TWO APPLICATIONS OF DERIVED CATEGORIES TO THE STUDY OF SINGULARITIES

by

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Abstract

This dissertation concerns itself with two distinct projects applying homotopical and derived category techniques to the study of singularities in equal characteristic. The first concerns maps of commutative noetherian local rings containing a field of positive characteristic. Given such a map φ of finite flat dimension, the results relate homological properties of the relative Frobenius of φ to those of the fibers of φ. The focus is on the complete intersection property and the Gorenstein property.

The second concerns derived splinter characterizations of singularities in characteristic zero. Let X be a normal, excellent, noetherian scheme over $\operatorname{Spec} \mathbb{Q}$ with a dualizing complex. In this note, we find an alternate characterization of the multiplier ideal of X, as defined by de Fernex-Hacon, by considering maps $\pi_* \omega_Y \to \mathcal{O}_X$ where $\pi \colon Y \to X$ ranges over all regular alterations. As a corollary to this result, we give a derived splinter characterization of klt singularities, akin to the characterization of rational singularities given by Kovács and Bhatt. We also give an analogous description of the test ideal in characteristic $p > 2$ as a corollary to a result of Epstein-Schwede.

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Chapter 1

Introduction

While smooth varieties have many nice properties, we are often forced to deal with varieties that are singular. For example, the minimal model program, focused on classifying algebraic varieties, often identifies a singular variety as the "simplest" representative of a birational equivalence class. In moduli theory, it is generally necessary to consider singular objects when compactifying the moduli space of smooth objects. For example, the compactification of the moduli space of smooth stable curves requires we include nodal curves. In commutative algebra, singular rings arise naturally when studying group actions on a polynomial ring or rings arising from combinatorially from objects such as graphs or matroids.

The methods used to study singularities are often characteristic dependent. In 1964, Hironaka [Hir64] proved that given a variety X over an algebraically closed field of characteristic zero, there exists a *resolution of singularities*, that is a birational map $\pi: Y \to X$ where Y is smooth. Many classes of singularities in characteristic zero are defined using the relationship between the variety X and its resolutions of singularities, including terminal, canonical, Kawamata log terminal (klt), log canonical, rational, and du Bois singularities. In positive characteristic, resolutions of singularities aren't known to exist in general and we are forced to use different tools. A 1969 theorem of Kunz characterizing smooth varieties as those whose Frobenius endomorphism is flat, so we can instead turn to the Frobenius to classify singularities.

This dissertation concerns itself with two distinct projects applying homotopical and derived category techniques to the study of singularities in equal characteristic. In Chapter 3, we discuss homological properties of the relative Frobenius in positive characteristic. Given a map $\varphi: R \to S$ of commutative noetherian rings of characteristic $p > 0$, the relative Frobenius is the natural map $F_{S/R}: S \otimes_R F_* R \to F_* S$ factoring the Frobenius on S. In this setting, Radu [Rad92] and André [And93] give a relative version of Kunz's theorem, which says φ is regular if and only if the relative Frobenius is flat. This was soon modified by Dumitrescu [Dum96] who showed that if we assumed φ is flat, then φ is regular if and only if the relative Frobenius has finite flat dimension.

Our work comes from weakening the assumption that φ is flat to simply requiring that φ has finite flat dimension. In this setting, we generalize Dumitrescu's result that φ is regular if and only if the relative Frobenius is finite flat dimension, and prove a similar result for complete intersection dimension, generalizing a result of Blanco-Majadas [BM98]. We also discuss a similar result for Gorenstein dimension.

These results rely on studying properties of the Frobenius on the (derived) fiber of the map $R \to S$, which requires an understanding of simplicial algebras. Chapter 2 is devoted to this background material, with some additional background on model categories for the sake of completeness.

In Chapter 4, we study (derived) splinter characterizations of singularitities in characteristic zero. Thanks to the Direct Summand Theorem [And18], we can consider (derived) splinters as a class of singularities, and work of Kovács [Kov00] and Bhatt [Bha12] shows that in characteristic zero being a derived splinter is equivalent to having rational singularities. If we consider klt singularities, which are rational, we expect that adding a condition to the derived splinter property will allow us to characterize klt singularities. If X is a normal excellent scheme over a field of characteristic zero with a dualizing complex and $\pi: Y \to X$ is a sufficiently large regular alteration, we show that the relevant condition is requiring that the splitting $R\pi_*\mathcal{O}_Y \to \mathcal{O}_X$ must locally factor through $\pi_8 * \omega_Y$. This follows from a new characterization of the multiplier ideal in terms of maps $\pi_*\omega_Y \to \mathcal{O}_X$.

Chapter 2

Simplicial methods in commutative algebra

The purpose of this section is to introduce simplicial rings for commutative algebraists. Roughly speaking, a simplicial ring can be thought of as a collection of rings $\{R_n\}_{n\geq 0}$ with maps between adjacent degrees satisfying certain simplicial identities. This provides a way of discussing homotopy theory in the setting of rings.

Via the Dold-Kan correspondence, simplicial rings may be related to differential graded rings, which may be more familiar to commutative algebraists. However, there are often real reasons to prefer the simplicial to the differential graded setting. In particular, the Frobenius has a natural definition in the simplicial setting - it's simply the Frobenius applied degreewise. In contrast, the p^{th} power map is not a map of differential graded rings, as it is not degree preserving.

To begin, we will introduce the framework of model categories, introduced by Quillen in [Qui67]. We will then use the example of chain complexes over a commutative ring to illustrate these concepts before moving on to simplicial rings. We end with a discussion of the Dold-Kan correspondence to move between the simplicial and differential graded settings.

2.1 Model categories

Oftentimes, we are working in a category where there are certain maps that we wish were isomorphisms. For example, commutative algebraists often want to look at chain complexes of modules over a commutatitive ring up to quasi-isomorphism and topologists want to consider (pointed) topoloical spaces up to homotopy equivalence. To rememdy this, one can formally invert this class of maps, though this risks a poorly-behaved quotient category. Specifically, the class of maps between two objects may no longer be a set.

Model categories, introduced by Quillen in [Qui67], offer one solution to the localization problem. The localization of a model category is particularly well-behaved, coming with a clear description of maps between objects and other additional structures that allow us to use tools from homotopy theory. One particularly important idea is that of lifting:

Definition 2.1.1. Suppose $i : A \to B$ and $p : X \to Y$ are maps in a category $\mathcal{C}$. We say that i *has the left lifting property with respect to* p and p *has the right lifting property with respect to* i if, for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

there is a lift $h : B$ such that $hi = f$ and $ph = g$.

We now introduce the definition of a model category, following [Hov99] who provides an updated perspective on the original axioms of Quillen:.

Definition 2.1.2 ([Hov99] Definition 1.1.3)**.** A *model structure* on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, cofibrations, and fibrations satisfying the following properties:

1. (2-out-of-3) If f and g are two composable morphisms of $\mathcal{C}$ and two of f, g, and gf are weak equivalences, then so is the third.

2. (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

3. (Lifting) Define a map to be a trivial (co)fibration if it is both a (co)fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

4. (Factorization) For any morphism f, f can be factored as a cofibration followed by a trivial fibration and a trivial cofibration followed by a fibration. Moreover, these factorizations can be chosen to be functorial.

A *model category* is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.

Definition 2.1.3. Let $\mathcal{C}$ be a model category. By considering the limit and colimit of the empty diagram, we will have an initial object and a terminal object. Let X and Y be objects of $\mathcal{C}$. We say that X is *cofibrant* if the map to X from the initial object is a cofibration and we say that Y is *fibrant* if the map from Y to the terminal object is a fibration. We say that $\mathcal{C}$ is a *pointed* model category if the map from the initial object to the terminal object is an isomorphism.

Note that by taking the functorial factorization of the map from the initial object to X, we get a cofibrant object QX such that the map $QX \to X$ is a trivial fibration. We call Q the cofibrant replacement functor of $\mathcal{C}$. Dually, by taking the functorial factorization of the map from X to the terminal object, we get a fibrant object RX such that the map $X \to RX$ is a trivial cofibration. We call R the fibrant replacement functor.

Remark 2.1.4 ([Hov99] Lemma 1.1.10)**.** If $\mathcal{C}$ is a model category, then a map is a cofibration (resp. trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (resp. fibrations). Dually, a map is a fibration (resp. trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (resp. cofibrations).

Given a model category $\mathcal{C}$, we can form the homotopy category $\mathrm{Ho}\,\mathcal{C}$ by inverting the class of weak equivalences. This gives us a functor $\gamma\colon \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ that is universal in the sense that any other functor $F : \mathcal{C} \to \mathcal{D}$ sending weak equivalences to isomorphisms must factor uniquely as $\mathrm{Ho}(F) \circ \gamma$ where $\mathrm{Ho}(F)\colon \mathrm{Ho}\,\mathcal{C} \to \mathcal{D}$.

With some work, we can show that this is equivalent to $\mathrm{Ho}\,\mathcal{C}_c$, $\mathrm{Ho}\,\mathcal{C}_f$ and $\mathrm{Ho}\,\mathcal{C}_{cf}$ [Hov99, Proposition 1.2.3]. Here, $\mathcal{C}_c$ is the full subcategory of cofibrant objects, $\mathcal{C}_f$ is the full subcategory of fibrant objects, and $\mathcal{C}_{cf}$ is the full subcategory of objects that are both fibrant and cofibrant. Additionally, $\mathrm{Ho}\,\mathcal{C}$ is equivalent to $\pi(\mathcal{C}_{cf})$ which has the same objects as $\mathcal{C}_{cf}$ with $\mathrm{Hom}_{\pi(\mathcal{C}_{Cf})}(X,Y) = \mathrm{Hom}_{\mathcal{C}_{cf}}(X,Y)/\sim$ where $f \sim g$ if they are right (equivalently left) homotopic. We have opted not to dive into the notion of homotopy here and refer the reader to [Qui67, §I.1], [GJ99, Ch. 2], or [Hov99, §1.2] for details.

Remark 2.1.5 ([Hov99] Theorem 1.2.10)**.** Let $\mathcal{C}$ be a model category with objects X and Y. Then there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(QRX, QRY)/\sim\,\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(\gamma X, \gamma Y) \cong \mathrm{Hom}_{\mathcal{C}}(RQX, RQY)/\sim$$

as well as a natural isomorphism $\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(\gamma X, \gamma Y) \cong \mathrm{Hom}_{\mathcal{C}}(QX, RY)/\sim$.

2.1.1 An example: Chain complexes over a commutative ring

Let R be a commutative noetherian ring. In this section, we'll consider a variety of model structures on a variety of categories of chain complexes of R-modules. We'll use the notation $\mathcal{C}(R)$ for chain complexes of R-modules and write $\mathcal{C}_{\geq 0}(R)$ (resp. $\mathcal{C}_{\leq 0}(R)$) for the category of chain complexes of R-modules concentrated in non-negative (resp. non-positive) homological degree.

The projective model structure on $\mathcal{C}_{\geq 0}(R)$

We can define a model structure on $\mathcal{C}_{\geq 0}(R)$ as follows:

- A map $X \to Y$ is a weak equivalence if it is a quasi-isomorphism, that is, an isomorphism of homology groups
- A map $X \to Y$ is a cofibration if it is injective with a degreewise projective cokernel
- A map $X \to Y$ is a fibration if it has the right lifting property with respect to the class of trivial cofibrations

With this setup, it is straightforward to show that fibrations are degreewise surjective in positive homological degree. To see this, consider the following diagram where i is a trivial cofibration and p is degreewise surjective in positive homological degree.

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

Let P be the cokernel of i, a complex of projectives with no homology. By design, we can write $B \cong A \oplus P$ and so to construct our lift $B \to X$ we simply need to construct a lift $P \to X$. We get a map $P_1 \to X_1$ using the fact that $X_1 \to Y_1$ is surjective, and from here we can inductively construct the lift in all higher degrees. The only question is what the map $P \to X$ looks like in degree zero. This is where we use that P has no homology. This implies that P_1 surjects onto P_0 and thus that P_0 is a summand of P_1. We then have a map $P_0 \to X_1 \to X_0$ which completes our construction of the map $P \to X$ and thus $B \to X$.

Remark 2.1.6. In this model structure, every object is fibrant and the cofibrant objects are complexes of projective modules.

The injective model structure on $\mathcal{C}_{\leq 0}(R)$

We can define a model structure on $\mathcal{C}_{\geq 0}(R)$ as follows:

- A map $X \to Y$ is a weak equivalence if it is a quasi-isomorphism, that is, an isomorphism of homology groups
- A map $X \to Y$ is a fibration if it is surjective with a degreewise injective kernel
- A map $X \to Y$ is a cofibration if it has the left lifting property with respect to the class of trivial fibrations

With this setup, it is straightforward to show that cofibrations are degreewise injective in negative homological degree. To see this, consider the following diagram where i is degreewise injective in positive homological degree and p is a trivial fibration.

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

Let I be the kernel of p, a complex of injectives with no homology. by design $X \cong I \oplus Y$. We already have a map $B \to Y$ so we just need to define a map $B \to I$. Using the map $A \to I$ and the fact the $A \to B$ is injective in negative homological degrees, we get a map $B_i \to I_i$ for $i \leq -1$. The only question is what the map $B \to I$ looks like in degree zero. This is where we use that I has no homology. This implies that I_0 injects into I_{-1} and thus that I_0 is a summand of I_{-1}. We then have a map $B_0 \to B_{-1} \to I_{-1} \to I_0$ which completes our construction of the map $B \to I$ and thus $B \to X$.

Remark 2.1.7. In this model structure, every object is cofibrant and the fibrant objects are complexes of injective modules.

Model structures on $\mathcal{C}(R)$

The above model structures will be relevant when we discuss (co)simplicial algebras in a few pages. However, one issue with this perspective in general is that by enforcing such a rigid cutoff in degree zero, we restrict our ability to shift our complexes. In other words, their homotopy categories are not stable and thus not triangulated. If we want to use tools for triangulated categories, such as thick subcategories, we need to consider unbounded chain complexes. In this setting, small modifications to the model structures above will give us both a projective and an injective model structure on $\mathcal{C}(R)$.

Proposition 2.1.8 ([Hov99] Theorem 2.3.11)**.** *Let R be a ring. Then there is a cofibrantly generated, proper, stable model structure on* $\mathrm{Ch}(R)$ *where a morphism* $f: X \to Y$ *is a:*

- *weak equivalence if it is a quasi-isomorphism;*
- *fibration if it is a degreewise epimorphism;*
- *cofibration if it is a degreewise split injection with cofibrant cokernel.*

We will call this the projective model structure. All objects are fibrant in this model structure. The cofibrant objects are those complexes which can be written as an increasing union of complexes such that the associated quotients are complexes of projectives with zero differential.

Proposition 2.1.9 ([Hov99] Theorem 2.3.11)**.** *Let R be a ring. Then there is a cofibrantly generated, proper, stable model structure on* $\mathrm{Ch}(R)$ *where a morphism* $f: X \to Y$ *is a:*

- *weak equivalence if it is a quasi-isomorphism;*
- *fibration if it is a degreewise split epimorphism with injectively fibrant kernel;*
- *cofibration if it is a degreewise monomorphism.*

We will call this the injective model structure. All objects are cofibrant in this model structure.

Remark 2.1.10 ([Bal21] Lemma 7.2.9)**.** The identity functor on $\mathrm{Ch}(R)$ gives an equivalence between the homotopy categories of $\mathrm{Ch}(R)$ under each model structure.

2.1.2 Derived functors

The framework of a model category also offers a good setup for computing derived functors, such as $- \otimes_R^{\mathbb{L}} M$ and $\mathrm{RHom}(M, -)$, and $\mathrm{RHom}(-M)$. We give a streamlined discussion of this here, but for a more complete reference see [Hov99].

Definition 2.1.11. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. We say that $F: \mathcal{C} \to \mathcal{D}$ is a *left Quillen functor* if F is a left adjoint and preserves cofibrations and trivial cofibrations.
2. We say that $U: \mathcal{D} \to \mathcal{D}$ is a *right Quillen functor* if U is a right adjoint and preserves fibrations and trivial fibrations.

3. Suppose (F, U, φ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$. We say that (F, U, φ) is a *Quillen adjunction* if F is a left Quillen functor (equiv. if U is a right Quillen functor [Hov99, Lemma 1.3.4]).

