solving the Hopf conjecture in the case of complex surfaces. □

6 Partial Results for Symplectic Manifolds

Another family of manifolds we can restrict ourselves to further are symplectic manifolds; that is, even-dimensional manifolds with a nondegenerate 2-form, ω. We take a look at one method of classification of symplectic manifolds: the Symplectic Kodaira dimension.

We begin with the definition of the Kodaira dimension, from [**Li15**]:

Definition 6.1. *Let (M, ω) be a symplectic manifold. Let $[\omega]$ be the cohomology class of the 2-form ω and K_ω be the first Chern class of the cotangent bundle of M. The symplectic Kodaira dimension of (M, ω) is defined to be the symplectic Kodaira dimension of its minimal model. If M is minimal, then the symplectic Kodaira dimensions is defined as*

$$\kappa^s(M, \omega) = \begin{cases} -\infty & K_\omega \cdot [\omega] < 0 \textit{ or } K_\omega \cdot K_\omega < 0 \\ 0 & K_\omega \cdot [\omega] = 0 \textit{ and } K_\omega \cdot K_\omega = 0 \\ 1 & K_\omega \cdot [\omega] > 0 \textit{ and } K_\omega \cdot K_\omega = 0 \\ 2 & K_\omega \cdot [\omega] > 0 \textit{ and } K_\omega \cdot K_\omega > 0. \end{cases}$$

Unlike the Enriques-Kodaira dimension above, the symplectic Kodaira dimension is not quite as easy to work with, as symplectic manifolds behave badly compared to complex surfaces. However, it is easy to prove the Hopf conjecture for symplectic Kodaira dimension $-\infty$ and 0.

Proposition 6.2. *If M is a symplectic manifold with $\kappa^s(M) = -\infty, 0$ then $\chi(M) \geq 0$ if M is aspherical.*

Proof. In the case that $\kappa^s(M) = -\infty$, the symplectic manifolds are in fact more restricted than the complex surfaces; the only such symplectic manifolds are ruled manifolds and rational manifolds [**Li15**]. However, from Theorem 5.4 we know that rational manifolds satisfy the conjecture, and therefore we know there exist no aspherical ruled manifolds from Theorem 5.3.

Second, in the case that $\kappa^s(M) = 0$, there exists a classification of such symplectic manifolds up to homology. It is known that M is either a homology K3 surface, a homology Enriques surface, or an elliptic bundle over an elliptic surface. In the first two cases, we can take a look at the table from page 5 it is known that the Euler characteristic of the K3 and Enriques surface both have Euler characteristics of 0, and an elliptic fiber has Euler characteristic $\chi(\Sigma_1) \cdot \chi(\Sigma_1) = 0$. So the conjecture holds in this case as well. $\square$

7 Partial Results for Kähler Manifolds

Kähler manifolds are a particularly restricted class of well-behaved 4-manifolds. The structure imposed on them grants us many tools to study their local and global properties. In this section, we will look at some restrictions one can place on these manifolds and prove that they satisfy the Hopf conjecture.

We begin with a definition for Kähler manifolds.

Definition 7.1. *A Kähler manifold is a manifold M with triple (g, ω, J) with g a Riemannian metric on M, ω a non-degenerate 2-form on M, and J an almost complex structure on M (that is, a function $J : TM \to TM$ such that $J^2 = -Id$ on tangent spaces). Specifically, M must not only admit these structures, but they must satisfy the compatibility relation: $g(X, Y) = \omega(JX, Y)$.*

7.1 Cohomological Einstein-Kähler manifolds

One type of restriction we can put on a Kähler manifold is for its first Chern class $c_1 \in H^2(M, \mathbb{R})$ to be definite; that is, positive everywhere, negative everywhere, or $c_1 = 0$. In this case, the Euler characteristic is always non-negative.

Theorem 7.2. *Let M be a closed, aspherical, 4 dimensional Kähler manifold with definite first Chern class. Then $\chi(M) \geq 0$.*

Proof. As discussed before, we break up the proof into the case when c_1 is positive definite, c_1 is negative definite, and when c_1 is zero.
Case 1: c_1 is negative definite: In the case that $c_1 < 0$, it was proved in theorem 7.15 in [**Aub82**] that a metric exists for M that is both Einstein and Kähler. By an Einstein metric, we mean a metric that is a scalar multiple of the Ricci tensor of the manifold M (that is, $g(X, Y) = \lambda \cdot \mathrm{Ric}(X, Y)$).

However, in [**JK93**] it is shown in theorem 1 that if a compact manifold admits an Einstein metric, then $|\sigma(M)| \leq \frac{2}{3}\chi(M)$. Thus, $\chi(M) \geq 0$ (indeed, it satisfies Winkelnkemper's inequality) and the statement holds in the case of $c_1 < 0$.

