Worksheet 12 - Intro to Series

1. (a) $a_n = \dfrac{4(-1)^{n-1}}{5^{n-1}} = 4\left(-\frac{1}{5}\right)^{n-1}$

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 4\left(-\frac{1}{5}\right)^{n-1}$

But, $-\dfrac{4}{5^{n-1}} \leq a_n \leq \dfrac{4}{5^{n-1}}$ as $-1 \leq (-1)^{n-1} \leq 1$

$\lim\limits_{n\to\infty} -\dfrac{4}{5^{n-1}} = 0$, $\lim\limits_{n\to\infty} \dfrac{4}{5^{n-1}} = 0$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0.$ (squeeze thm)

(c) $r = -\frac{1}{5}$, $|r| < 1$ so this geometric series converges.

$a = 4$ $\therefore$ Sum $= \dfrac{a}{1-r} = \dfrac{4}{1+1/5} = \dfrac{4}{6/5} = \dfrac{20}{6} = \dfrac{10}{3}.$

2. (a) k^{th} partial sum $S_k = \sum\limits_{n=2}^{k} \dfrac{1}{n(n-1)}.$

By partial fractions, $\dfrac{1}{n(n-1)} = \dfrac{1}{n-1} - \dfrac{1}{n}$

$\therefore S_k = \sum\limits_{n=2}^{k} \left(\dfrac{1}{n-1} - \dfrac{1}{n}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{k-2}} - \cancel{\tfrac{1}{k-1}}\right) + \left(\cancel{\tfrac{1}{k-1}} - \dfrac{1}{k}\right)$

$= 1 - \dfrac{1}{k}$

So $\lim\limits_{k\to\infty} S_k = \lim\limits_{k\to\infty} 1$, hence $\sum\limits_{n=2}^{\infty} \dfrac{1}{n(n-1)} = \lim\limits_{k\to\infty} S_k = 1.$

(converges).

(b) k^{th} partial sum : $\sum_{n=1}^{k} \ln\left(\frac{n+1}{n}\right)$.

$$\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$$

$$\therefore S_k = \sum_{n=1}^{k} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{k} \left(\ln(n+1) - \ln(n)\right)$$

$$= (\cancel{\ln 2} - \ln 1) + (\cancel{\ln 3} - \cancel{\ln 2}) + \dots + (\cancel{\ln(k)} - \cancel{\ln(k-1)}) + (\ln(k+1) - \cancel{\ln(k)})$$

$$= -\ln 1 + \ln(k+1) = \ln(k+1)$$

$$\therefore \lim_{n\to\infty} S_k = \lim_{k\to\infty} \ln(k+1) = \infty$$

So, $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges!

3. (a) $S = \sum_{k=0}^{12} 2^k \Rightarrow 2S = \sum_{k=0}^{12} 2\cdot 2^k = \sum_{k=0}^{12} 2^{k+1}$

So,

$$S = 2^0 + 2^1 + 2^2 + \dots + 2^{12}$$

$$2S = \quad 2^1 + 2^2 + \dots + 2^{12} + 2^{13}$$

$$\Rightarrow S - 2S = 2^0 - 2^{13}$$

$$\therefore -S = 1 - 2^{13} \Rightarrow S = 2^{13} - 1 = 1024 \cdot 8 - 1$$

$$= 8192 - 1 = 8191$$

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(b) $S = \sum_{n=1}^{10} \left(\frac{4}{7}\right)^n$, $\frac{4}{7} S = \sum_{n=2}^{10} \left(\frac{4}{7}\right)^{n+1}$

$$S = \frac{4}{7} + \left(\frac{4}{7}\right)^2 + \dots + \left(\frac{4}{7}\right)^{10}$$

$$\frac{4}{7} S = \left(\frac{4}{7}\right)^2 + \dots + \left(\frac{4}{7}\right)^{10} + \left(\frac{4}{7}\right)^{11}$$

$$S - \frac{4}{7} S = \frac{4}{7} - \left(\frac{4}{7}\right)^{11}$$

$$\frac{3}{7} S = \frac{4}{7} - \left(\frac{4}{7}\right)^{11} \Rightarrow S = \left[\frac{\frac{4}{7} - \left(\frac{4}{7}\right)^{11}}{3/7}\right].$$

(c) First one has $r = 2$, $|r| \geq 1 \longrightarrow$ diverges

Second one has $r = \frac{4}{7}$, $|r| < 1 \rightarrow$ converges

4. (a) $$\sum_{n=1}^{\infty} \frac{(-4)^n}{9^n}$$

first few terms : $-\frac{4}{9}, \frac{16}{81}, \dots$

$$a = -\frac{4}{9}, \; r = -\frac{4}{9}$$

Converges as $|r| = \frac{4}{9} < 1$, and sum is $\frac{a}{1-r} = \frac{-\frac{4}{9}}{1+\frac{4}{9}} = \frac{-4}{13}$.