Definition 2.1.12. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. If $F: \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, the *total left derived functor* $LF: \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ is the composite
$$\mathrm{Ho}\,\mathcal{C} \xrightarrow{\mathrm{Ho}\,Q} \mathrm{Ho}\,\mathcal{C}_c \xrightarrow{\mathrm{Ho}\,F} \mathcal{D}$$
2. If $U: \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, the *total right derived functor* $RF: \mathrm{Ho}\,\mathcal{D} \to \mathrm{Ho}\,\mathcal{C}$ is the composite
$$\mathrm{Ho}\,\mathcal{D} \xrightarrow{\mathrm{Ho}\,R} \mathrm{Ho}\,\mathcal{C}_f \xrightarrow{\mathrm{Ho}\,U} \mathcal{C}$$

Example 2.1.13. Let R be a commutative ring and consider $\mathrm{Ch}_{\geq 0}(R)$ with the projective model structure. Let M be a cofibrant object of $\mathrm{Ch}_{\geq 0}(R)$. Then we claim that $-\otimes_R M$ is left Quillen. Let $A \to B$ be a cofibration so that we have a short exact sequence of complexes
$$0 \to A \to B \to P \to 0$$
where P is degreewise projective. Applying $-\otimes_R M$ we see that $A \otimes_R M \to B \otimes_R M$ remains injective with cokernel $P \otimes_R M$ degreewise projective, as M is degreewise projective. Thus, we get that the total left derived functor of $-\otimes_R M$ is the functor $-\otimes_R^{\mathbb{L}} M$ that sends
$$X \mapsto QX \otimes_R M.$$
Note that because M is cofibrant, the natural map $QX \otimes_R M \to X \otimes_R M$ is a quasi-isomorphism, so in practice we only need to take the cofibrant replacement of one of the arguments in our tensor product to compute the derived tensor product, and so given any two objects M and N of $\mathrm{Ch}_{\geq 0}(R)$ we will write $M \otimes_R^{\mathbb{L}}$ for the derived tensor product.

Given a Quillen adjunction, this always descends to an adjunction on the corresponding homotopy categories.

Lemma 2.1.14 ([Hov99] Lemma 1.3.10)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and (F, U, φ) be a Quillen adjunction. Then LF and RU are part of an adjunction $L(F, U, \varphi) = (LF, RU, R\gamma)$ that we call the* derived adjunction*.*

Sometimes, this adjunction on homotopy categories is an equivalence (even if the original adjunction is not an equivalence) leading us to the following definition.

Definition 2.1.15. A Quillen adjunction $(F, U, \varphi): \mathcal{C} \to \mathcal{D}$ is called a *Quillen equivalence* if and only if, for every cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f: FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $\varphi(f): X \to UY$ is a weak equivalence in $\mathcal{C}$.

Proposition 2.1.16 ([Hov99] Prop. 1.3.13 and Cor. 1.3.14)**.** *Suppose $(F, U, \varphi): \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent.*

1. *(F, U, φ) is a Quillen equivalence*
2. *$L(F, U, \varphi)$ is an adjoint equivalence of categories*

2.2 Simplicial algebras

We are now ready to introduce our main construction of interest - simplicial algebras. As discussed above, the reason for this is that simplicial algebras come equipped with a natural Frobenius map in a way DG algebras do not. Simplicial algebras are also used to compute Andé-Quillen homology, though this application is outside the scope of this dissertation.

To begin, we introduce the simplex category Δ, largely following [Rie11]. This is the category of finite, non-empty, totally ordered sets with order-preserving maps. We can then immediately make the following definition:

Definition 2.2.1. A *simplicial set* is a functor from $\Delta^{op} \to \text{Set}$. More generally, given a category $\mathcal{C}$, a *simplicial object* of $\mathcal{C}$ is a functor from $\Delta^{op} \to \mathcal{C}$. We use the notation $s\mathcal{C}$ to denote the category of simplicial objects in $\mathcal{C}$, that is, the functor category $\text{Fun}(\Delta^{op}, \mathcal{C})$.

The simplicity of this definition belies the combinatorial implications the come from working with the category Δ. We will denote by $[n]$ the set $\{0, \ldots, n\}$. Any map in Δ can be factored into co-face maps $d^i : [n-1] \to [n]$ and co-degeneracy $s^i : [n+1] \to [n]$ defined as follows

$$d^i(k) = \begin{cases} k & , k < i \\ k+1 & , k \geq i \end{cases} \qquad s^i(k) = \begin{cases} k & , k \leq i \\ k-1 & , k > i \end{cases}$$

These maps satisfy the following identities

$$d^j d^i = d^i d^{j-1}, i < j$$

$$s^j s^i = s^i s^{j+1}, i \leq j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & i < j \\ 1 & i = j, j+1 \\ d^{i-1} s^j & i > j+1 \end{cases}$$

Thus we can think of a simplicial object in a category $\mathcal{C}$ both as a functor $X: \Delta^{op} \to \mathcal{C}$ and as collection of objects $\{X_n\}_{n\geq 0}$ with face maps $d_i: X_n \to X_{n-1}$ and degeneracy maps $s_i: X_n \to X_{n+1}$ satisfying identities dual to the ones above.

Remark 2.2.2. Two important things to note when working with simplicial objects in a category $\mathcal{C}$ are

1. $\mathcal{C}$ embeds in $s\mathcal{C}$ by taking an object X in $\mathcal{C}$ to the object in $s\mathcal{C}$ which has X in all degrees and the identity for all face and degeneracy maps

2. Given an object X in $s\mathcal{C}$, there is a unique map from $X_0 \to X_n$ for all n coming from the unique map $[n] \to [0]$

Given a commutative ring k, let $sk-Alg$ be the category of simplicial k-algebras. If we take $k = \mathbb{Z}$ we get the category of simplicial rings. We will want to talk about derived functors in this setting, so we want to put a model structure on $sk-Alg$.

The original model structure is due to Quillen [Qui67, §II.4 Theorem 4] and can be obtained by transferring along the free-forgetful adjunction between $sSet$ and $sk-Alg$. Without getting into the details, we give a model structure on $sk-Alg$ where a morphism $f: X \to Y$ is a

- weak equivalence if the underlying morphism of simplicial sets is a weak equivalence in $sSet$;

- fibration if the underlying morphism of simplicial sets is a fibration;

- cofibration if it has the left-lifting property with respect to the class of trivial fibrations.

Quillen also showed that the cofibrations are well-understood.

Proposition 2.2.3 ([Qui67] p. 4.11)**.** *A morphism in $sk-Alg$ is a cofibration if and only if it is a retract of a free map.*

Here, a free map means the cokernel is a free k-algebra in each degree, and so a cofibration will have cokernal that is a projective k-algebra in each degree.

Given a simplicial ring A, we will use the notation of $\mathrm{Mod}(A)$ for simplicial modules over A. The model structure on $\mathrm{Mod}(A)$, discussed in [Qui67, §II.6], is essentially the same, where a map is a weak equivalence (equiv. fibration) if the map of underlying simplicial sets is a weak equivalence (resp. fibration) and cofibrations have the left lifting property against trivial fibrations. Again, cofibrations are retracts of free maps. We will write $\mathrm{Ho}(A)$ for the corresponding homotopy category.

Given a simplicial abelian group A (such as a module over a simplicial ring), we have an abelian group structure on the set

$$\pi_n(A,0) = [(\Delta^n, \partial\Delta^n), (A,0)]$$

of homotopy classes of pairs of maps which satisfies an interchange law with respect to the canonical group structure for the homotopy group. It follows that the homotopy group structure and the induced abelian group structure coincide. In particular, there is a natural isomorphism

$$\pi_n(A,0) \cong H_n(NA).$$

We will often just write $\pi_n(A)$.

Remark 2.2.4. Given a simplicial ring A and an integer $n \geq 0$, there is a simplicial ring B and a map of simplicial rings $\varphi\colon A \to B$ with the following properties:

(1) $\pi_i(B) = 0$ for $i \geq n+1$;

(2) $\pi_i(\varphi)$ is bijective for $i \leq n$.

The map can be obtained by a process of killing the homology in A in degree $n+1$ and higher; see also the discussion on [Toë10, pp. 162] for the construction of φ. This is part of the data of a Postnikov tower for A.

2.3 The Dold-Kan correspondence

We end this section with a discussion of a powerful tool that will allow is to transition between the (co)simplicial and (co)chain/DGA settings. Known as the *Dold-Kan correspondence*, this classical theorem gives an equivalence of categories between simplicial objects in an abelian category $\mathcal{A}$ and chain complexes of objects of $\mathcal{A}$ concentrated in nonnegative homological degrees. Moreover, in many relevant cases, the equivalence is also a Quillen equivalence.

Let R be a classical commutative ring. The Dold-Kan correspondence then says that there is an equivalence of categories

$$\mathrm{Mod}(sR) \underset{N}{\overset{\Gamma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathrm{Ch}_{\geq 0}(R)$$

that preserves the natural (projective) model structures. Moreover, the homotopy of an object M of $\mathrm{Mod}(sR)$ is isomorphic to the homology of $N(M)$.

We first review the construction of these functors in this setting. For a more complete reference see [GJ99, §III.2]. Let M be an object of $\mathrm{Mod}(sR)$. Then we define $N(M)$, the normalized chain complex of M, to be the chain complex with $N(M)_k := \bigcap_{i=1}^{k} \ker d_i$ and differential given by the restriction of d_0.

In the other direction, let C be an object of $\mathrm{Ch}_{\geq 0}(R)$. Then define $\Gamma(V)$ to be the simplicial R-module with

$$\Gamma(C)_n := \bigoplus_{[n]\twoheadrightarrow[k]} C_k.$$

The degeneracy maps from $\Gamma(C)_n \to \Gamma(C)_{n+1}$ send C_k indexed by the surjection $[n] \twoheadrightarrow [k]$ so the surjection $[n+1] \to [n] \twoheadrightarrow [k]$ where the map $[n+1] \to [n]$ is the corresponding co-degeneracy map in Δ. To determine where the face maps from $\Gamma(C)_n \to \Gamma(C)_{n-1}$ send the summand C_k indexed by $[n] \twoheadrightarrow [k]$, we consider the composition $[n-1] \hookrightarrow [n] \twoheadrightarrow [k]$ with the corresponding co-face map. If this map is surjective, then we are done. Otherwise, you can check that it will miss exactly one element of [k], which means it can be factored uniquely into a surjection from $[n-1] \to [k-1]$ and a coface map $[k-1] \to [k]$. Then the map $C_k \to C_{k-1}$ is given by $(-1)^n\partial$ if the coface map is d^0 and is zero otherwise.

We will often need to upgrade to the setting to a simplicial ring A, in which case NA is a differential graded ring concentrated in nonnegative homological degree. Luckily, we have the following result of Schwede and Shipley [SS03, Theorem 1.1] upgrades the Dold-Kan correspondence to this setting. Here, *connective* means concentrated in non-negative homological degrees.

Theorem 2.3.1 (Theorem 1.1 [SS03])**.** *1. Given a connective differential graded ring R, there is a Quillen equivalence between the categories of connective differential graded R-modules and simplicial modules over the simplicial ring ΓR*

$$\mathrm{Mod}(R) \simeq_Q \mathrm{Mod}(\Gamma R)$$

where Γ is the right adjoint of the Quillen equivalence

2. Given a simplicial ring A, there is a Quillen equivalence between the categories of connective differential-grade NA-modules and simplicial modules over A

$$\mathrm{Mod}(NA) \simeq_Q \mathrm{Mod}(A)$$

where normalization is the right adjoint of the Quillen equivalence.

3. *Given a commutative ring k, there is a Quillen equivalence between the categories of connective differential graded k-algebras and simplicial k-algebras*

$$DGA_k \simeq_Q sk - Alg$$

where normalization is the right adjoint of the Quillen equivalence.

4. *Given a simplicial commutative ring A, there is a Quillen equivalence between the categories of connective differential graded NA-algebras and simplicial A-algebras*

$$NA - Alg \simeq_Q A - Alg$$

where normalization is the right adjoint of the Quillen equivalence.

Note that the left adjoints are not the usual Dold-Kan functors and are discussed in [SS03, §3.3].

One unfortunate consequence of working in this setting is that because the normalization functor is only a right adjoint, it doesn't necessarily preserve cofibrations. This means if we want to compute say, the homology of a derived tensor product in $\mathrm{Mod}(A)$, we can't just compute the derived tensor product and normalize it. That is to say, if we have X and Y in $\mathrm{Mod}(A)$, it's not necessarily the case that the following map is weak equivalence:

$$NQ(X) \otimes_{NA} NQ(Y) \to N(QX \otimes_A QY) \cong N(X \otimes^{\mathbb{L}}_A Y).$$

Luckily, the following result of Avramov will allow us to compute the homology of simplicial modules over a simplicial ring by instead working over the corresponding DG ring.

Proposition 2.3.2 (Prop. 2.2 [Avr99])**.** *Let Y be a simplicial left module over a simplicial ring A, X a cofibrant simplicial right module over A, and let X' be a right differential graded module over the differential graded ring NA such that $(X')^\natural$ is free over $NA^\natural$, where $(-)^\natural$ is the functor that forgets differentials. If $\mu: X' \to NX$ is a weak equivalence, then the composition*

$$X' \otimes_{NA} NY \xrightarrow{\mu} NX \otimes_{NA} NY \longrightarrow N(X \otimes_A Y)$$

is a weak equivalence.

Chapter 3

Homological properties of the relative Frobenius

Given a ring R of characteristic $p > 0$, the Frobenius endomorphism is the map $F\colon R \to R$ sending r to r^p. Given an R-module M, the R-module structure on M via restriction of scalars along the e-fold Frobenius is denoted by $F_*^e M$. Studying properties of this module structure allows us to understand the singularities of R, thanks to a theorem of Kunz [Kun69] which says that R is regular if and only if $F^e\colon R \to R$ is flat for some (equivalently all) $e > 0$. Work of Rodicio [Rod88] generalizes Kunz's result to the case where the Frobenius has finite flat dimension, and work of Blanco-Majadas [BM98], Takahasi-Yoshino [TY04], and Iyengar-Sather-Wagstaff [ISW04] does the same for other homological dimensions.

In this paper, we work instead in the relative setting. Given a map of commutative rings of positive characteristic $\varphi\colon R \to S$, we consider the following diagram:

$$\begin{array}{ccccc} R & \xrightarrow{\ \varphi\ } & S & & \\ \downarrow{\scriptstyle F} & & \downarrow & \searrow{\scriptstyle F} & \\ F_*R & \longrightarrow & S \otimes_R F_*R & \xrightarrow[F_{S/R}]{} & F_*S \end{array}$$

where $F_{S/R}$ is the *relative Frobenius* of φ induced by the universal property of the tensor product. It sends $s \otimes r$ to $s^p r$. Recall that a map $\varphi\colon R \to S$ is regular if it flat and has geometrically regular fibers. We have a relative version of Kunz's theorem due to Radu and André [Rad92, And93, And94] which says that φ is regular if and only if $F_{S/R}$ is flat for some (equivalently, all) $e \geq 1$. This implies Kunz's theorem by considering the following diagram:

$$\begin{array}{ccccc} \mathbb{F}_p & \xrightarrow{\ \varphi\ } & R & & \\ \downarrow{\scriptstyle F = 1_{\mathbb{F}_p}} & & \| & \searrow{\scriptstyle F} & \\ \mathbb{F}_p & \xrightarrow{\ \varphi\ } & R & \longrightarrow & F_*R \end{array}$$

Taking this result as inspiration, we investigate how homological properties of the relative Frobenius are reflected in homological properties of the Frobenius on the (derived) fibers of $\varphi\colon R \to S$. In this direction, work of Dumitrescu [Dum96] shows that the original result is equivalent to φ being flat and $F_{S/R}$ having finite flat dimension, and recent work of Alvite-Barral-Majadas [ABM22] extends this to the non-Noetherian setting.

Assuming R is a local ring with residue field k, the derived fiber of φ refers to $S \otimes^{\mathbb{L}}_R k$, thought of as a simplicial k-algebra. When φ is flat, this is simply $S \otimes_R k$. The Frobenius extends naturally to the setting of simplicial rings by applying the classical Frobenius degreewise, and so we can consider $F\colon S \otimes^{\mathbb{L}}_R k \to S \otimes^{\mathbb{L}}_R k$. This leads us to our first result.

Theorem 3.0.1. *Let R and S be F-finite local rings of positive characteristic. If $\varphi\colon R \to S$ is a local homomorphism of finite flat dimension, the Betti numbers of the relative Frobenius $F_{S/R}$ grow at the same rate as the Betti numbers of the Frobenius on $S \otimes^L_R k$.*

A precise formulation of this result can be found in Theorem 3.2.1. As a corollary, we recover the results of Radu [Rad92], André [And93, And94] and Dumitrescu [Dum96] (concerning regularity) and Blanco-Majadas [BM98] (concering the complete intersection property) while also weakening the hypothesis that φ is flat to requiring that φ has finite flat dimension.

Corollary 3.0.2. *For φ as in Theorem 3.2.1, the fibers of φ are regular (resp. complete intersection) if and only if $F_{S/R}$ has finite flat (resp. CI-) dimension.*

We also prove a similar result for the Gorenstein property.

Theorem 3.0.3. *Let R and S be F-finite local rings of positive characteristic. If $\varphi\colon R \to S$ is a flat local homomorphism, then the fibers of φ are Gorenstein if and only if $F_{S/R}$ has finite G-dimension.*

Note that we require φ to be flat here rather than finite flat dimension as in the previous results. The reason for this is that for φ of finite flat dimension, we must consider the *derived* fibers of φ. We plan to take this up on a later occasion.