Case 2: c_1 is positive definite: Next, in the case that $c_1 > 0$ the Ricci curvature Ric^g is positive on M. We use Myer's theorem: if $Ric^g \geq (m-1)k$ everywhere (for some $k \in \mathbb{R}_{>0}$), then the length between any geodesic is bounded above by $\frac{\pi}{\sqrt{k}}$. First note that since M is compact, there exists an $\varepsilon > 0$ that bounds Ric^g uniformly from below. Now consider the universal covering space $p : \widetilde{M} \to M$. Then $\widetilde{M}$ has a metric on it, specifically the pullback p^*g. However, since the Ricci curvature of g is positive, the Ricci curvature of p^*g is positive. Furthermore, p^*g is bounded below by some $\eta > 0$ because g is bounded uniformly below. Letting $k = \frac{\eta}{3}$, $\widetilde{M}$ and Ric^g satisfy the hypotheses of Myer's theorem. So any two points in the ambient space they have distance at most $\frac{\pi}{\sqrt{k}}$ and $\widetilde{M}$ is bounded and compact.

Now, given an open set $U \subset M$, the set $p^{-1}(U)$ is a collection of disjoint open sets of $\widetilde{M}$ and so it is a finite collection. But this means that p is a finite covering of M, which tells us that the fundamental group $\pi_1(M)$ is finite. But since M is aspherical, theorem 4.8 tells us that $\pi_1(M)$ must be trivial. Since all the homotopy groups of M are trivial, so too are the homology groups of M (by Hurewicz) and so $\chi(M) = 0$.

Case 3: c_1 is the zero form: We lastly consider the case when $c_1 = 0$. In this case, the manifold is minimal and the (classical) Kodaira dimension is at most 0; that is, $\kappa(M) = -\infty$ or $\kappa(M) = 0$ [**Bau08**]. However, these manifolds are precisely those manifolds which are considered in Theorem 5.4. So when $c_1 = 0$ the conjecture holds, and this finished the proof for cohomological Einstein-Kähler manifolds. □

7.2 Homogeneous Kähler Manifolds

Another restriction we can place on a Kähler manifold is for it to be homogeneous; this condition forces certain symmetries on the manifold.

Definition 7.3. *We call a Riemannian manifold homogeneous if its automorphism group $Aut(M)$ acts transitively on the points of M.*

This restriction has found use in geometric topology and many simple manifolds satisfy this restriction, like $\mathbb{R}^n$, S^n, $\mathbb{RP}^n$, and the special orthogonal group $SO(n)$. This restriction greatly simplifies the structure of Kähler manifolds and make the asphericality of the space easy to verify and the Euler characteristic easy to calculate.

Theorem 7.4. *The Hopf conjecture holds for homogeneous Kähler manifolds.*

Proof. The main tool we use here is the fundamental conjecture of homogeneous Kähler manifolds (which is not a conjecture anymore). From [**DN88**], we get that every homogeneous Kähler manifold is a fiber bundle

$$F \to M \to B$$

where $F = F_0 \times F_1$, F_0 a flat homogeneous Kähler manifold, F_1 a compact simply connected Kähler manifold, and B a homogeneous bounded domain.

In order to satisfy the hypothesis of the Hopf conjecture, B must have dimension zero. To see this, if B has nonzero dimension, it would be a bounded subset of $\mathbb{C}^1$ or $\mathbb{C}^2$ with boundary $\partial B \neq \emptyset$. Taking a point $b \in \partial B$ and looking at the local trivialization of the bundle at an open set U around b, we obtain a product $F \times U$ which has boundary $\partial(F \times U) = (\partial F) \times U \cup F \times (\partial U) \neq$ and so the bundle M is not a closed manifold. So we can assume that F has dimension 4 so that $M = F_0 \times F_1$.

Next, we break up by cases. Since F_0 and F_1 are Kähler they have even dimension and their dimension sums to 4. So we have three cases: $\dim F_0 = 0$, $\dim F_0 = 2$, and $\dim F_0 = 4$.

Case 1: F_0 has dimension zero:
This first case is trivial. Since F_0 has dimension 0, $F_0 = \{0\}$ and $M \cong F_1$. So M is a simply connected space. But this implies that all its homology groups are trivial and thus $\chi(M) = 0$, which finishes the case.

Case 2: F_0 has dimension two:
In this case we have that both F_0 and F_1 have dimension 2. However, F_0 and F_1 are determined completely by the properties given to them by the fundamental conjecture. Any flat Riemannian surface has diffeomorphism type of $\mathbb{R}^2$, $\mathbb{R}^1 \times S^1$, $S^1 \times S^1$, the Klein bottle, or the Möbius strip. F_1 is a simply connected compact Riemannian surface, which forces it to be S^2. Since M is orientable F_0 is neither the Klein bottle nor the Möbius strip and since $M = F_0 \times S^2$ is compact, F_0 cannot be the plane or the cylinder. So $M = \Sigma_1 \times S^1$. But this is not aspherical, so the hypotheses of the Hopf conjecture are not satisfied in this case.