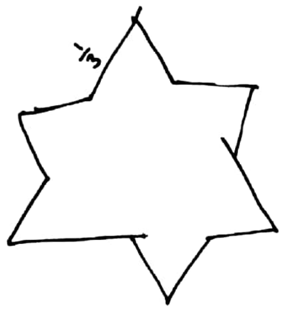
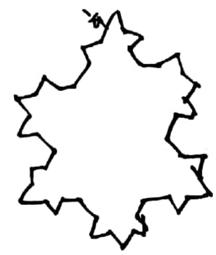
(b) $$\sum_{n=1}^{\infty} \frac{(-4)^{2n}}{3^n} = \sum_{n=1}^{\infty} \left(\frac{(-4)^2}{3}\right)^n = \sum_{n=1}^{\infty} \left(\frac{16}{3}\right)^n.$$

$a = \frac{16}{3}$, $r = \frac{16}{3}$, $|r| > 1 \rightarrow$ series diverges.

5. $0.\overline{456} = 0.456456456456\ldots$

$$= \frac{456}{1000} + \frac{456}{1000^2} + \frac{456}{1000^3} + \ldots$$

$$= \frac{456}{\left(1-\frac{1}{1000}\right)} = \frac{\overset{152}{\cancel{456}}}{\underset{333}{\cancel{999}}}\cdot 1000 = \frac{152000}{\cancel{999}\;333}$$

6. I_0 :  , I_1 =  , I_2 = 

$L_0 = 3$ $\quad L_1 = 12\cdot\frac{1}{3} = 3\cdot\frac{4}{3}$, $\quad L_3 = 48\cdot\frac{1}{9} = 3\cdot\frac{16}{9}, \ldots$

To obtain ~~L_{n+1} from L_n we first~~

So we guess the general term $L_n = 3\cdot\left(\frac{4}{3}\right)^{n-1}$.

Then $\lim_{n\to\infty} L_n = \infty$ as $\frac{4}{3} > 1$.

$$A_0 = \frac{\sqrt{3}}{4}(1)^2 = \frac{\sqrt{3}}{4}$$

$$A_1 = \frac{\sqrt{3}}{4} + 3\,(\text{area of smaller triangle})$$
$$= \frac{\sqrt{3}}{4} + 3\cdot\frac{\sqrt{3}}{4}\cdot\left(\frac{1}{3}\right)^2$$
$$= \frac{\sqrt{3}}{4}\left(1+\frac{1}{3}\right) = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}$$

$$A_2 = \frac{\sqrt{3}}{4}\cdot\frac{4}{3} + 12\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^2$$
$$= \frac{\sqrt{3}}{4}\left(\frac{4}{3}+\frac{12}{81}\right)$$
$$= \frac{\sqrt{3}}{4}\left(\frac{4}{3}+\frac{4}{27}\right)$$
$$= \frac{\sqrt{3}}{4}\left(\frac{40}{27}\right)$$

$$A_3 = A_2 + 48\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{27}\right)^2$$
$$= \frac{\sqrt{3}}{4}\left(\frac{4}{3}+\frac{4}{27}+\frac{16}{243}\right)$$

$$\Rightarrow A_n = \frac{\sqrt{3}}{4}\left(\frac{4}{3}+\frac{4}{27}+\frac{4}{243}+\ldots\right)$$
$$= \frac{\sqrt{3}}{4}(1) + \frac{\sqrt{3}}{4}\left(\frac{1}{3}+\left(\frac{4}{9}\right)\cdot\frac{1}{3}+\left(\frac{4}{9}\right)^2\cdot\frac{1}{3}+\ldots\right)$$

So $\lim_{n\to\infty} A_n = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\cdot\frac{1/3}{1-4/9} = \frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\cdot 3/5$

$$= \frac{2\sqrt{3}}{5}.$$

Worksheet-13

1. $\lim_{k\to\infty} a_k = 1 \neq 0 \rightarrow \sum_{k=1}^{\infty} a_k$ diverges.

2. $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So this is false.

3. p-test: $\sum_{k=10}^{\infty} \frac{1}{k^p}$ converges if $0<p<1$
diverges if $p \geq 1$ (or $p \leq 0$).

4. (a) $\sum_{k=1}^{\infty} \frac{5}{k} = 5\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

(b) $\sum_{k=1}^{\infty} \frac{k^3-1}{k^3+1}$: $\lim_{k\to\infty} \frac{k^3-1}{k^3+1} = 1$, so the series diverges.

