# Math 181 Extra Problems: $u$-substitution

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**Problem 1.** Find the following antiderivative:

$$\int \frac{x^2-1}{x(x^2+1)}\, dx.$$

*Solution.* We shall use $u$-substitution after noting that

$$\int \frac{x^2-1}{x(x^2+1)}\, dx = \int \frac{(x^2-1)x}{x^2(x^2+1)}\, dx,$$

Which suggests us to substitute $u = x^2$. Indeed, this means $du = 2xdx$, and thus $dx = \frac{du}{x}$. Then, we have to find

$$\int \frac{(u-1)}{u(u+1)}\, du.$$

To find this, we need to use partial fractions! Note that

$$\int \frac{u-1}{u(u+1)}\, du = \int \frac{2u-(u+1)}{u(u+1)}\, du = \int \left(\frac{2}{u+1} - \frac{1}{u}\right)\, du,$$

Which tells us

$$\int \frac{x^2-1}{x(x^2+1)}\, dx = 2\ln(|u+1|) - \ln(|u|) + c = 2\ln(x^2+1) - \ln(x^2) + c.$$

□

**Problem 2.** Find

$$\int \frac{x^2-1}{x^2(x^2+3x+1)}\, dx.$$

*Solution.* This is a hard integral. You need to use a very clever substitution that is almost impossible to notice! But, notice that

$$\int \frac{x^2-1}{x(x^2+3x+1)}\, dx = \int \frac{\frac{x^2-1}{x}}{\frac{x(x^2+3x+1)}{x}}\, dx = \int \frac{x-\frac{1}{x}}{x\left(x+3+\frac{1}{x}\right)}\, dx.$$

Let $u = x + \frac{1}{x}$, then $du = (1-\frac{1}{x^2})dx$, ie $du = \frac{x^2-1}{x^2}dx$. Magically, $\frac{x-\frac{1}{x}}{x} = \frac{x^2-1}{x^2}$!

This point on it's just algebra that tells us

$$\int \frac{x-\frac{1}{x}}{x\left(x+3+\frac{1}{x}\right)}\, dx = \int \frac{du}{u+3} = \ln|u+3| + c,$$

So we have

$$\int \frac{x^2-1}{x^2(x^2+3x+1)}\, dx = \ln\left|x+\frac{1}{x}+3\right| + c.$$

□

**Problem 3.** Find

$$\int \tan^3 x \ dx.$$

*Solution.* There are several ways of approaching this problem. Let's try the following:

$$\int \tan^3 x \ dx = \int \frac{\sin^3 x}{\cos^3 x} \ dx = \int \frac{\sin^2 x \cdot \sin x}{\cos^3 x} \ d\,x.$$

This suggests us to take $u = \cos x$, and thus $du = -\sin x dx$. However, there is an extra $\sin^2 x$ in the numerator, which we need to express in terms of $u = \cos^2 x$. The key to this is to use the trig identity $\sin^2 x = 1 - \cos^2 x$, which means $\sin^2 x = 1 - u^2$. Then our integral just becomes

$$\int \frac{1 - u^2}{u^3} \ (-du) = -\int \left( \frac{1}{u^3} - \frac{1}{u} \right) \ du = -\int (u^{-3} - u^{-1}) \ du = -\left( \frac{u^{-2}}{-2} - \ln u \right) + c,$$

So

$$\int \tan^3 x \ dx = \frac{1}{2}(\cos x)^{-2} + \ln|\cos x| + c.$$

□