3.1 Simplicial rings

Going forward, our rings will be commutative and noetherian. Given a local ring R, we will write $\mathfrak{m}_R$ for its maximal ideal and k_R for its residue field. Given a map of local rings $\varphi\colon R \to S$ of positive characteristic, the proof of the Radu-André theorem relies, in part, on investigating the properties of the Frobenius on the fibers of φ.

In general, we do not assume φ is flat, so we instead want to consider its derived fiber, $S \otimes^{\mathbb{L}}_R k_R$. In order to talk about the Frobenius on $S \otimes^{\mathbb{L}}_R k_R$, we think of $S \otimes^{\mathbb{L}}_R k_R$ as a simplicial k_R-algebra, in which case the Frobenius is simply the usual Frobenius applied degreewise. Compare this to the differential graded setting, where the p^{th} power map is not a map of differential graded algebras because it is not degree-preserving.

We discuss the generalization of familiar tools for modules over a classical ring to modules over a simplicial ring. Given a simplicial ring A, we write $\mathrm{Mod}(A)$ for the category of simplicial A-modules with the projective model structure and $\mathrm{Ho}(A)$ for the corresponding homotopy category.

3.1.1 The Koszul complex

One construction we will often use is the Koszul complex. To start, consider the ordinary ring $\mathbb{Z}[x]$. The *Koszul complex on* $\mathbb{Z}[x]$ *with respect to* x is the complex

$$0 \longrightarrow \mathbb{Z}[x] \xrightarrow{x} \mathbb{Z}[x] \longrightarrow 0$$

which we denote $K[\mathbb{Z}[x]; x]$. Passing along the Dold-Kan correspondence, we can consider this complex as a simplicial module over the discrete simplicial ring $\mathbb{Z}[x]$, which we also call $K[\mathbb{Z}[x]; x]$. Note that $N(K[\mathbb{Z}[x]; x]) = K[\mathbb{Z}[x]; x]$.

Given a simplicial ring A with degree zero ring A_0 a local ring with maximal ideal $\mathfrak{m}$ and residue field k, consider $f \in \mathfrak{m}$. Consider $\mathbb{Z}[x] \to A_0$ sending $x \mapsto f$. This extends to $\mathbb{Z}[x] \to A$ and define the Koszul complex of A with respect to f to be

$$K[A; f] := A \otimes_{\mathbb{Z}[x]} K[\mathbb{Z}; x]$$

Given a sequence $f_1, \ldots, f_k \in \mathfrak{m}$, we can define the Koszul complex of A with respect to $f_1, \ldots, f_k$ by either iterating the above construction for maps $\mathbb{Z}[x_{i+1}] \to K[A; f_1, \ldots, f_i]$ sending $x_{i+1} \mapsto f_{i+1}$ or by considering $\mathbb{Z}[x_1, \ldots, x_k] \to A$ sending $x_i \mapsto f_i$ and taking the appropriate tensor product as above.

Given an A-module M, the *Koszul complex on* M *with respect to* $f_1, \ldots, f_k$ is

$$K[M; f_1, \ldots, f_k] := M \otimes_A K[A; f_1, \ldots, f_k].$$

We write K^A for the Koszul complex on a minimal generating set for the ideal $\mathfrak{m}$ and K^M for $M \otimes_A K^A$.

Remark 3.1.1. Let A be as above with $f \in \mathfrak{m}$. The Koszul complex on A with respect to f is directly inherited from the Koszul complex on A_0 with respect to f via the isomorphism

$$A \otimes_{\mathbb{Z}[x]} K[\mathbb{Z}[x];x] \cong A \otimes_{A_0} A_0 \otimes_{\mathbb{Z}[x]} K[\mathbb{Z}[x];x] \cong A \otimes_{A_0} K[A_0;f].$$

This also implies

$$K[M;f] \cong M \otimes_{A_0} K[A_0,f]$$

3.1.2 Betti numbers

Over ordinary commutative rings, Betti numbers are important homological invariants. In this section, we define them for modules over simplicial rings and prove several lemmas about how they behave along maps of rings.

Definition 3.1.2. Let A be a commutative simplicial ring. We say that A is *local* if A_0 is a noetherian local ring. If k is the residue field of A_0, we write (A,k). Given $M \in \mathrm{Mod}(A)$, we say that M is a *finite* A-module if $\pi_i(M)$ is a finite $\pi_0(A)$-module for each i and $\pi_i(M) = 0$ for $i \gg 0$. We write $\mathrm{mod}(A)$ for the category of finite A-modules.

Definition 3.1.3. Let (A,k) be a local simplicial ring and $M \in \mathrm{mod}(A)$. Define the i^{th} *Betti number* of M as

$$\beta_i^A(M) := \mathrm{rank}_k\, \pi_i(M \otimes_A^{\mathbb{L}} k)$$

where $M \otimes_A^{\mathbb{L}} k$ is computed by applying $- \otimes_A k$ to a cofibrant replacement of M (equivalently applying $M \otimes_A -$ to a cofibrant replacement of k).

The formal power series

$$P_M^A(t) := \sum_{n=0}^{\infty} \beta_n^A(M) t^n$$

is the *Poincaré series* of M over A.

The homological properties we are interested in are captured in the asymptotic properties of Betti numbers. Specifically, we are interested in the notion of curvature, first introduced for modules over an ordinary commutative ring by Avramov in [Avr96].

Definition 3.1.4. Let (A,k) be a local simplicial ring and $M \in \mathrm{mod}(A)$. The *curvature* of M is the number

$$\mathrm{curv}_A M = \limsup_n \sqrt[n]{\beta_n^A(M)}$$

This is the inverse of the radius of convergence of the Poincaré series of M over A.

While we prefer to work in the simplicial setting in order to use the Frobenius, we will occasionally use the following lemma to pass along the Dold-Kan correspondence to the dg algebra setting for computations involving Betti numbers.

Lemma 3.1.5. *Let (A, k) be a local simplicial ring and $M \in \operatorname{mod}(A)$. Then*

$$\beta_i^A(M) = \beta_i^{N(A)}(N(M)) = \operatorname{rank}_k \operatorname{Tor}_i^{N(A)}(N(M), k)$$

Proof. Let $F \to M$ be a cofibrant replacement and let $F' \to N(F)$ be a free resolution. Letting $(-)^\natural$ be the functor that forgets differentials, $(F')^\natural$ is a free $N(A)^\natural$-module and so, by [Avr99, Proposition 2.2], the following map is a quasi-isomorphism

$$F' \otimes_{N(A)} N(k) \to N(F) \otimes_{N(A)} N(k) \to N(F \otimes_A k)\,. \qquad \square$$

The benefit of this is that we can view $\operatorname{Mod}_{\geq 0}(N(A))$ inside of $\operatorname{Mod}(N(A))$ which has a stable homotopy category which we denote $\operatorname{Ho}(N(A))$. We again use the projective model structure, which restricts to the projective model structure on $\operatorname{Mod}_{\geq 0}(N(A))$.

Lemma 3.1.6. *Let (A, k) be a local simplicial ring and $M \in \operatorname{mod}(A)$. Suppose $N(M') \in \operatorname{thick}(N(M))$ (as a subcategory of $\operatorname{Ho}(N(A))$), then $\operatorname{curv}_A(M') \leq \operatorname{curv}_A(M)$.*

Proof. By Lemma 3.1.5, we can compute Betti numbers and hence curvature as $N(A)$-modules. Since $N(M) \otimes^{\mathbb{L}}_{N(A)} k$ is independent of whether we work in the category of unbounded or nonnegatively graded dg $N(A)$-modules, we can work in the category of unbounded modules. We now claim that the following is a thick subcategory of $\operatorname{Ho}(N(A))$:

$$\mathcal{C}_M := \left\{ X \in \operatorname{Ho}(N(A)) : \ \operatorname{curv}_{N(A)}(X) \leq \operatorname{curv}_{N(A)}(N(M)) \right\}.$$

It is clearly closed under suspensions and summands. Suppose $X \to Y \to Z \to$ is a triangle in $\operatorname{Ho}(N(A))$ with $X, Y \in \mathcal{C}_M$. Applying $- \otimes^{\mathbb{L}}_{N(A)} k$ to this triangle and considering the long exact sequence in homology we get

$$\operatorname{curv}_{N(A)}(Z) \leq \max\{\operatorname{curv}_{N(A)}(X), \operatorname{curv}_{N(A)}(Y)\} \leq \operatorname{curv}_{N(A)}(N(M)). \qquad \square$$

We work in the relative setting with $\varphi\colon A \to B$ a map of simplicial rings and $M \in \operatorname{mod}(B)$. While this could pose problems, as a finite module over the target ring may no longer be finite over the source, we will avoid this by only treating the case where φ is finite. A definition for relative Betti numbers and curvature in the setting of ordinary commutative rings without the assumption that φ is finite can be found in [AIM06] and can be generalized to the simplicial setting. However, we include the simplifying assumption that φ is finite to avoid clouding the essential parts of our argument with technical details.

These next two results discuss how curvature changes along certain maps. This first lemma is adapted, with similar proof, from [AHIY12, Proposition 3.3].

Lemma 3.1.7. *Let (A, k_A) and (B, k_B) be local simplicial rings and $M, N \in \operatorname{mod}(B)$. Let $A \to B$ be a finite map of local simplicial rings. Then*

$$\operatorname{curv}_A(M \otimes_B^{\mathbb{L}} N) \leq \max\{\operatorname{curv}_A(M), \operatorname{curv}_B(N)\}$$

Proof. By Theorem II.6.6 of [Qui67], we have a spectral sequence

$$E^2_{p,q} := \pi_p(N \otimes_B^{\mathbb{L}} \pi_q(M \otimes_A^{\mathbb{L}} k_A)) \implies \pi_{p+q}((M \otimes_B^{\mathbb{L}} N) \otimes_A^{\mathbb{L}} k_A)$$

Because $A \to B$ is finite, $\mathfrak{m}_B \pi_q(M \otimes_A^{\mathbb{L}} k_A) \otimes_B k_B = 0$ and so

$$\pi_p(N \otimes_B^{\mathbb{L}} \pi_q(N \otimes_A^{\mathbb{L}} k_A)) \cong \pi_p(N \otimes_B^{\mathbb{L}} k_B) \otimes_{k_A} \pi_q(M \otimes_A^{\mathbb{L}} k_A))$$

From this, we get the following coefficientwise inequality

$$P^A_{M \otimes_B^{\mathbb{L}} N}(t) \leq P^A_M(t) P^B_N(t)$$

and thus that

$$\operatorname{curv}_A(M \otimes_B^{\mathbb{L}} N) \leq \max\{\operatorname{curv}_A(M), \operatorname{curv}_B(N)\}. \qquad \square$$

This next lemma is reminiscent of [AIM06, Theorem 9.3.2] which discusses how curvature changes along complete intersection maps.

Lemma 3.1.8. *Let (A, k_A) and (B, k_B) be local simplicial rings and $M \in \operatorname{mod}(B)$. Let $A \to B$ be a finite map of local simplicial rings. Then*

$$\operatorname{curv}_A(M) \leq \max\{\operatorname{curv}_A(B), \operatorname{curv}_B(M)\}$$
$$\operatorname{curv}_B(M) \leq \max\{\operatorname{curv}_A(M), \operatorname{curv}_{\bar{B}}(k_B)\}$$

where $\bar{B} := B \otimes_A^{\mathbb{L}} k_A$.

Proof. The first inequality follows from Lemma 3.1.7 as

$$\operatorname{curv}_A(M) = \operatorname{curv}_A(B \otimes_B^{\mathbb{L}} M) \leq \max\{\operatorname{curv}_A(B), \operatorname{curv}_B(M)\}$$

We now work to prove the second inequality. We have

$$k_B \otimes^{\mathbb{L}}_{\bar{B}} (k_A \otimes^{\mathbb{L}}_A M) \cong k_B \otimes^{\mathbb{L}}_{\bar{B}} (k_A \otimes^{\mathbb{L}}_A B) \otimes^{\mathbb{L}}_B M \cong k_B \otimes^{\mathbb{L}}_B M$$

By [Qui67, Theorem II.6.6], we have a spectral sequence

$$E^2_{p,q} := \pi_p(k_B \otimes^{\mathbb{L}}_{\bar{B}} \pi_q(k_A \otimes^{\mathbb{L}}_A M)) \implies \pi_{p+q}(k_B \otimes^{\mathbb{L}}_B M)$$

From this we get the inequality

$$\beta^B_n(M) \le \sum_{i=0}^{n} \beta^{\bar{B}}_{n-i}(\pi_i(k_A \otimes^{\mathbb{L}}_A M)) \le \sum_{i=0}^{n} \beta^{\bar{B}}_{n-i}(k_B)\beta^A_i(M)$$

Note that the second inequality holds because $\pi_i(k_A \otimes^{\mathbb{L}}_A M)$ is a finite k_B-vector space. Then

$$P^B_M(t) \le P^{\bar{B}}_k(t) P^A_M(t)$$

and we get

$$\operatorname{curv}_B(M) \le \max\{\operatorname{curv}_A(M), \operatorname{curv}_{\bar{B}}(k_B)\}. \qquad \square$$

We will generally use this lemma with more specific hypotheses on φ and we record these use-cases in the following corollaries.

Definition 3.1.9 ([TV08] Lemma 2.2.2.2)**.** A map $\varphi: A \to B$ of simplicial rings is flat if $\pi_0(B)$ is a flat $\pi_0(A)$-module and the natural map $\pi(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi(B)$ is an isomorphism.

Corollary 3.1.10. *Let (A, k_A) and (B, k_B) be local simplicial rings and let $M \in \operatorname{mod}(B)$. If $\varphi: A \to B$ is a flat map of local simplicial rings then*

$$\operatorname{curv}_A(M) \le \operatorname{curv}_B(M) \le \max\{\operatorname{curv}_A(M), \operatorname{curv}_{\bar{B}}(k_B)\}$$

where $\bar{B} := B \otimes^{\mathbb{L}}_A k_A \cong \pi_0(B) \otimes_{\pi_0(A)} k_A$.

Proof. The discussion in the proof of [TV08, Lemma 2.2.2.2] implies $\bar{B} \cong \pi_0(B) \otimes_{\pi_0(A)} k_A$, which also implies $\operatorname{curv}_A(B) = 0$. The result follows immediately from Lemma 3.1.8. $\square$

Corollary 3.1.11. *Let (A, k) be a local simplicial ring and let B be a Koszul complex on $x_1, \dots, x_r \in \mathfrak{m}$, the maximal ideal of A_0. Then*

$$\operatorname{curv}_A(M) \le \operatorname{curv}_B(M) \le \max\{\operatorname{curv}_A(M), 1\}.$$

Proof. It suffices to consider the case where $B = K[A; x]$, the Koszul complex on a single element. By construction, $\operatorname{curv}_A(B) = 0$. We claim $\operatorname{curv}_{\bar{B}}(k_B) = 1$. By Lemma 3.1.5, we can compute this in the DG setting, where $\bar{B}$ is an exterior algebra over k on x in degree 1. Then by [Avr98, Proposition 6.1.7] or, more directly, [AI18, Lemma 1.5], the divided power algebra $\bar{B}\langle X \mid \partial(X) = x\rangle$, is a resolution of k over $\bar{B}$ and so $k \otimes^{\mathbb{L}}_{\bar{B}} k \cong k\langle X\rangle$ which has curvature 1. □

3.1.3 Simplicial rings in positive characteristic

In this section, we collect some results about simplicial rings in positive characteristic that will help contextualize our later results on homological dimension. We first prove the following result which is simply a re-framing of [BIL+23, Theorem 2.1] for simplicial rings. The proof is exactly the same aside from checking that various statements remain true when we replace a classical ring R with a simplicial ring A. We include it here for the convenience of the reader.

Proposition 3.1.12. *Let (A, k) be a local simplicial ring of characteristic $p > 0$. Then there is a natural number c such that for any A-module M and any $p^e > c$ there is an isomorphism*

$$F^e_* K^M \simeq \pi(F^e_* K^M)$$

in $\operatorname{Ho}(A)$. *In particular, k is a summand of $F^e_* K^M$ when $\pi(K^M) \neq 0$.*

Proof. Let $\mathfrak{m} = \ker(A \to k)$. We can complete A at $\mathfrak{m}$ by computing $\lim_n A/\mathfrak{m}^n$, noting that limits are computed degree-wise. Let $\hat{A}$ be the completion and note that because $\mathfrak{m}\pi(K^M) = 0$, the natural map

$$K^M \to \hat{A} \otimes_A K^M \simeq K^{\hat{A} \otimes_A M}$$

is an isomorphism and so we can assume A and thus A_0 are complete. Let $B \to A_0$ be a minimal Cohen presentation of A_0. Let $\rho\colon B\{X\} \xrightarrow{\sim} A$ and $B\{Y\} \xrightarrow{\sim} k$ be simplicial free resolutions of A and k respectively as B-algebras. Then

$$B\{X, Y\} := B\{X\} \otimes_B B\{Y\}$$

is a simplicial free resolution of K^A over $B\{X\}$. We also get that $B\{X, Y\} \xrightarrow{\simeq} k\{X\}$ by applying $B\{X\} \otimes_B -$ to the quasi-isomorphism $B\{Y\} \xrightarrow{\simeq} k$.