Case 3: F_0 has dimension four:
In this case F_1 is a point so that $M \cong F_0$. Then M has a flat Riemannian metric. Since the metric is flat, M has zero sectional curvature. In particular, it has non-positive curvature. By a result of Chern [**Che55**], all 4 dimensional manifolds with non-positive curvature have $\chi(M) \geq 0$ (and in particular all such manifolds are aspherical). This finishes the case and the proof. □

7.3 Symplectic Kodaira dimension revisited

As mentioned in the previous section, for a given symplectic manifold with symplectic Kodaira dimension $\kappa^s(M) = -\infty$ or $\kappa^s(M) = 0$, M satisfies the Hopf conjecture. However, given the restraint of our manifold being Kähler, we can trivially prove that a Kähler manifold M satisfies the Hopf conjecture if $\kappa^2(M) = 2$.

Theorem 7.5. *Let M be a closed aspherical Kähler manifold. If $\kappa^s(M) = 2$, then $\chi(M) \geq 0$.*

Proof. We know from the definition of symplectic Kodaira dimension that $K_\omega \cdot K_\omega > 0$. However, this is equivalent to $c_1^2 = 2\chi(M) + 3\sigma(M) > 0$ for M. Now, according to [**Li15**], it is known that Kähler manifolds of general symplectic type satisfy the BMY inequality; $c_1^2 \leq 3c_2 = 3\chi(M)$. Thus, we have that $3\sigma(M) \leq \chi(M)$. In the case that $\chi(M) < 0$, we have that $3\sigma(M) < 0$ and $2\chi(M) + 3\sigma(M) = K_\omega \cdot K_\omega < 0$ which contradicts $\kappa^s(M) = 2$. So $\chi(M) \geq 0$. □

Out of all the subcases presented in this section, this is by far the most promising in terms of proving the Hopf conjecture for Kähler manifolds. One only has to prove it in the case that $\kappa^s(M) = 1$ (that is, when $2\chi(M) + 3\sigma(M) = 0$), and it is an open conjecture that in fact *every* symplectic manifold M with $\kappa^s(M) = 1$ satisfies $\chi(M) \geq 0$.

8 Conclusion

The Hopf conjecture, while simple to state, is difficult to actually attack. Looking at the two dimensional case, we see there is not any particular reason for it to work. We simply classify all aspherical manifolds and calculate their Euler characteristics. Further, as mentioned previously there is no such classification for 4-manifolds and so this method of attack is not possible. The positive result obtained in this paper in fact mirrors the 2 dimensional case; complex 4-manifolds do admit a useful classification which allowed us to tackle the problem case by case.

In the scope of this thesis, one particular desideratum is a proof of the conjecture for Kähler manifolds. Given their restrictions compared to symplectic or Riemannian manifolds, they should be a simpler class of spaces to tackle. As mentioned at the end of section 7, the Kähler case only needs to be solved for those Kähler manifolds with symplectic Kodaira dimension 1. However, the properties of spaces with positive Kodaira dimension (even those which are Kähler) are not well understood. However, another strategy is to look at the fundamental group of compact Kähler manifolds. These groups have several restrictions on them; for example, the rank of such a group must be even [**Amo96**]. Studying these groups restricted to aspherical Kähler manifolds along with tools like Winkelnkemper's inequality may yield positive results.

One other tool which was not covered in this thesis is the L^2 Betti numbers of Hermann Lück [**Lüc02**]. In a sentence, we can create the space of bounded functions on the set of L^2 bounded sum of formal sums over $\pi_1(M)$ with coefficients in $\mathbb{C}$ (that is, elements of the form $\sum \lambda_g \cdot g, g \in \pi_1(M)$ and $\sum |\lambda_g|^2 < \infty$). Taking the tensor product of this set with the groups in the cellular chain complex of $\widetilde{M}$ and then creating a homology out of this complex gives us the L^2 Betti numbers $b_i^{(2)}(\widetilde{M})$. While being fairly technical, this tool gives us new invariants which describe the universal covering map as well as satisfying properties like homotopy invariance, Künneth's formula and, most importantly, the formula

$$\chi^{(2)}(\widetilde{M}) = \sum_{i \geq 0} (-1)^i b_i^{(2)}(\widetilde{M}) = \chi(M).$$

This reduces the problem from studying aspherical manifolds to studying the L^2 Betti numbers of contractible manifolds. Moreover, this expands the study of the Hopf conjecture from smoothable manifolds, which we restricted ourselves to in our look at the problem, to general closed and orientable 4-manifolds.

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