Math 180 Worksheet 15: Extra Problems!

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Problem (1)**.** Sketch the following function using the steps mentioned in the worksheet!

$$f(x) = \frac{x^2 - x - 2}{(x-1)^2}.$$

Solution. Let's start off by noting some stuff about the function f.

- $f(x) = 0$ at $x = -1$ and $x = 2$.
- Now we calculate $f'(x)$ and $f''(x)$, which turn out to be:[1]

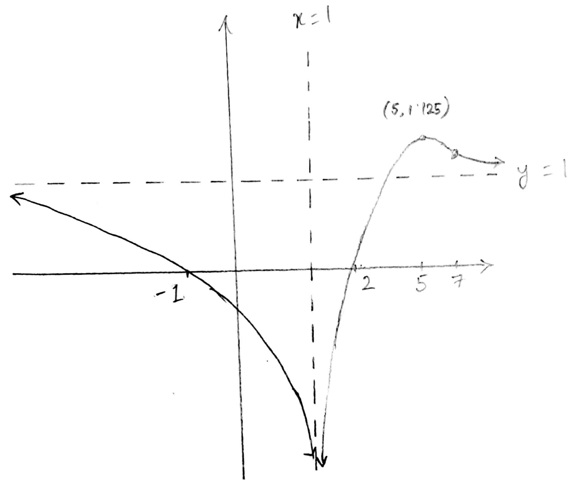
$$f'(x) = \frac{5-x}{(x-1)^3},\ f''(x) = \frac{2(x-7)}{(x-1)^4}.$$

- We have a vertical asymptote at $x = 1$. The limit as $x \to 1^-$ and $x \to 1^+$ are both $-\infty$, and the limits $\lim\limits_{x\to\pm\infty} f(x)$ are both 1. Therefore f has the horizontal asymptote $y = 1$ at both ends.
- Critical points. $f'(x) = 0$ at $x = 5$, and $f'(x)$ is undefined at $x = 1$. Therefore 1 and 5 are our critical points.
- $f'(x) < 0$ on the intervals $(-\infty, 1) \cup (5, \infty)$ and $f'(x) > 0$ on the interval $(1, 5)$. So we know the intervals where f is decreasing and where it's decreasing.
- $f''(x) < 0$ on the intervals $(-\infty, 1) \cup (1, 7)$ and $f''(x) > 0$ on $(7, \infty)$. So we know the intervals where f is concave up and down, and it has a point of inflection at $x = 7$.

With all this information, we're all set to draw the graph of f.

□

[1] To calculate these, don't foil out $(x-1)^2$, and factor out the highest power of $(x-1)$ you can, after doing the quotient rule, instead.



Problem (3)**.** You want to enclose a right triangular pen with a fixed area of 800 ft^2 with a fence. What is the minimum length of fence required?

Solution. Let b and h be the base and height of the right triangle, then the area is $\frac{1}{2}bh = 800$, therefore $b = \frac{1600}{h}$. The perimeter of the triangle, which is the objective function, is exactly

$$p = b + h + \sqrt{b^2 + h^2} = h + \frac{1600}{h} + \sqrt{h^2 + \left(\frac{1600}{h}\right)^2}.$$

The recommended way is to take the derivative of this function, equate it to zero, and find the root. There is another way we can go about this, though, which is, to complete the squares!

Note that,

$$p(h) = \left(\sqrt{h} - \frac{40}{\sqrt{h}}\right)^2 + 80 + \sqrt{\left(h - \frac{1600}{h}\right)^2 + 2 \cdot 1600}$$

Each of the square terms in the brackets are ≥ 0. That means,

$$p(h) \geq 80 + \sqrt{2 \cdot 1600} = 80 + 40\sqrt{2}.$$

So the perimeter is at least the value above, and this smallest value can only be attained when $\sqrt{h} = \frac{40}{\sqrt{h}}$ **and** $h = \frac{1600}{h}$. Fortunately, these two conditions give us the common solution of $h = 40$. And so the minimum value of the perimeter is attained at $h = 40$, and therefore $b = 40$. The minimum perimeter is $80 + 40\sqrt{2}$. □