Consider $J = \ker(k\{X\} \to k)$. $k\{X\}$ is a free simplicial k-algebra, so [Qui70, Theorem 6.12] says that for every integer $n \geq 0$ $\pi_i(J^{n+1}) = 0$ for $i \leq n$. In particular,

$k\{X\} \to k\{X\}/J^{n+1}$ will be bijective on homology in degrees $\leq n$. Because $k\{X\} \simeq K^A$ we know that for $c > \sup\left\{i : \pi_i\left(K^A\right) \neq 0\right\}$

$$\pi_i\left(k\{X\}\right) \cong \pi_i\left(K^A\right) = 0 \text{ for } i > c$$

By Remark 2.2.4 we have a map of simplicial rings $k\{X\}/J^{c+1} \to C$ that is bijective on homology in degree $\leq c$ and with $\pi_i(C) = 0$ for $i > c$. Then the composition

$$k\{X\} \longrightarrow k\{X\}/J^{c+1} \longrightarrow C$$

is a quasi-isomorphism by construction. Let $I := \ker\left(\varepsilon : B\{X\} \to k\right)$ and let e be such that $p^e \geq c+1$. Then the composition

$$B\{X\} \xrightarrow{F^e} B\{X\} \longrightarrow B\{X,Y\} \longrightarrow k\{X\}$$

takes I into J^{c+1}, and so the map $B\{X\} \to C$ factors through ε, yielding the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{F^e} & A & \longrightarrow & K^A & & \\ \simeq\uparrow\rho & & \simeq\uparrow & & \simeq\uparrow & & \\ B\{X\} & \xrightarrow{F^e} & B\{X\} & \longrightarrow & B\{X,Y\} & \xrightarrow{\simeq} & k\{X\} \\ \downarrow\varepsilon & & & & & & \downarrow\simeq \\ k & & & \xrightarrow{\Psi} & & & C \end{array}$$

Given a C-module M, the A-module $\rho^*\varepsilon_*\Psi_*(M)$ must be isomorphic to its homology after normalizing, as this is true for any k-module. Thus, by the above commutative diagram we get that any simplicial K^A-module M', the simplicial A-module $F^e_*(M')$ is isomorphic to its homology after normalizing, and letting M' be K^M we prove the result. The final statement comes from the fact that if $\pi(K^M) \neq 0$ then $\pi(F^e_* K^M)$ has $F^e_* k$, and thus k, as a summand. □

From this we can derive a simplicial version of Theorem 1.1 of [AHIY12], itself a generalization of [Kun69, Theorem 2.1] and [Rod88, Theorem 2].

Proposition 3.1.13. *Let (A,k) be a local simplicial ring and $M \in \operatorname{mod}(A)$. If $\operatorname{curv}_A(F^e_* M) = 0$, then A has finite global dimension and thus $A \simeq \pi_0(A)$ is a regular ring.*

Proof. If $\operatorname{curv}_A(F^e_* M)$, then $\operatorname{curv}_{N(A)} N(F^e_* M) = 0$ by Lemma 3.1.5. Then $N(F^e_* M)$ and $N(F^e_* K^M)$ are in $\operatorname{thick}(N(A))$. Then, by Proposition 3.1.12, $k \in \operatorname{thick}(N(A))$. Then, by [Jør10, Theorem A], $A \simeq \pi_0(A)$ and so $\pi_0(A)$ is regular. □

We can also use Proposition 3.1.12 to extend, with similar proof, [AHIY12, Theorem 5.1].

Lemma 3.1.14. *Let (A,k) be a local simplicial ring and $M \in \operatorname{mod}(A)$. Then* $\operatorname{curv}(F_*^e M) = \operatorname{curv}(k)$ *for all* $e \geq 1$.

Proof. First, note that for $e \gg 0$ and any M a finitely generated A-module

$$\operatorname{curv}(k) \leq \operatorname{curv}(F_*^e K^M) = \operatorname{curv}(F_*^e M) \leq \operatorname{curv}(k).$$

The first inequality follows from Proposition 3.1.12 and Lemma 3.1.6. The second equality follows because [Avr99, Proposition 2.2] implies that $N(K^M) \cong N(K^A) \otimes_{N(A)} N(M)$. Because $N(K^A)$ is a perfect complex, $\operatorname{curv}_{N(A)}(N(K^M)) = \operatorname{curv}_{N(A)}(N(M))$ and so the equality follows from Lemma 3.1.5. The third inequality holds because $\operatorname{curv}(N) \leq \operatorname{curv}(k)$ for any finite A-module N. Thus we get that $\operatorname{curv}(F_*^e M) = \operatorname{curv}(k)$ for $e \gg 0$.

We now show that this holds for $e \geq 1$. Define

$$M^{(1)} := M$$
$$M^{(n+1)} := M^{(n)} \otimes_R^{\mathbb{L}} F_* M$$

Tautologically, $\operatorname{curv}(F_* M^{(1)}) \leq \operatorname{curv}(F_* M)$, so assume that $\operatorname{curv}_A(F_*^n M^{(n)}) \leq \operatorname{curv}_A(F_* M)$. Applying Lemma 3.1.7 to the maps $A \to F_*^n A \to F_*^{n+1} A$, we get

$$\begin{aligned}
\operatorname{curv}_A(F_*^n M^{(n+1)}) &= \operatorname{curv}_A\left(F_*^n M^{(n)} \otimes_{F_*^n R}^{\mathbb{L}} F_*^{n+1} M\right) \\
&\leq \max\left\{\operatorname{curv}_A(F_*^n M^{(n)}), \operatorname{curv}_{F_*^n A}(F_*^{n+1} M)\right\} \\
&= \max\left\{\operatorname{curv}_A(F_*^n M^{(n)}), \operatorname{curv}_A(F_* M)\right\} \\
&\leq \operatorname{curv}_A(F_* M).
\end{aligned}$$

Then we get that for $e \gg 0$

$$\operatorname{curv}(k) = \operatorname{curv}(F_*^e M^{(e)}) \leq \operatorname{curv}(F_* M) \leq \operatorname{curv}(k). \qquad \square$$

3.2 Homological dimension

This section contains the main results connecting the homological properties of the relative Frobenius of a map φ of local rings to the homological properties of the (derived) fibers of φ. We first focus on regularity and the complete intersection property, as these are characterized in terms of Betti number growth, before turning to the Gorenstein property.

3.2.1 Growth of Betti numbers

Throughout, all rings will be commutative, noetherian of characteristic $p > 0$. We will also assume all rings are F-finite. Fix a local homomorphism $\varphi\colon (R, \mathfrak{m}_R, k_R) \to (S, \mathfrak{m}_S, k_S)$ of finite flat dimension. In this setting, a result of Peskine-Szpiro [PS73] gives that $S \otimes^{\mathbb{L}}_R F_* R \cong S \otimes_R F_* R$. Our goal is to understand how the homological properties of the relative Frobenius relate to those of the (derived) fibers of φ. Recall the following diagram

$$\begin{array}{ccccc} R & \xrightarrow{\varphi} & S & & \\ \downarrow{\scriptstyle F} & & \downarrow & \overset{F}{\searrow} & \\ F_* R & \longrightarrow & A := S \otimes_R F_* R & \xrightarrow[F_{S/R}]{} & F_* S \end{array}$$

where $F_{S/R}$ is the relative Frobenius of φ induced by the universal property of the tensor product. The main idea that we exploit to investigate this relationship is that the Frobenius on the derived closed fiber, $F\colon S \otimes^{\mathbb{L}}_R k_R \to F_*\left(S \otimes^{\mathbb{L}}_R k_R\right)$, can be factored as follows:

$$\begin{array}{ccccc} & & S \otimes_R F_* R & \xrightarrow{F_{S/R}} & F_* S \\ & & \downarrow & & \downarrow \\ S \otimes^{\mathbb{L}}_R k_R & \xrightarrow{(1)} & S \otimes^{\mathbb{L}}_R F_* k_R & \xrightarrow{(2)} & F_*(S \otimes^{\mathbb{L}}_R k_R) \end{array}$$

That is, the Frobenius on the derived closed fiber is the base change of the relative Frobenius up to a field extension. More generally, we can consider the perfect fibers $S \otimes^{\mathbb{L}}_R k'$ for k' a finite, purely inseparable extension of k_R and use that $k' \subseteq F^e_* k_R$ for $e \gg 0$ to identify an analogous factorization of the Frobenius on $S \otimes^{\mathbb{L}}_R k'$. We are ready to state our main result.

Theorem 3.2.1. *Let R and S be F-finite local rings of positive characteristic with residue fields k_R and k_S respectively. Let $\varphi\colon R \to S$ be a map of local rings. If φ has finite flat dimension, then*

$$\operatorname{curv}_{\bar{S}'}(F_* \bar{S}') \leq \operatorname{curv}_A(F_* S) \leq \max\{\operatorname{curv}_{\bar{S}'}(F_* \bar{S}'), 1\}$$

where $\bar{S}' := S \otimes^{\mathbb{L}}_R k'$ for k' any finite, purely inseparable extension of k_R and A is the localization of $S \otimes_R F_* R$ at the contraction of the maximal ideal of $F_* S$.

Proof. Let k' be a finite, purely inseparable field extension of k_R and let e be such that $k' \subseteq F^e_* k_R$. Let A^e be the localization of $S \otimes_R F^e_* R$ at the contraction of the maximal ideal of $F^e_* S$, let k_{A^e} be the residue field of A^e, and fix the following notation:

$$\begin{aligned}\bar{S} &:= S \otimes^{\mathbb{L}}_R k_R \\ \bar{S}' &:= \bar{S} \otimes_{k_R} k' \\ \bar{A}^e &:= A^e \otimes^{\mathbb{L}}_{F^e_* R} F^e_* k_R \simeq S \otimes^{\mathbb{L}}_R F^e_* k_R\end{aligned}$$

Our goal is to compare the Betti numbers of the relative Frobenius to those of the Frobenius on the perfect fibers of $\varphi: R \to S$, so we first look at the Betti numbers of $F^e_{S/R}: A^e \to F^e_* S$. Note that

$$\begin{aligned}F^e_* S \otimes^{\mathbb{L}}_{A^e} k_{A^e} &\simeq F^e_* S \otimes^{\mathbb{L}}_{A^e} \left(\bar{A}^e \otimes^{\mathbb{L}}_{\bar{A}^e} k_{A^e}\right) \\ &\simeq \left(F^e_* S \otimes^{\mathbb{L}}_{F^e_* R} F^e_* k_R\right) \otimes^{\mathbb{L}}_{\bar{A}^e} k_{A^e} \\ &\simeq F^e_* \bar{S} \otimes^{\mathbb{L}}_{\bar{A}^e} k_{A^e}.\end{aligned}$$

Thus

$$\beta_i^{A^e}(F^e_* S) = \beta_i^{\bar{A}^e}(F^e_* \bar{S})$$

and so

$$\operatorname{curv}_{A^e}(F^e_* S) = \operatorname{curv}_{\bar{A}^e}(F^e_* \bar{S}).$$

We now turn to the Betti numbers of the Frobenius on the perfect fibers. Notice that $F^e: S \otimes^{\mathbb{L}}_R k' \to F^e_*(S \otimes^{\mathbb{L}}_R k')$ factors as

$$S \otimes^{\mathbb{L}}_R k' \xrightarrow{(1)} S \otimes^{\mathbb{L}}_R F^e_* k_R \xrightarrow{(2)} F^e_*(S \otimes^{\mathbb{L}}_R k_R) \xrightarrow{(3)} F^e_*(S \otimes^{\mathbb{L}}_R k')$$

which in our notation is

$$\bar{S}' \xrightarrow{(1)} \bar{A}^e \xrightarrow{(2)} F^e_* \bar{S} \xrightarrow{(3)} F^e_* \bar{S}'$$

Now we note that

$$F^e_* \bar{S}' \otimes^{\mathbb{L}}_{\bar{A}^e} k_A \cong \left(F^e_* \bar{S} \otimes_{F^e_* k_R} F^e_* k'\right) \otimes^{\mathbb{L}}_{\bar{A}^e} k_{A^e} \cong \left(F^e_* \bar{S} \otimes^{\mathbb{L}}_{\bar{A}^e} k_A\right) \otimes_{F^e_* k_R} F^e_* k'$$

which gives us

$$\operatorname{curv}_{\bar{A}^e}(F^e_* \bar{S}) = \operatorname{curv}_{\bar{A}^e}(F^e_* \bar{S}').$$

Finally, consider $\bar{S}' \to \bar{A}^e$. This is flat, and because $\bar{A}^e \otimes^{\mathbb{L}}_{\bar{S}'} k_{S'} \cong k_{A^e} \otimes_{k'} k_{S'}$, which is complete intersection, Lemma 3.1.8 tells us

$$\operatorname{curv}_{\bar{S}'}(F^e_* \bar{S}') \leq \operatorname{curv}_{\bar{A}^e}(F^e_* \bar{S}') \leq \max\{\operatorname{curv}_{\bar{S}'}(F^e_* \bar{S}'), 1\}.$$

Combining this with the fact that $\operatorname{curv}_{A^e}(F^e_*S) = \operatorname{curv}_{\bar{A}^e}(F^e_*\bar{S}) = \operatorname{curv}_{\bar{A}^e}(F^e_*\bar{S}')$ we get

$$\operatorname{curv}_{\bar{S}'}(F^e_*\bar{S}') \leq \operatorname{curv}_{A^e}(F^e_*S) \leq \max\{\operatorname{curv}_{\bar{S}'}(F^e_*\bar{S}'), 1\}.$$

Invoking Lemma 3.1.14 we immediately get

$$\operatorname{curv}_{\bar{S}'}(F_*\bar{S}') \leq \operatorname{curv}_{A^e}(F^e_*S) \leq \max\{\operatorname{curv}_{\bar{S}'}(F_*\bar{S}'), 1\}.$$

We would like to remove e from the middle term, as right now e depends on the extension k' of k_R. This is immediate when $\operatorname{curv}_A(F_*S) \geq 1$, so we only need to consider the cases where $\operatorname{curv}_A(F_*S) = 0$. In this case, we claim $\operatorname{curv}_{A^e}(F^e_*S) = 0$ for all $e \geq 1$. We prove this by induction. It is tautological for $e = 1$ so suppose it's true up to $e = n$. Note that $S \otimes_R F^{n+1}_*R \to F^{n+1}_*S$ factors as

$$(S \otimes_R F_*R) \otimes_{F_*R} F^n_*R \longrightarrow F_*S \otimes_{F_*R} F^{n+1}_*R \longrightarrow F^{n+1}_*S$$

The first map is simply $F_{S/R} \otimes_{F_*R} F^n_*R$ and so has curvature zero. The second map is simply $F_*(F^n_{S/R})$ and so has curvature zero. Thus, by Lemma 3.1.7 $\operatorname{curv}_{A^{n+1}}(F^{n+1}_*S) = 0$. Thus

$$\operatorname{curv}_{\bar{S}'}(F_*\bar{S}') \leq \operatorname{curv}_A(F_*S) \leq \max\{\operatorname{curv}_{\bar{S}'}(F_*\bar{S}'), 1\}. \qquad \square$$

From this, we recover the results of Radu, André and Dumitrescu. First, recall the definition of a regular homomorphism of rings.

Definition 3.2.2. A map $\varphi\colon R \to S$ of noetherian rings is *regular* if it is flat and all the fibers are geometrically regular, meaning for every $\mathfrak{p} \in \operatorname{Spec} R$ and every finite purely inseparable field extension $k(\mathfrak{p}) \subseteq k'$, the ring $S \otimes_R k'$ is regular.

Corollary 3.2.3. *Let R and S be F-finite local rings of positive characteristic. Let $\varphi\colon R \to S$ be a map of local rings of finite flat dimension. Then $F_{S/R}$ has finite flat dimension if and only if φ is regular.*

Proof. Suppose $F_{S/R}$ has finite flat dimension. Regularity is a local condition, so we can work in the setting of Theorem 3.2.1 and assume R and S are local. Let k' be a finite, purely inseparable field extension of k_R, the residue field of R and let $\bar{S}' := S \otimes^{\mathbb{L}}_R k'$. Because $\beta^A_i(F_*S) = 0$ for $i \gg 0$, Theorem 3.2.1 tells us $\beta^{\bar{S}}_i(F_*\bar{S}) = 0$ for $i \gg 0$. By Proposition 3.1.13, $\bar{S}' \cong S \otimes_R k'$ is regular and flatness of φ follows immediately. The reverse direction follows from [Rad92, And93]. $\square$

From here, we move on to consider CI-dimension, so-named because a ring R is complete intersection if and only if the CI-dimension of every finite R-module is finite. This was first defined in [AGP97].

Definition 3.2.4. A finite R-module M is said to have *finite CI-dimension* if there is a local flat homomorphism $R \to R'$ and a surjective homomorphism $Q \to R'$ with kernel generated by a regular sequence such that $\mathrm{pd}_Q(M \otimes_R R') < \infty$.

Definition 3.2.5. Let $\varphi\colon (R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of commutative local rings and let $\grave{\varphi}$ denote the composition of φ with the completion map $S \to \hat{S}$. A *Cohen factorization* of $\grave{\varphi}$ is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad\grave{\varphi}\quad} & \hat{S} \\ & {\scriptstyle\dot{\varphi}}\searrow \quad \nearrow{\scriptstyle\varphi'} & \\ & R' & \end{array}$$

of local homomorphisms such that

(i) $\dot{\varphi}$ is flat

(ii) R' is complete and $R'/\mathfrak{m}_R R'$ regular

(iii) φ' is surjective

Definition 3.2.6 ([Avr99] §1)**.** We say that a map of local rings $\varphi\colon R \to S$ is *complete intersection* at the maximal ideal of S if for some (equiv. any) Cohen factorization $R \to R' \to \hat{S}$ of φ, the ideal $\ker(R' \to \hat{S})$ is generated by a regular sequence. $\mathbf{f}'$.

Finally, we define the minimal model of a map of local rings. For more details, see [Avr98]

Definition 3.2.7. Let $\varphi\colon R \to S$ be a map of local rings. A minimal model for φ is a factorisation $R \to A \to S$, where A is a dg R-algebra with the following properties:

(1) $A = R[X]$ is the free strictly graded commutative R-algebra on a graded set $X = X_1, X_2, \ldots$, each X_i being a set of degree i variables;

(2) the differential of A satisfies $\partial(\mathfrak{m}A) \subseteq \mathfrak{m}_R A + \mathfrak{m}_A^2$;

(3) $A \to S$ is a quasi-isomorphism

Note that $A \otimes_R k_R \cong S \otimes_R^{\mathbb{L}} k_R$.

We are now ready to state and prove the following corollary to Theorem 3.2.1.

Corollary 3.2.8. *Let R and S be F-finite local rings of positive characteristic. Let $\varphi\colon R \to S$ be a map of local rings of finite flat dimension. Then $F_{S/R}$ has finite CI dimension if and only if φ is complete intersection at $\mathfrak{m}_S$.*

Proof. Let A be the localization of $S \otimes_R F_*S$ at the contraction of the maximal ideal of F_*S. Let $\bar{S} := S \otimes^{\mathbb{L}}_R k_R$ where k_R is the residue field of R. Since $F_{S/R}$ has finite CI-dimension, $\operatorname{curv}_A(F_*S) \le 1$, by [AGP97, Theorem 5.3], and hence $\operatorname{curv}_{\bar{S}}(k_S) \le 1$, by Theorem 3.2.1. We claim that this implies φ is complete intersection at $\mathfrak{m}_S$; to deduce this we use the results from [Avr99, Theorem 3.4]; see also [AI03, Theorem 5.4].

First, we reduce to the case where $\varphi\colon R \to S$ is surjective and S is complete. Note that $S \otimes^{\mathbb{L}}_R k_R \to \hat{S} \otimes^{\mathbb{L}}_R k_R$ is flat and both rings have common residue field k_S. Lemma 3.1.10 tells us that curvature stays constant along this map, and so we can assume $S = \hat{S}$.

Take a Cohen presentation of φ:

$$\begin{array}{ccc} R & \xrightarrow{\quad \dot{\varphi} \quad} & S \\ & {\scriptstyle \dot{\varphi}}\searrow \qquad \nearrow{\scriptstyle \varphi'} & \\ & R' & \end{array}$$

Let $\bar{R}' = R' \otimes_R k_R$. This is a regular local ring so its residue field, k_S, is resolved by the Koszul complex $K^{\bar{R}'}$ and

$$\left(S \otimes^{\mathbb{L}}_R k_R\right) \otimes_{\bar{R}'} K^{\bar{R}'} \cong \left(S \otimes^{\mathbb{L}}_{R'} \bar{R}'\right) \otimes^{\mathbb{L}}_{\bar{R}'} k_S \cong S \otimes^{\mathbb{L}}_{R'} k_S.$$

Thus $S \otimes^{\mathbb{L}}_{R'} k_S$ is a Koszul complex on $S \otimes^{\mathbb{L}}_R k_R$ and so by Corollary 3.1.11 we see that $\operatorname{curv}_{S \otimes^{\mathbb{L}}_R k_R}(k_S) \le 1$ if and only if $\operatorname{curv}_{S \otimes^{\mathbb{L}}_{R'} k_S}(k_S) \le 1$.

Thus, we can assume $R \to S$ is a surjective map of complete local rings. Let k be their common residue field. We know that $\operatorname{curv}_{\bar{S}}(k) = \operatorname{curv}_{\bar{S}}(F_*\bar{S}) \le 1$ and we aim to show the $\varepsilon_n(\varphi) = 0$ for $n \ge 3$, which will show $R \to S$ is c.i. at $\mathfrak{m}_S$ by [Avr99, Theorem 3.4]. Here $\varepsilon_n(\varphi)$ is the n^{th} deviation of φ, defined originally in [Avr99], though a typo in the formula was corrected in [AI03, 2.5]. The important fact for us will be that for a minimal model $R \to R[X] \to S$, $\varepsilon_n(\varphi) = \operatorname{card}(X_{n-1})$ for $n \ge 3$.

Let $R \to R[X] \to S$ be a minimal model and note that for $n \ge 3$ one has

$$\varepsilon_n(\varphi) = \operatorname{rank}_k \pi_{n-1}(S \otimes^{\mathbb{L}}_R k).$$

Then, because $k[X] \subseteq \operatorname{Tor}^{\bar{S}}(k,k)$ and $\operatorname{curv}_{\bar{S}}(k) \le 1$, $\lim\limits_{n\to\infty} \sqrt[n]{\varepsilon_n(\varphi)} \le 1$, and so, by [AI03, Corollary 5.5], φ is complete intersection at $\mathfrak{m}_S$.

For the reverse direction, suppose $R \to S$ is complete intersection at $\mathfrak{m}_S$. Then $\ker(R \to S)$ is generated by a regular sequence $\mathbf{x}$ and $K[R;\mathbf{x}] \simeq S$. Then $\bar{S} \cong K[R;\mathbf{x}] \otimes_R k$, which is an exterior algebra on $\mathbf{x}$ with zero differential. Then by [Avr98, Proposition 6.1.7] or, more directly, [AI18, Lemma 1.5], the divided power algebra $\bar{S}\langle X \mid \partial(X) = x\rangle$, is a resolution of k over $\bar{S}$ and so $k \otimes^{\mathbb{L}}_{\bar{S}} k \cong k\langle X\rangle$ which has curvature 1. □

3.2.2 G-dimension

With a Radu-André-type result for flat and CI-dimension, it is natural to ask whether a similar result can be proven for G-dimension, an analogue of CI-dimension for the Gorenstein property. This was first defined for finite modules in [AB69] and was generalized to modules over local homomorphisms in [ISW04].

Definition 3.2.9. Given M a homologically finite complex of R-modules, meaning $H(M)$ is degreewise finite and bounded, we say M has finite G-dimension if the following natural map is an isomorphism

$$M \xrightarrow{\sim} \mathrm{RHom}_R(\mathrm{RHom}_R(M, R), R).$$

That is to say, M is *derived reflexive.*

In the following result, we restrict to a flat homomorphism $\varphi\colon R \to S$, rather than requiring φ to have finite flat dimension as in the previous section. This is to avoid discussing the definition of finite G-dimension in the setting of derived rings which we plan to take up on a later occasion.

Theorem 3.2.10. *Let R and S be F-finite local rings of positive characteristic. Let $\varphi\colon R \to S$ be a flat map of local rings. Then the relative Frobenius $F_{S/R}\colon S \otimes_R F_*R \to F_*S$ has finite G-dimension if and only if φ is Gorenstein at $\mathfrak{m}_S$.*

Proof. Recall the following diagram

$$\begin{array}{ccccc} R & \xrightarrow{\varphi} & S & & \\ \downarrow{\scriptstyle F} & & \downarrow & \searrow^{F} & \\ F_*R & \longrightarrow & A := S \otimes_R F_*R & \xrightarrow[F_{S/R}]{} & F_*S \end{array}$$

where $F_{S/R}$ is the relative Frobenius of φ induced by the universal property of the tensor product. The main idea that we exploit to investigate this relationship is that the Frobenius on the derived closed fiber, $F\colon S\otimes^{\mathbb{L}}_R k_R \to F_*\left(S\otimes^{\mathbb{L}}_R k_R\right)$, can be factored as follows:

$$\begin{array}{ccccc} & & S\otimes_R F_*R & \xrightarrow{F_{S/R}} & F_*S \\ & & \downarrow & & \downarrow \\ S\otimes^{\mathbb{L}}_R k_R & \xrightarrow{(1)} & S\otimes^{\mathbb{L}}_R F_*k_R & \xrightarrow{(2)} & F_*(S\otimes^{\mathbb{L}}_R k_R) \end{array}$$

Since the (1) is just the base change of a field extension, if we can show the (2) has finite G-dimension, we will be done. So, we simply want to show that finite G-dimension is preserved when we base change along $F_*R \to F_*k_R$. Let $A\colon= S\otimes_R F_*R$. Since the relative Frobenius has finite G-dimension, we consider the following isomorphism

$$F_*S \xrightarrow{\sim} \operatorname{RHom}_A(\operatorname{RHom}_A(F_*S, A), A).$$

Applying $-\otimes^{\mathbb{L}}_{F_*R} F_*k_R$, we get

$$F_*(S/\mathfrak{m}_R S) \xrightarrow{\sim} \operatorname{RHom}_A(\operatorname{RHom}_A(S, A), A)\otimes^{\mathbb{L}}_{F_*R} F_*k_R.$$

Then, because φ is flat,

$$F_*(S/\mathfrak{m}_R S) \xrightarrow{\sim} \operatorname{RHom}_{\bar{A}}(\operatorname{RHom}_{\bar{A}}(F_*(S/\mathfrak{m}_R S), \bar{A}), \bar{A})$$

where again $\bar{A}\colon= S\otimes_R F_*k_R$.

For the reverse direction, consider the exact triangle in $\operatorname{Ho}(R)$

$$F_*S \to \operatorname{RHom}_A(\operatorname{RHom}_A(F_*S, A), A) \to C$$

where C is the cone of the natural map $F_*S \to \operatorname{RHom}_A(\operatorname{RHom}_A(F_*S, A), A)$. If, after applying $-\otimes^{\mathbb{L}}_R F_*k_R$ to the triangle, we get that the first map is a quasi-isomorphism, this implies $C\otimes^{\mathbb{L}}_R F_*k_R$ is acyclic. By Nakayama's Lemma, this implies C is acyclic.

Thus we have that $S_{S/R}$ has finite G-dimension if and only if $F: S\otimes_R k_R \to F_*(S\otimes_R k_R)$ has finite G-dimension, and the result follows from [TY04, Theorem 6.2]. □

Chapter 4

Multiplier ideals and klt singularities via (derived) splittings

Let X be a normal, excellent, noetherian scheme over $\operatorname{Spec}\mathbb{Q}$ with a dualizing complex. For $\Delta \geq 0$ a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, we can consider the multiplier ideal $\mathcal{J}(X, \Delta) \subseteq \mathcal{O}_X$ as a measure of the severity of the singularities of the pair (X, Δ). Introduced in the analytic setting by Nadel [Nad90] and further developed by Demailley, Siu and others [Dem93,Siu93,AS95,Siu98], Esnault and Viehweg [EV92, Chapter 7] independently developed the theory in the algebro-geometric setting. Lipman also encountered multiplier ideals in connections with his work on the Briancon-Skoda theorem [Lip94], and their applications to algebra were studied by many others [EL97, Kaw99a, Kaw99b, EL99, DEL00, ELS01].

While $\mathcal{J}(X, \Delta)$ depends on the geometry of the pair (X, Δ), de Fernex and Hacon introduced in [dFH09] an object $\mathcal{J}(X)$ whose definition doesn't require the choice of a boundary divisor while also showing that $\mathcal{J}(X)$ is the unique maximal element in the collection $\{\mathcal{J}(X, \Delta)\}$ where Δ ranges over all effective $\mathbb{Q}$-divisors on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We call $\mathcal{J}(X)$ the multiplier ideal of X and say that X is has *klt type* if $\mathcal{J}(X) = \mathcal{O}_X$.

We set out to prove an alternate characterization of the multiplier ideal by considering maps $\pi_*\omega_Y \to \mathcal{O}_X$ where $\pi : Y \to X$ is a regular alteration. This leads us to the main result of this paper:

Theorem 4.0.1. *Let X be a normal, excellent, noetherian scheme over* $\operatorname{Spec}\mathbb{Q}$ *with a dualizing complex and $I = \prod \mathcal{J}_k^{a_k}$ be a formal effective $\mathbb{Q}$-linear combination of ideal sheaves on X. The multiplier ideal $\mathcal{J}(X, I)$ can be realized as*

$$\mathcal{J}(X, I) = \sum_{\pi : Y \to X} \operatorname{Im}\left(\mathcal{H}om_X(\pi_*\omega_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \pi_*\omega_Y(-E_Y) \to \mathcal{O}_X\right)$$

where $\pi : Y \to X$ *ranges over all log regular alterations of* (X, I) *with* $E_Y = \sum a_k E_k$ *for* $\mathcal{J}_k \mathcal{O}_Y = \mathcal{O}_Y(-E_k)$ *and the map* $\mathcal{H}om_X(\pi_*\omega_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \pi_*\omega_Y(-E_Y) \to \mathcal{O}_X$ *is the evaluation map.*

Roughly speaking, we can interpret this as saying, at least locally, that an element $f \in \mathcal{J}(X, I)$ if and only if it is in the image of some map $\phi : \pi_*\omega_Y \to \mathcal{O}_X$ restricted to $\pi_*\omega_Y(-E_Y)$. Showing that $\mathcal{J}(X, I)$ is contained in this sum of images is fairly straightforward. It is in showing the reverse containment where the work is done by proving the following key lemma inspired by Lemma 1.1 of [FG12], though the proof technique is quite different.

Lemma 4.0.2. *Let* $\rho : \operatorname{Spec} S \to \operatorname{Spec} R$ *be a finite map of normal, excellent, noetherian domains containing* $\mathbb{Q}$ *with dualizing complexes and let* $I = \prod \mathcal{J}_k^{a_k}$ *be a formal effective* $\mathbb{Q}$*-linear combination of ideal sheaves on* $\operatorname{Spec} R$*. Suppose we have a* $\mathbb{Q}$*-Cartier divisor* $\Gamma \geq \rho^* K_R$ *with* $n\Gamma = \operatorname{div} g$*. Then*

$$\operatorname{Tr}_{S/R}(\rho_*\mathcal{J}(\omega_S, \Gamma, IS)) \subseteq \mathcal{J}(R, I).$$

In the case where $I = R$ *and* $\operatorname{Tr}_{S/R}(\rho_*\mathcal{J}(\omega_S, \Gamma)) = R$*, this implies that* R *has klt type.*

The original motivation for this work was to develop a derived splinter characterization of klt singularities, akin to the following characterization of rational singularities by Bhatt and Kovács [Bha12, Kov00]:

Theorem 4.0.3 ([Bha12] Theorem 2.12, [Kov00] Theorem 3)**.** *A scheme* X *of essentially finite type over a field of characteristic* 0 *has rational singularities if and only if it is a derived splinter, meaning for ever proper surjective map* $\pi : Y \to X$*, the natural map* $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$ *splits in the derived category of coherent sheaves on* X*.*

If X has klt type, then it has rational singularities and hence $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$ splits for any proper surjective map $\pi : Y \to X$. Thus, we might expect that some extra conditions on the splitting $R\pi_*\mathcal{O}_Y \to \mathcal{O}_X$ may characterize schemes having klt type. Ultimately, such a characterization follows as a corollary to Theorem 4.0.1:

Corollary 4.0.4. *Let* X *be a normal, excellent, noetherian scheme over* $\operatorname{Spec} \mathbb{Q}$ *with a dualizing complex and* $I = \prod \mathcal{J}_k^{a_k}$ *be a formal effective* $\mathbb{Q}$*-linear combination of ideal sheaves on* X*. The following are equivalent*

1. (X, I) *has klt type*

2. *For all sufficiently large regular alterations* $\pi : Y \to X$*, the natural map* $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$ *splits and locally factors through* $R\pi_*\omega_Y(-E_Y) = \pi_*\omega_Y(-E_Y)$

3. *For all sufficiently large regular alterations* $\pi : Y \to X$*,* $\mathcal{O}_X$ *is locally a summand of* $R\pi_*\omega_Y(-E_Y) = \pi_*\omega_Y(-E_Y)$

Here, $E_Y := \sum a_k E_k$ *where* $\mathcal{O}_Y(-E_Y) = \mathcal{J}_k\mathcal{O}_Y$*.*

In positive characteristic, the test ideal plays a role analogous to the multiplier ideal and can be used to a define *strongly* F*-regular* singularities, an analog of klt singularities in positive characteristic. In [BST15], Blickle, Schwede and Tucker give a characteristic-free description of an ideal $J(X, \Delta)$ that specializes to the multiplier ideal in characteristic zero and the test ideal in characteristic $p > 0$, so we might expect that our characterization of the multiplier ideal might carry over to a characterization of the test ideal in positive characteristic. We achieve such a result in characteristic $p > 2$ as a corollary to a result of Epstein and Schwede [ES14] combined with the existence of quasi-Gorenstein finite covers.

Proposition 4.0.5. *Let* R *be a Noetherian,* F*-finite reduced ring of characteristic* $p > 2$*. The test ideal* $\tau(R)$ *can be realized as*

$$\tau(R) = \sum_{R \to S} \mathrm{Im}\left(\mathrm{Hom}_R(\omega_S, R) \otimes_R \tau(\omega_S) \to R\right)$$

where the sum ranges over all finite extensions $R \to S$ *and* $\mathrm{Hom}_R(\omega_S, R) \otimes_R \tau(\omega_S) \to R$ *is the evaluation map.*

Here $\tau(\omega_S) \subseteq \omega_S$ is the parameter test submodule which plays an analogous role to $\pi_*\omega_Y$, which is sometimes called the Grauert-Riemenschneider sheaf or multiplier submodule, in characteristic zero.

Remark 4.0.6. The above results could be stated for regular alterations by a similar argument outlined in the characteristic zero case for the multiplier ideal using the fact that $\tau(X) = \sum \tau(X, \Delta)$ for finitely many log-$\mathbb{Q}$-Gorenstein pairs (X, Δ) (see [Sch11]). However, we leave the result in this format as the local statement statement is more readily applicable to questions in local algebra.

This also gives us the following splinter characterization of strongly F-regular singularities as a byproduct.

Corollary 4.0.7. *Let* R *be a Noetherian,* F*-finite reduced ring of characteristic* $p > 2$*. Then* R *is strongly* F*-regular if and only if* R *is a summand of* $\tau(\omega_S)$ *for any sufficiently large finite cover* $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$*.*

4.1 Characteristic zero

4.1.1 Preliminaries

Throughout this section, all schemes are noetherian, normal and integral over a field of characteristic zero. We will often additionally require our schemes be excellent with a dualizing complex, but will always make this explicit. Before we discuss our main object of study, the multiplier ideal, we will review some preliminaries about canonical modules and the trace map. We assume that the reader is familiar with canonical modules at the level of [Har77] and [KM98]. This discussion largely follows Section 2 in [BST15], but we include it here for the convenience of the reader.

Given a normal integral scheme X with canonical sheaf ω_X, we say an integral divisor K_X is a *canonical divisor* for X if $\mathcal{O}_X(K_X) \cong \omega_X$. Given $\pi : Y \to X$ a proper generically finite map of normal integral schemes over a field k, we can consider the trace map

$$\mathrm{Tr}_{Y/X} : \pi_*\omega_Y \to \omega_X$$

Since any generically finite map can be factored as a proper birational map followed by a finite map, we discuss the trace map in these contexts.

Let $\pi : Y \to X$ be a proper birational map of normal integral schemes and fix a canonical divisor K_Y on Y and set $K_X = \pi_* K_Y$ (which ensures K_Y and K_X agree on the locus where π is an isomorphism). Because π is an isomorphism outside a codimension 2 subset of X, $\pi_*\mathcal{O}_Y(K_Y)$ is a torsion-free sheaf whose reflexification is $\mathcal{O}_X(K_X)$ and the trace map is simply the natural reflexification map $\pi_*\mathcal{O}_Y(K_Y) \hookrightarrow \mathcal{O}_X(K_X)$.

If $\pi : Y \to X$ is a finite surjective map of normal integral schemes, then $\pi_*\omega_Y \cong \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \omega_X)$. We can then identify the trace map with the evaluation-at-1 map, $\mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y, \omega_X) \to \omega_X$. Assuming additionally that $\pi : Y \to X$ is a finite separable map of normal integral schemes with ramification divisor Ram_π, we fix a canonical divisor K_X on X and set $K_Y = \pi^* K_X + \mathrm{Ram}_\pi$. Then the field-trace map

$$\mathrm{Tr}_{K(Y)/K(X)} : K(Y) \to K(X)$$

restricts to a map $\pi_*\mathcal{O}_Y(K_Y) \to \mathcal{O}_X(K_X)$ which can be identified with the Grothendieck trace map. Throughout the rest of the paper, whenever we have a generically finite map of normal integral schemes $\pi : Y \to X$, we will always choose K_X and K_Y compatibly according to the above discussion.

We are now ready to introduce the concept of pairs. A $\mathbb{Q}$-divisor Γ on X is a formal linear combination of prime Weil divisors with coefficients in $\mathbb{Q}$. Writing $\Gamma = \sum a_i Z_i$ where the Z_i are distinct prime divisors, we use $\lceil \Gamma \rceil = \sum \lceil a_i \rceil Z_i$ and $\lfloor \Gamma \rfloor = \sum \lfloor a_i \rfloor Z_i$ to denote the round up and round down of Γ, respectively. We say that Γ is $\mathbb{Q}$-Cartier if there exists an integer $n > 0$ such that $n\Gamma$ is an integral Cartier divisor, and the smallest such n is called the *index* of Γ.

Definition 4.1.1. A *pair* (X, Δ) is the combined data of a normal integral scheme X together with a $\mathbb{Q}$-divisor Δ on X. The pair (X, Δ) is called *log-$\mathbb{Q}$-Gorenstein* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Definition 4.1.2. Given a log-$\mathbb{Q}$-Gorenstein pair (X, Δ), a *log resolution of singularities* of the pair (X, Δ) is a resolution of singularities $\pi : Y \to X$ such that $\operatorname{except}(\pi)$ is a divisor and $\pi_*^{-1}(\Delta) + \operatorname{except}(\pi)$ has simple normal crossing support.

More generally, let $I = \prod \mathcal{J}_k^{a_k}$ be an effective formal $\mathbb{Q}$-linear combination of ideal sheaves on X. A *log resolution* of (X, I) is a proper birational morphism $\pi : Y \to X$ with Y smooth such that for every k the sheaf $\mathcal{J}_k \mathcal{O}_Y \cong \mathcal{O}_Y(-E_k)$ for $E_k \geq 0$ $\mathbb{Q}$-Cartier, $\operatorname{except}(\pi)$ is also a divisor, and $\operatorname{except}(\pi) + E_Y$ has simple normal crossing support where $E_Y = \sum a_k E_k$. Log resolutions exist when X is quasi-excellent by [Tem08, Theorem 2.3.6].

For our purposes, we will also want to move beyond resolutions of singularities to consider regular alterations.

Definition 4.1.3. A map $\pi : Y \to X$ of schemes is a regular alteration if it is surjective, proper, and generically finite and Y is nonsingular.

The main benefit to this is the Stein factorization. If $\pi : Y \to X$ is a regular alteration, then π factors as

$$\begin{array}{ccc} Y & \xrightarrow{\quad\pi\quad} & X \\ & \searrow^{\tau} \quad {}^{\rho}\nearrow & \\ & Z & \end{array}$$

where $\rho : Z \to X$ is a finite surjective map and $\tau : Y \to Z$ is a resolution of singularities.

Given a log $\mathbb{Q}$-Gorenstein pair (X, Δ), we will say that $\pi : Y \to X$ is a log regular alteration if it is a regular alteration, $\operatorname{except}(\tau)$ is a divisor, and $\pi_*^{-1}(\Delta) + \operatorname{except}(\tau)$ has simple normal crossing support.

This then leads us to the object of interest: the multiplier ideal. The theory of multiplier ideals was largely developed by Esnault and Viehweg in our setting [EV92], and more details on the theory can be found in [Laz04].

Definition 4.1.4. Let X be a normal, excellent, noetherian scheme over $\operatorname{Spec}\mathbb{Q}$ with a dualizing complex. Given a log-$\mathbb{Q}$-Gorenstein pair (X,Δ) with $\Delta \geq 0$, the *multiplier ideal* of the pair (X,Δ) is

$$\mathcal{J}(X,\Delta) := \pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X+\Delta)\rceil) \subseteq \mathcal{O}_X$$

where $\pi : Y \to X$ is a log resolution of singularities of the pair (X,Δ). Note that [Mur21, Theorem A] ensures that $R\pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X+\Delta)\rceil) = \pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X+\Delta)\rceil)$.

More generally, we can let $\pi : Y \to X$ be a log regular alteration of (X,Δ) and consider

$$\mathcal{J}(X,\Delta) = \operatorname{Tr}_{Y/X}(\pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X+\Delta)\rceil)) \subseteq \mathcal{O}_X$$

via Theorem 8.1 of [BST15].

We will also use the related concept of *multiplier submodules*, sometimes called Grauert-Riemenschneider sheaves as they were first suggested as objects of study by Grauert and Riemenschneider [GR70]. As far as we know, the first instance of the name multiplier submodule appears in [Bli04].

Definition 4.1.5 ([Bli04]). Let X be a normal, excellent, noetherian scheme over $\operatorname{Spec}\mathbb{Q}$ with a dualizing complex $\omega_X^\bullet$. Let ω_X be the canonical module, the last non-vanishing cohomology sheaf of $\omega_X^\bullet$, and fix K_X a canonical divisor with $\mathcal{O}_X(K_X) \cong \omega_X$. Given an effective $\mathbb{Q}$-Cartier divisor Δ, we define the multiplier submodule $\mathcal{J}(\omega_X,\Delta)$, also known as the Grauert-Riemenschneider sheaf, as

$$\mathcal{J}(\omega_X,\Delta) := \pi_*\mathcal{O}_Y\left(\lceil K_Y - \pi^*\Delta\rceil\right)$$

where $\pi : Y \to X$ is a log resolution of (X,Δ). Once again, [Mur21, Theorem A] ensures $R\pi_*\mathcal{O}_Y\left(\lceil K_Y - \pi^*\Delta\rceil\right) = \pi_*\mathcal{O}_Y\left(\lceil K_Y - \pi^*\Delta\rceil\right)$.

The hypothesis that $K_X + \Delta$ be $\mathbb{Q}$-Cartier is included because there is a well-defined theory of pullbacks for Cartier divisors. Work of de Fernex and Hacon [dFH09] removes this hypothesis by defining a pullback operation that uses the fractional ideal sheaf corresponding to a divisor.

Assume the same hypotheses as 4.1.5. Given a divisor D on X and $\pi : Y \to X$ proper, birational, de Fernex and Hacon define the natural pullback of D along π to be

$$\pi^\natural(D) := \operatorname{div}_Y(\mathcal{O}_X(-D)\cdot\mathcal{O}_Y).$$

In particular, $\mathcal{O}_Y(-\pi^\natural D) = (\mathcal{O}_X(-D)\cdot\mathcal{O}_Y)^{\vee\vee}$ where $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F},\mathcal{O}_Y)$ for any quasi-coherent sheaf on Y. Applying this operation to multiples of K_X, we come to the following definition.

Definition 4.1.6 ([dFH09] 2.6, 3.1)**.** Given a proper birational morphism of normal, excellent, noetherian schemes over $\operatorname{Spec}\mathbb{Q}$ with dualizing complexes $\pi : Y \to X$, the m^{th} limiting relative canonical divisor $K_{m,Y/X}$ is

$$K_{m,Y/X} := K_Y - \frac{1}{m}\pi^\natural(mK_X)$$

If $I = \prod \mathcal{J}_k^{a_k}$ be an effective formal $\mathbb{Q}$-linear combination of ideal sheaves on X, define

$$\mathcal{J}_m(X,I) := \pi_*\mathcal{O}_Y(\lceil K_{m,Y/X} - E_Y\rceil)$$

where $\pi : Y \to X$ is a log resolution of $(X, I + \mathcal{O}_X(-mK_X))$.

Proposition 4.1.7 (cf. [dFH09] Proposition 4.7)**.** *Let X be a normal, excellent, noetherian scheme over* $\operatorname{Spec}\mathbb{Q}$ *with a dualizing complex and $I = \prod \mathcal{J}_k^{a_k}$ be a formal effective $\mathbb{Q}$-linear combination of ideal sheaves on X. The collection $\{\mathcal{J}_m(X,I)\}_{m\geq 1}$ has a unique maximal element.*

Proof. Fix $\pi : Y \to X$ a proper birational morphism of normal, excellent, noetherian schemes over $\operatorname{Spec}\mathbb{Q}$ with dualizing complexes. We first note that $K_{m,Y/X} \leq K_{mq,Y/X}$ for all m, q positive integers as $mv^\natural(D) \geq v^\natural(mD)$. Thus $\mathcal{J}_m(X,\Delta) \subseteq \mathcal{J}_{mq}(X,\Delta)$ and the existence of a unique maximal element follows by noetherianity. □

Definition 4.1.8. Let X be a normal, excellent, noetherian scheme over $\operatorname{Spec}\mathbb{Q}$ with a dualizing complex and $I = \prod \mathcal{J}_k^{a_k}$ be a formal effective $\mathbb{Q}$-linear combination of ideal sheaves on X. We call the unique maximal element of $\{\mathcal{J}_m(X,I)\}_{m\geq 1}$ the *multiplier ideal of the pair* (X,I) and denote it $\mathcal{J}(X,I)$. When $I = \mathcal{O}_X$, we call this unique maximal element the *multiplier ideal* of X and denote it by $\mathcal{J}(X)$.

Remark 4.1.9. For sufficiently divisible n, $\mathcal{J}(X) = \mathcal{J}(\omega_X, (\omega_X^{(-n)})^{1/n})$. To see this, let $\pi : Y \to X$ be a log resolution of $(X, \mathcal{O}_X(-nK_X))$ and note that

$$\begin{aligned}\mathcal{J}(X) &= \pi_*\mathcal{O}_Y\left(\left\lceil K_Y - \frac{1}{n}\pi^\natural(nK_X))\right\rceil\right)\\ &= \pi_*\mathcal{O}_Y\left(\left\lceil K_Y - \frac{1}{n}\operatorname{div}_Y(\mathcal{O}_X(-nK_X)\cdot\mathcal{O}_Y)\right\rceil\right)\\ &= \mathcal{J}(\omega_X, (\omega_X^{(-n)})^{1/n}).\end{aligned}$$

If we additionally assumed that X is a variety, [dFH09, Corollary 5.5] shows that $\mathcal{J}(X, I)$ is also the unique maximal element of $\{\mathcal{J}((X, \Delta); I)\}$. This follows from [dFH09, Proposition 5.2], which says that $\mathcal{J}(X, I) = \mathcal{J}_m(X, I) = \mathcal{J}((X, \Delta); I)$ whenever Δ is what is called *m-compatible*, and [dFH09, Theorem 5.4], which shows that an m-compatible Δ exists for all $m \geq 0$. We will introduce a variant of their result in our more general setting, which will require defining an m-compatible boundary and a Bertini theorem from [LM22].

Definition 4.1.10 ([dFH09] Definition 5.1)**.** Let X be a normal, excellent, noetherian scheme over $\operatorname{Spec} \mathbb{Q}$ with a dualizing complex and $I = \prod \mathcal{J}_k^{a_k}$ be a formal effective $\mathbb{Q}$-linear combination of ideal sheaves on X. Fix an integer $m \geq 2$. Given a log resolution $\pi : Y \to X$ of $(X, I\mathcal{O}_X(-mK_X))$, we say that the log $\mathbb{Q}$-Gorenstein pair (X, Δ) is said to be m-compatible for (X, I) with respect to π if

(i) $m\Delta$ is integral and $\lfloor \Delta \rfloor = 0$,

(ii) no component of Δ is contained in Z_k for any k where Z_k is the subscheme defined by $\mathcal{J}_k$

(iii) π is a log resolution of the pair $((X, \Delta); I\mathcal{O}_X(-mK_X))$

(iv) $K_Y + \Delta_Y - \pi^*(K_X + \Delta) = K_{m,Y/X}$ where Δ_Y is the proper transform of Δ on Y

Theorem 4.1.11 (Theorem 10.1 [LM22])**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local domain containing $\mathbb{Q}$. Fix an integer $N \geq 1$. Let $f : X \to \mathbb{P}_R^N$ be a separated morphism of finite type from a regular Noetherian scheme X. Assume that every closed point of X lies over the unique closed point of* $\operatorname{Spec} R$*.*

Let $T_0, T_1, \ldots, T_N$ be a basis of $H^0(\mathbb{P}_R^N, \mathcal{O}(1))$ as a free R-module. Then, there exists a nonempty Zariski open subset $W \subseteq \mathbb{A}_k^{N+1}$ with the following property: For all $a_0, a_1, \ldots, a_N \in R$, if

$$(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_N) \in W(k),$$

then the section

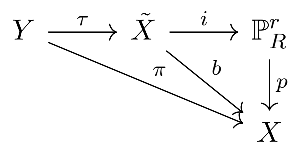
$$h = a_0T_1 + a_1T_1 + \cdots + a_NT_N$$

is such that $f^{-1}(V(h))$ is regular.

We will want a slightly more specific application of this result for our purposes, so we include that statement here, making no claims over its ownership.

Lemma 4.1.12. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local domain containing $\mathbb{Q}$. Let $J = (g_0, \dots, g_r)$ be an ideal of R, let $I = \prod \mathfrak{J}_k^{a_k}$ be an effective $\mathbb{Q}$-linear combination of ideal sheaves on $X = \operatorname{Spec} R$, and let $\pi : Y \to X$ be a log resolution of (X, I, J). Then for a general choice of $(a_0, \dots, a_r) \in \mathbb{A}_k^{n+1}$ we have that, for $f := a_0 g_0 + \cdots a_r g_r$, $\operatorname{div}_X f$ is reduced and avoids the components of I and $\pi : Y \to X$ is a log resolution of $(X, I, \operatorname{div}_X f)$.*

Proof. Let $\tilde{X}$ be the blowup of J in R. Consider the following diagram coming from the universal property of the blowup



where i comes from the surjection of graded rings $R[T_0, \dots, T_r] \to Bl_I(R)$ sending $T_i \mapsto g_i$ in degree one. Let $h = a_0 T_0 + \cdots + a_r T_r$. Then $\operatorname{div}_{\tilde{X}} f = \operatorname{div}_{\tilde{X}} h$. By Theorem 4.1.11, $\operatorname{div}_Y f$ is smooth for a Zariski dense subset of $(a_0, \dots, a_r) \in \mathbb{A}_k^{r+1}$, and by [LM22, Remark 10.2], we can also ensure that $\operatorname{except}(\pi) + E_Y + \operatorname{div}_Y f$ is simple normal crossing (as $\pi : Y \to X$ is already a log resolution of (R, I)) and that $\operatorname{div}_X f$ avoids the components of I. □

We now show that, working sufficiently locally, m-compatible boundaries exist in our setting and thus the multiplier ideal agrees with the multiplier ideal of some pair.

Proposition 4.1.13 (cf. [dFH09] Theorem 5.4, Corollary 5.5)**.** *Let R be a local, normal, excellent, noetherian domain containing $\mathbb{Q}$ with a dualizing complex, let $X = \operatorname{Spec} R$ and $I = \prod \mathfrak{J}_k^{a_k}$ be a formal effective $\mathbb{Q}$-linear combination of ideal sheaves on X. Then*

$$\mathfrak{J}(X, I) = \mathfrak{J}((X, \Delta); I)$$

for some log $\mathbb{Q}$-Gorenstein pair (X, Δ).

Proof. This proof is essentially identical to the work of de Fernex-Hacon but we include it here for the convenience of the reader. We first claim that $\mathfrak{J}((X, I); \Delta) \subseteq \mathfrak{J}(X, I)$ for any effective log $\mathbb{Q}$-Gorenstein pair (X, Δ) [dFH09, cf. Remark 5.3]. Suppose m is the $\mathbb{Q}$-Cartier index for $K_X + \Delta$. It is straightforward to show that $\pi^\natural(C + D) = \pi^\natural(C) + \pi^\natural(D)$ for C Cartier, so $\pi^\natural(mK_X) = \pi^\natural(m(K_X + \Delta) - m\Delta) = m\pi^*(K_X + \Delta) + \pi^\natural(-m\Delta)$. This implies $K_Y + \Delta_Y - \pi^*(K_X + \Delta) \leq K_{m,Y/X} = K_Y - \frac{1}{m}\pi^\natural(mK_X)$ and thus that $\mathfrak{J}((X, \Delta); I) \subseteq \mathfrak{J}_m(X, I) \subseteq \mathfrak{J}(X, I)$.

We next claim that $\mathcal{J}((X,I);\Delta) = \mathcal{J}_m(X,I)$ for any $m \geq 2$ and any m-compatible boundary Δ [dFH09, cf. Proposition 5.2]. This is because, since Δ shares no common components with Z and $\lfloor \Delta \rfloor = 0$ we have, for any log resolution $\pi : Y \to X$ of $((X,\Delta);I)$,

$$\begin{aligned}\mathcal{J}((X,I);\Delta) &= \pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X+\Delta) - E_Y\rceil)\\ &= \pi_*\mathcal{O}_Y(\lceil K_Y + \Delta_Y - \pi^*(K_X+\Delta) - E_Y\rceil)\\ &= \mathcal{J}_m(X,I).\end{aligned}$$

Finally, we claim that we can find an m-compatible log $\mathbb{Q}$-Gorenstein pair (X,Δ) for every $m \geq 2$ [dFH09, cf. Theorem 5.4]. Let D be an effective divisor such that $K_X - D$ is $\mathbb{Q}$-Cartier and let $\pi : Y \to X$ be a log resolution of $(X, \mathcal{O}_X(-mK_X) + \mathcal{O}_X(-mD))$ and let $E = \pi^\natural(mD)$. By Lemma 4.1.12, we can find $g \in \mathcal{O}_X(-mD)$ such that $\operatorname{div}_X g := G = M + mD$ is reduced and shares no common components with D or I. Set $\Delta := \frac{1}{m}M$. Note that by design, $K_X + \Delta = K_X - D + \frac{1}{m}G$ is $\mathbb{Q}$-Cartier and $\frac{1}{m}\pi^*G = \Delta_Y + \frac{1}{m}E$. To se that (X,Δ) is an m-compatible log $\mathbb{Q}$-Gorenstein pair, note that $m\Delta$ is integral and $\lfloor \Delta \rfloor = 0$. Working generally enough, we can assume π is also a log resolution of $((X,\Delta);I)$. Finally, note that

$$\begin{aligned}K_Y + \Delta_Y - \pi^*(K_X+\Delta) &= K_Y + \Delta_Y - \pi^*(K_X + \Delta - \frac{1}{m}G) - \frac{1}{m}\pi^*G\\ &= K_Y - \pi^*(K_X - D) - \frac{1}{m}E\\ &= K_{m,Y/X}.\end{aligned}$$

□

The last result we will need is a variant of [Laz04, Proposition 9.2.28].

Proposition 4.1.14. *Let $X = \operatorname{Spec} R$ be a local, normal, excellent, noetherian domain containing $\mathbb{Q}$ with a dualizing complex and let $I = \prod \mathcal{J}_k^{a_k}$ an effective formal $\mathbb{Q}$-linear combination of ideal sheaves on* $\operatorname{Spec} R$. *Let $J \subseteq R$ be an ideal and fix $c > 0$ a rational number. For $k > c$, we can find find $f_1, \ldots, f_k \in J$ such that, for $D = \frac{1}{k}\sum_i \operatorname{div}_X f_i$,*

$$\mathcal{J}(\omega_R, I, J^c) = \mathcal{J}(\omega_R, I, c \cdot D).$$

Proof. Let $\pi : Y \to X$ be a log resolution of (X, I, J^c) and let $J\mathcal{O}_Y = \mathcal{O}_Y(-B)$ for $B \geq 0$ $\mathbb{Q}$-Cartier. By repeated applications of Lemma 4.1.12, we can choose $f_1, \ldots, f_k$ generally so that $\pi : Y \to X$ is a log resolution of $(X, I, c \cdot D)$.

Write $\operatorname{div}_Y f_i = B + A_i$ for A_i effective and reduced. Then

$$\begin{aligned}\mathcal{J}(\omega_R, I, c \cdot D) &= \pi_* \mathcal{O}_Y \left(\lceil K_Y - E_Y - c\pi^* D \rceil\right) \\ &= \pi_* \mathcal{O}_Y \left(\left\lceil K_Y - E_Y - c \cdot B - \frac{c}{k} \sum A_i \right\rceil\right) \\ &= \pi_* \mathcal{O}_Y (\lceil K_Y - E_Y - c \cdot B \rceil) \\ &= \mathcal{J}(\omega_R, I, J^c).\end{aligned}$$

□

4.1.2 Results

Throughout this section, fix X, a normal, excellent, noetherian scheme over $\operatorname{Spec} \mathbb{Q}$ with a dualizing complex and $I = \prod \mathcal{J}_k^{a_k}$ be an effective formal $\mathbb{Q}$-linear combination of ideal sheaves on X. Whenever we have a log resolution of singularities or log regular alteration of the pair (X, I), we will use the notation $E_Y = \sum a_k E_k$ where $\mathcal{J}_k \mathcal{O}_Y = \mathcal{O}_Y(-E_k)$ is the ideal sheaf of an effective Cartier divisor E_k on Y for each k. We begin by showing one direction of the containment.

Proposition 4.1.15. *Let X be a normal, excellent, noetherian scheme over* $\operatorname{Spec} \mathbb{Q}$ *with a dualizing complex. Then the multiplier ideal $\mathcal{J}(X, I)$ satisfies*

$$\mathcal{J}(X, I) \subseteq \sum_{\pi : Y \to X} \operatorname{Im}\left(\mathcal{H}om_X(\pi_* \omega_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \pi_* \omega_Y(-E_Y) \to \mathcal{O}_X\right)$$

where $\pi : Y \to X$ ranges over all regular log alterations of (X, I) and the map

$$\mathcal{H}om_X(\pi_* \omega_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \pi_* \omega_Y(-E_Y) \to \mathcal{O}_X$$

is the evaluation map.

Proof. We check containment locally. By Proposition 4.1.13, let $\mathcal{J}(X, I) = \mathcal{J}((X, \Delta); I)$ for some $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\tau : \tilde{X} \to X$ be a log resolution of the triple (X, Δ, I). By [BST15, Lemma 4.5] we can finite a finite cover $\rho : Y \to \tilde{X}$ with $\pi = \tau \circ \rho$ such that $\pi * (K_X + \Delta)$ and $\rho^* E_{\tilde{X}} = E_Y$ are both effective Cartier divisors. Let $\pi^*(K_X + \Delta) = \operatorname{div}(g) \in \mathcal{O}_Y$. Using the projection formula we get that

$$\begin{aligned}\operatorname{Tr}(g \cdot \pi_* \omega_Y(-E_Y)) &= \operatorname{Tr}(\pi_* \mathcal{O}_Y(K_Y - \operatorname{div} g - E_Y)) \\ &= \operatorname{Tr}(\pi_* \mathcal{O}_Y(K_Y - \pi^*(K_X + \Delta) - E_Y)) \\ &= \mathcal{J}((X, \Delta); I) \\ &= \mathcal{J}(X, I)\end{aligned}$$

with the penultimate equality coming from Theorem 8.1 of [BST15]. □

For the reverse containment, multiplier submodules on finite covers will play a key role. The key idea is that for $\pi : Y \to X$ a regular alteration, any map $\phi : \pi_*\omega_Y \to \mathcal{O}_X$ factors through the multiplier submodule $\mathcal{J}(\omega_Z, \Gamma)$ of some divisor Γ on X, where $\rho : Z \to X$ is the finite part of the Stein factorization of π. This ultimately implies, thanks to a key lemma, $\operatorname{Im}\phi \subseteq \mathcal{J}(X)$ (as well as the analogous statement for pairs (X, I)). Before we proceed, we need the following fact about how multiplier submodules transform when we enlarge finite covers of our base.

Lemma 4.1.16. *Let* $\operatorname{Spec} S' \xrightarrow{\phi} \operatorname{Spec} S \xrightarrow{\psi} \operatorname{Spec} R$ *be finite surjective maps of normal, excellent, noetherian affine schemes over* $\operatorname{Spec}\mathbb{Q}$ *with dualizing complexes with* $\Gamma \geq 0$ *a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* $\operatorname{Spec} S$ *and* I *an effective formal* $\mathbb{Q}$*-linear combination of ideal sheaves on* $\operatorname{Spec} S$*. Then*

$$\operatorname{Im}\operatorname{Tr}_{S/R}(\psi_*\mathcal{J}(\omega_S, \Gamma, I)) = \operatorname{Im}\operatorname{Tr}_{S'/R}\left(\psi_*\phi_*\mathcal{J}(\omega_{S'}, \phi^*\Gamma, IS')\right).$$

Proof. Use Proposition 4.1.14 to replace I with some divisor D and IS' with ϕ^*D. Because any regular alteration of $\operatorname{Spec} S'$ will also be a regular alteration of $\operatorname{Spec} S$, [BST15, Theorem 8.1] gives

$$\operatorname{Tr}_{S'/S}\left(\phi_*\mathcal{J}(\omega_{S'}, \phi^*\Gamma, IS')\right) = \mathcal{J}(\omega_S, \Gamma, I).$$

[BST15, Lemma 2.3] tells us that $\operatorname{Tr}_{S'/R} = \operatorname{Tr}_{S/R} \circ \psi_* \operatorname{Tr}_{S'/S}$ and so we get that

$$\begin{aligned}\operatorname{Im}\operatorname{Tr}_{S/R}(\psi_*\mathcal{J}(\omega_S, \Gamma, I)) &= \operatorname{Im}\operatorname{Tr}_{S/R} \circ \psi_* \operatorname{Tr}_{S'/S}\left(\phi_*\mathcal{J}(\omega_{S'}, \phi^*\Gamma, IS')\right) \\ &= \operatorname{Im}\operatorname{Tr}_{S'/R}\left(\psi_*\phi_*\mathcal{J}(\omega_{S'}, \phi^*\Gamma, IS')\right).\end{aligned}$$

□

Lemma 4.1.17. *Let* $\rho : \operatorname{Spec} S \to \operatorname{Spec} R$ *be a finite surjective map of normal, excellent, noetherian affine schemes over* $\operatorname{Spec}\mathbb{Q}$ *with dualizing complexes and let* $I = \prod \mathfrak{J}_k^{a_k}$ *be a formal effective* $\mathbb{Q}$*-linear combination of ideal sheaves on* $\operatorname{Spec} R$*. Suppose we have a* $\mathbb{Q}$*-Cartier divisor* $\Gamma \geq \rho^* K_R$ *with* $n\Gamma = \operatorname{div} g$*. Then*

$$\operatorname{Tr}_{S/R}(\rho_*\mathcal{J}(\omega_S, \Gamma, IS)) \subseteq \mathcal{J}(R, I).$$

In the case where $I = R$ *and* $\operatorname{Tr}_{S/R}(\rho_*\mathcal{J}(\omega_S, \Gamma)) = R$*, this implies that* R *has klt type.*

Proof. Without lost of generality, we can assume $R \to S$ is a map of local rings. Fix $K_R \geq 0$ and note that we can assume $R \to S$ is generically Galois. Indeed, let S' be a the normalization of S inside a Galois closure of $K(S)$ over $K(R)$. This a finite extension, so by replacing Γ with its pullback along this extension and invoking Lemma 4.1.16, we can assume $R \to S$ is generically Galois.

Because $\operatorname{div} g \geq n\rho^* K_R$ we know that $g \in S(-n\rho^* K_R)$. Choosing a general element $f \in S(-n\rho^* K_R)$, Proposition 4.1.14 implies

$$\mathcal{J}(\omega_S, g^{1/n}, IS) \subseteq \mathcal{J}(\omega_S, S(-n\rho^* K_R)^{1/n}, IS) = \mathcal{J}(\omega_S, f^{1/n}, IS).$$

Let $G = \operatorname{Gal}(K(S)/K(R))$ and let

$$h = \prod_{\sigma \in G} \sigma(f)$$

Then we claim that

$$\mathcal{J}(\omega_S, h^{1/(n|G|)}, IS) = \mathcal{J}(\omega_S, f^{1/n}, IS)$$

To see this, let Y be a G-equivariant log resolution of singularities of $(\operatorname{Spec} S, S(-n\rho^* K_R), I)$ (see [Tem23, Remark 2.1.5(ii)]). Then for any $\sigma \in G$ we have the following commutative square

$$\begin{array}{ccc} R(-nK_R)S & \longrightarrow & S(-n\rho^* K_R) \\ \downarrow \sigma & & \downarrow \sigma \\ R(-nK_R)S & \longrightarrow & S(-n\rho^* K_R) \end{array}$$

The horizontal maps are given by reflexification and are thus functorial and so since the lefthand vertical map is an isomorphism, so is the righthand vertical map. Thus, the ideal $S(-n\rho^* K_R)$ is stable under the Galois action. We note that for $S(-n\rho^* K_R) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, this implies that for any $\sigma \in G$,

$$\left\lfloor \frac{1}{n} F \right\rfloor = \left\lfloor \frac{1}{n} \sigma F \right\rfloor = \left\lfloor \frac{1}{n} \operatorname{div}_Y f \right\rfloor$$

and so the portion of $\frac{1}{n} \operatorname{div}_Y f$ that is not fixed by the Galois action will have coefficients < 1. This implies that $\left\lfloor \frac{1}{n|G|} \operatorname{div}_Y h \right\rfloor = \left\lfloor \frac{1}{n} \operatorname{div}_Y f \right\rfloor$, proving the claim that $\mathcal{J}(\omega_S, h^{1/(n|G|)}, IS) = \mathcal{J}(\omega_S, f^{1/n}, IS)$. Then because h is Galois invariant and thus in $R(-n|G|K_R)$, an application of Lemma 4.1.16 gives

$$\rho_* \mathcal{J}(\omega_S, h^{1/n|G|}, IS) \xrightarrow{\operatorname{Tr}_{S/R}} \mathcal{J}(\omega_R, h^{1/n|G|}, I) \subseteq \mathcal{J}(\omega_R, R(-n|G|K_R)^{1/n|G|}, I) \subseteq R.$$

and thus $\operatorname{Tr}_{S/R}(\rho_* \mathcal{J}(\omega_S, g^{1/n}, IS) \subseteq \mathcal{J}(\omega_R, R(-n|G|K_R)^{1/n|G|}, I)$. By Proposition 4.1.7 and the chain of equalities in Remark 4.1.9,

$$\mathcal{J}(\omega_R, R(-n|G|K_R)^{1/n|G|}, I) \subseteq \mathcal{J}(R, I),$$

with equality if we choose n such that $n|G|$ is sufficiently divisible. □

Our goal now is to show that given a $\pi : Y \to X$ a log regular alteration of the pair (X, I) and $\phi : \pi_*\omega_Y(-E_Y) \to \mathcal{O}_Y$, we can find a divisor on a finite cover of X such that the trace of the multiplier submodule corresponding to that divisor agrees with the image of ϕ.

Lemma 4.1.18. *Let $X = \operatorname{Spec} R$ be a normal, excellent, noetherian scheme over $\operatorname{Spec} \mathbb{Q}$ with a dualizing complex and consider $\pi : Y \to X$ a log regular alteration of the pair (X, I) and $\phi : \pi_*\omega_Y \to \mathcal{O}_X$. Let $\rho : Z \to X$ be the finite portion of the Stein factorization. Then we can find a principal divisor $\Gamma = \operatorname{div}(h) \geq \rho^* K_X$ on Z such that*

$$\operatorname{Im}\left(\operatorname{Tr} : \rho_* \mathcal{J}(\omega_Z, \Gamma, I\mathcal{O}_Z) \to \mathcal{O}_X\right) = \operatorname{Im}\left(\phi : \pi_*\omega_Y(-E_Y) \to \mathcal{O}_X\right).$$

Proof. Fix $K_X \geq 0$. Because X is S_2, ϕ factors through $(\pi_*\omega_Y)^{S_2} \cong \rho_*\omega_Z$, so by abuse of notation, we consider $\phi : \rho_*\omega_Z \to \mathcal{O}_X$. By a standard Grothendieck duality argument we see

$$\begin{aligned}
\rho_*\mathcal{O}_Z(-\rho^*K_X) &\simeq \rho_* \mathcal{H}om_Z(\omega_Z \otimes_Z \rho^*\omega_X, \omega_Z) \\
&\simeq \mathcal{H}om_X(\rho_*\omega_Z \otimes_X \omega_X, \omega_X) \\
&\simeq \mathcal{H}om_X(\rho_*\omega_Z, \mathcal{O}_X)
\end{aligned}$$

Taking global sections, we get that $\Gamma(Z, \mathcal{O}_Z(-\rho^*K_X)) \simeq \operatorname{Hom}_X(\rho_*\omega_Z, \mathcal{O}_X)$. Tracing through the sequence of isomorphisms, we claim that this isomorphism sends $s \in \Gamma(Z, \mathcal{O}_Z(-\rho^*K_X))$ to $\operatorname{Tr}_{Z/X}(s-)$. To see this, note that, at the level of global sections, the first isomorphism above sends s to the multiplication-by-s map $\mu_s : \omega_Z \otimes_Z \rho^*\omega_X \to \omega_Z$. The second isomorphism sends μ_s to $\operatorname{Tr}_{Z/X} \circ \rho_*\mu_s = \operatorname{Tr}_{Z/X}(s-)$ via Grothendieck duality. The final isomorphism can be understood by viewing $\operatorname{Tr}_{Z/X}(s-) : K(Z) \to K(X)$ and thinking of the isomorphism as restriction to different $\mathcal{O}_X$-submodules of $K(Z)$.

Then, let $\phi = \operatorname{Tr}_{Z/X}(h-)$ for $h \in \Gamma(Z, \mathcal{O}_Z(-\rho^*K_X))$. Viewing h as an element of $\mathcal{O}_Z$, we get a divisor $\operatorname{div} h$ such that $\operatorname{div} h \geq \rho^*K_X$ and, by the projection formula, such that

$$\phi(\pi_*\omega_Y(-E_Y)) = \operatorname{Tr}(h \cdot \pi_*\omega_Y(-E_Y)) = \operatorname{Tr}(\rho_*\mathcal{J}(\omega_Z, \operatorname{div} h, I\mathcal{O}_Z)). \qquad \square$$

We are now ready to show the reverse containment holds.

Proposition 4.1.19. *With notation as above*

$$\mathcal{J}(X) \supseteq \sum_{\pi : Y \to X} \operatorname{Im}\left(\mathcal{H}om_X(\pi_*\omega_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \pi_*\omega_Y(-E_Y) \to \mathcal{O}_X\right)$$

Proof. Since both these objects are $\mathcal{O}_X$-submodules, we can check containment locally. By Lemma 4.1.18, given $\pi : Y \to X$ a log regular alteration of the pair (X, I) and

$\phi : \pi_*\omega_Y \to \mathcal{O}_X$, we can find a finite cover $Z \to X$ and a divisor Γ on Z such that $\operatorname{Tr}(\mathcal{J}(\omega_Z, \Gamma, I)) = \phi(\pi_*\omega_Y(-E_Y))$. Then by Proposition 4.1.17 we can find Δ on X such that $\operatorname{Im}\phi \subseteq \mathcal{J}(X, \Delta, I) \subseteq \mathcal{J}(X, I)$. □

Together, Lemmas 4.1.15 and 4.1.19 give us our main result.

Theorem 4.1.20 (Theorem 4.0.1)**.** *Let X be a normal, excellent, noetherian scheme over* $\operatorname{Spec}\mathbb{Q}$ *with a dualizing complex and $I = \prod \mathcal{J}_k^{a_k}$ be a formal effective $\mathbb{Q}$-linear combination of ideal sheaves on X. The multiplier ideal $\mathcal{J}(X, I)$ in the sense of [dFH09] can be realized as*

$$\mathcal{J}(X, I) = \sum_{\pi : Y \to X} \operatorname{Im}\left(\mathcal{H}om_X(\pi_*\omega_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \pi_*\omega_Y(-E_Y) \to \mathcal{O}_X\right)$$

where $\pi : Y \to X$ ranges over all log regular alterations of (X, I) with $E_Y = \sum a_k E_k$ for $\mathcal{J}_k\mathcal{O}_Y = \mathcal{O}_Y(-E_k)$ and the map $\mathcal{H}om_X(\pi_*\omega_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \pi_*\omega_Y(-E_Y) \to \mathcal{O}_X$ is the evaluation map.

As a corollary, we deduce the following derived splinter characterization of klt singularities:

Corollary 4.1.21. *Let X be a normal, excellent, noetherian scheme over* $\operatorname{Spec}\mathbb{Q}$ *with a dualizing complex and $I = \prod \mathcal{J}_k^{a_k}$ be a formal effective $\mathbb{Q}$-linear combination of ideal sheaves on X. The following are equivalent*

1. *(X, I) has klt type*
2. *For all sufficiently large regular alterations $\pi : Y \to X$, the natural map $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$ splits and locally factors through $R\pi_*\omega_Y(-E_Y) = \pi_*\omega_Y(-E_Y)$*
3. *For all sufficiently large regular alterations $\pi : Y \to X$, $\mathcal{O}_X$ is locally a summand of $R\pi_*\omega_Y(-E_Y) = \pi_*\omega_Y(-E_Y)$*

Here, $E_Y := s\sum a_k E_k$ where $\mathcal{O}_Y(-E_Y) = \mathcal{J}_k\mathcal{O}_Y$.

Proof. By Theorem 4.0.1, $(2) \Longrightarrow (1) \Longleftrightarrow (3)$. To see $(1) \Longleftrightarrow (3) \Longrightarrow (2)$, let $\pi : Y \to X$ a regular log alteration of (X, Δ, I) and localize so that $\mathcal{O}_X$ is a summand of $\pi_*\omega_Y$. Then because X has klt type and thus rational singularities we can find maps p_1, p_2, and i such that the following composition is the identity:

$$\mathcal{O}_X \longrightarrow R\pi_*\mathcal{O}_Y \xrightarrow{p_1} \mathcal{O}_X \xrightarrow{i} \pi_*\omega_Y(-E_Y) \xrightarrow{p_2} \mathcal{O}_X.$$

□

4.2 Positive characteristic

4.2.1 Preliminaries

Let R be a noetherian ring of characteristic $p > 0$. We will denote by $F : R \to R$ the Frobenius morphism defined by $F(r) = r^p$ and denote by F^e the e-fold composition of the Frobenius with itself. Given an R-module M, we denote by $F^e_* M$ the R-module where R acts via restriction of scalars along the e-th iterate of the Frobenius. We say that R is *F-finite* if $F : R \to R$ is a finite map.

Because we do not have resolutions of singularities in characteristic p, we are forced to use different methods to study singularities. The Frobenius turns out to be a valuable tool, allowing us to relate singularities of rings and schemes to properties of their Frobenius endomorphism. In particular, we are able to define an analog of the multiplier ideal using such methods.

Definition 4.2.1. Let R be an F-finite, reduced ring. The *test ideal* $\tau(R)$ is the unique smallest ideal $J \subseteq R$, not contained in any minimal prime of R, such that $\phi(F^e_* J) \subseteq J$ for every $\phi \in \mathrm{Hom}(F^e_* R, R)$ and every $e > 0$.

This leads to a definition of strongly F-regular singularities, the positive characteristic analog of klt singularities.

Definition 4.2.2. Let R be an F-finite, reduced ring. Then R is *strongly F-regular* if $\tau(R) = R$.

Similarly, we are able to define analogs of multiplier submodules. Suppose R is an F-finite, locally equidimensional reduced ring with canonical module ω_R. Consider the e-th iterate of the Frobenius $R \to F^e_* R$ and apply the functor $\mathrm{Hom}(-, \omega_R)$. This gives us a map

$$\mathrm{Hom}(F^e_* R, \omega_R) \to \omega_R$$

and using that $\mathrm{Hom}(F^e_* R, \omega_R) \cong F^e_* \omega_R$, we get a map $T^e : F^e_* \omega_R \to \omega_R$ that is dual to the Frobenius.

Definition 4.2.3 ([Smi95], [Bli04], [ST08])**.** Let R be an F-finite, locally equidimensional ring with canonical module ω_R and $T : F_* \omega_R \to \omega_R$ dual to the Frobenius. Then the *parameter test submodule* $\tau(\omega_R)$ is the unique smallest submodule $M \subseteq \omega_R$, nonzero at any minimal prime of R, such that

$$T(F_* M) \subseteq M$$

Ultimately, our result in characteristic $p > 2$ is a corollary to the following result of [ES14] combined with the existence of quasi-Gorenstein finite covers.

Proposition 4.2.4 ([ES14] Corollary 6.5)**.** *If R is an F-finite reduced ring of characteristic $p > 0$ and $R \subseteq S$ is a finite extension with S reduced, then*

$$\tau(R) = \sum_{e \geq 0} \sum_{\phi} \phi(F^e_* \tau(S))$$

where ϕ ranges over all elements of $\operatorname{Hom}_R(F^e_* S, R)$.

We give an overview of the construction of quasi-Gorenstein finite covers which we expect is well-known to experts. See, for example, the construction in the proof of [Kaw88, Theorem 8.5]. We include the argument here for the convenience of the reader.

Lemma 4.2.5. *Let R be a normal commutative Noetherian ring of essentially finite type over a field k of characteristic not equal to 2. Then $\operatorname{Spec} R$ has a quasi-Gorenstein finite cover.*

Proof. We first claim that we can find $H \sim -2K_R$ sufficiently general and hence reduced. In characteristic zero, this follows from Bertini's theorem for basepoint-free linear systems, here thinking of $|-2K_R|$ as a linear system on the regular locus of $\operatorname{Spec} R$. In characteristic $p > 0$, let $X \to \operatorname{Spec} R$ be the normalized blowup of $\omega_R^{(-2)}$. Then because $\omega_R^{(-2)} \mathcal{O}_X \cong \mathcal{O}_X(-G)$ is very ample, if k is infinite, Bertini's theorem tells us that a general divisor $H' \sim -G$ will be normal, hence reduced. Pushing H' down to H on $\operatorname{Spec} R$, we get that H is reduced and $H \sim -2K_R$, as this holds outside a codimension 2 subset of $\operatorname{Spec} R$ and R is normal. If k is not infinite, we can find such an H on $R \otimes_k \overline{\mathbb{F}_p}$ and thus on $R \otimes_k \mathbb{F}_{p^e}$ for some e. Since we are only looking for a finite cover of $\operatorname{Spec} R$ at the end of the day, we can assume we have such an H by working on $R \otimes_k \mathbb{F}_{p^e}$.

Given such an H with $H + 2K_R = \operatorname{div} f$, let

$$S = R \oplus \omega_R = R \oplus \omega_R t$$

and define multiplication as $(a, b)(x, y) = (ax + byf, ay + bx)$. Since R is regular outside of a set of codimension two, ω_R will be a line bundle outside of this set and $\omega_R^{(2)} \cong R(-H)$ (via multiplication by f) on the smooth locus. Because R is S_2, this isomorphism will hold everywhere and so we get an R-algebra that is finite as an R-module. Furthermore, since f is reduced, S is normal.

We then claim that S is quasi-Gorenstein. To see this, note that $\omega_S \simeq \operatorname{Hom}_R(S, \omega_R)$. However, since S is normal and hence S_2, $\omega_S = \operatorname{Hom}_R(S, \omega_R) \simeq S$ and thus S is quasi-Gorenstein. ∎

4.2.2 Results

In this section, we assume that R is a normal, F-finite, Noetherian ring of characteristic $p > 2$ and let $X = \operatorname{Spec} R$. By excluding characteristic 2, we have quasi-Gorenstein finite covers by Lemma 4.2.5, a key ingredient in showing that the multiplier ideal contains the image of any map $\pi_*\omega_Y \to \mathcal{O}_X$. We then have the following analogous description of the test ideal.

Proposition 4.2.6. *Let R be a Noetherian, F-finite reduced ring of characteristic $p > 2$. The test ideal $\tau(R)$ can be realized as*

$$\tau(R) = \sum_{R \to S} \operatorname{Im}\left(\operatorname{Hom}_R(\omega_S, R) \otimes_R \tau(\omega_S) \to R\right)$$

where the sum ranges over all finite extensions $R \to S$ and $\operatorname{Hom}_R(\omega_S, R) \otimes_R \tau(\omega_S) \to R$ is the evaluation map.

Proof. Consider $\phi : \tau(\omega_S) \to R$ and consider $S \to S'$ a quasi-Gorenstein finite cover. Then by a restatement of the main theorem of [ST14] we get that $\phi \circ \operatorname{Tr}_{S'/S}(\tau(\omega_{S'})) = \phi(\tau(\omega_S))$. Thus we can restrict ourself to only considering quasi-Gorenstein covers in the sum.

Checking equality locally, consider a quasi-Gorenstein cover S with an isomorphism $\omega_S \simeq S$ (and thus $\tau(\omega_S) \simeq \tau(S)$). $R \to F^e_* S$ is finite for all e and so by Proposition 4.2.4 and the above observation we have

$$\tau(R) = \sum_{e \geq 0} \sum_{\phi} \phi(F^e_* \tau(S)) = \sum_{e \geq 0} \sum_{\phi} \phi(F^e_* \tau(\omega_S)) = \sum_{R \to S'} \sum_{\phi} \phi(\tau(\omega_{S'}))$$

where the final sum again ranges over all $R \hookrightarrow S'$ finite and $\phi \in \operatorname{Hom}_R(\omega_{S'}, R)$.

□

As a corollary, we get the following characterization of strongly F-regular singularities.

Corollary 4.2.7. *Let R be a Noetherian, F-finite reduced ring of characteristic $p > 2$. Then R is strongly F-regular if and only if R is a summand of $\tau(\omega_S)$ for any sufficiently large finite cover $\operatorname{Spec} S \to \operatorname{Spec} R$.*

Proof. R is strongly F-regular if and only if $\tau(R) = R$. By the previous proposition, this implies there is some finite $R \to S$ and a surjection $\tau(\omega_S) \to R$ which must split as a map of R-modules. The reverse direction follows from Proposition 4.2.6 by letting $\pi : \tau(\omega_S) \to R$ be the projection onto the summand, implying $\tau(R) = R$. □

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