(c) $\sum_{k=2}^{\infty} \frac{k^e}{k^\pi} = \sum_{k=2}^{\infty} \frac{1}{k^{\pi-e}}$, $p=\pi-e \approx 3.14-2.7 < 1$, diverges!

5. (a) $\sum_{k=1}^{\infty} \frac{5}{\sqrt{k}} = 5\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 5\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$. $p=\frac{1}{2}<1$, diverges.

(b) $\sum_{k=1}^{\infty} \left(e^{\frac{1}{k}} - e^{\frac{1}{k+1}}\right)$

1st term $\downarrow$ $\quad$ $(n-1)^{th}$ term $\downarrow$ $\quad$ n^{th} term $\downarrow$

$S_n =$ n^{th} partial sum $= (e^1 - \cancel{e^{\frac{1}{2}}}) + (\cancel{e^{\frac{1}{2}}} - \cancel{e^{\frac{1}{3}}}) + \dots + (\cancel{e^{\frac{1}{n-1}}} - \cancel{e^{\frac{1}{n}}}) + (\cancel{e^{\frac{1}{n}}} - e^{\frac{1}{n+1}})$

$= e - e^{\frac{1}{n+1}}$

$\therefore \sum_{k=1}^{\infty} \left(e^{1/k} - e^{1/(k+1)}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} e - e^{\frac{1}{n+1}} = e-1.$ converges

(c) $\sum_{k=1}^{\infty} \frac{2^{k+1}+(-2)^k}{5^k} = \sum_{k=1}^{\infty} \frac{2^{k+1}}{5^k} + \sum_{k=1}^{\infty} \frac{(-2)^k}{5^k} = \sum_{k=1}^{\infty} 2\left(\frac{2}{5}\right)^k + \sum_{k=1}^{\infty}\left(\frac{-2}{5}\right)^k.$

Both are geometric series with $|r|<1$, so both converge.
Converges.

6. (a) True : divergence test

(b) False : $\{\frac{1}{n}\}$ converges but $\sum_{n\geq 1}^{\infty} \frac{1}{n}$ diverges

(c) False : $p = 1$, diverges

(d) True : $\lim_{n\to\infty} \frac{1}{n} = 0$, thus $\{\frac{1}{n}\}$ converges

(e) False : $r = 1/5$ in this problem.

$a_1 = \frac{3}{5^0} = 3$, $a_2 = \frac{3}{5^1} = \frac{3}{5} = 3 \cdot \frac{1}{5}$, $a_3 = \frac{3}{5^2} = 3 \cdot \left(\frac{1}{5}\right)^2$,

etc.

Worksheet - 14

1. (a)

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{(n+1)} = \frac{n+1}{2^n}$$

converges

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \underbrace{\frac{n+1}{2^n}}_{\frac{\infty}{\infty}} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0 < 1$$

(b)

$$\frac{a_{n+1}}{a_n} = \frac{3^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{3^{2n}} = \frac{3^{2n+2}}{\cancel{n!}\,(n+1)} \cdot \frac{\cancel{n!}}{3^{2n}}$$

$$= \frac{3^{2n+2-2n}}{n+1} = \frac{9}{n+1}.$$

converges

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{9}{n+1} = 0 < 1$$

(c)

$$\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{5^{n+1}} \cdot \frac{5^n}{(2n)!}$$

$$= \frac{\cancel{1.2 \ldots 2n}\,.\,(2n+1)(2n+2)}{\cancel{5^n}.5} \cdot \frac{\cancel{5^n}}{\cancel{1.2 \ldots 2n}}$$

$$= \frac{(2n+1)(2n+2)}{5}$$

diverges.

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty > 1$$

2. (a) $0 \le a_n < b_n$ for all n and Σa_n converges

$\Rightarrow$ Σa_n converges.

(The partial sums $\{s_k\}$ increase and are bounded above, so $\Sigma a_n = \lim_{k\to\infty} s_k$ exists.)

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(b) $a_n > b_n \geq 0$ for all n, then $\sum a_n$ may converge or diverge.

if the terms are still close to b_n but ~~[illegible]~~ bigger

if the terms get arbitrarily large

eg: $b_n = \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges

$a_n = \frac{2}{n^2} \Rightarrow \frac{2}{n^2} > \frac{1}{n^2}$ and $\sum a_n$ converges

$a_n = \frac{1}{n} \Rightarrow \frac{1}{n} > \frac{1}{n^2}$ and $\sum a_n$ diverges

3. $$\sum_{n=1}^{\infty} \frac{-1}{n^3-7n^2+17n+20} = -\sum_{n=1}^{\infty} \frac{1}{n^3-7n^2+17n+20} = -\sum_{n=1}^{\infty} a_n$$

Use limit comparison: $b_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^3-7n^2+17n+20} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{n^3}{n^3-7n^2+17n+20}$$

$$= 1$$

$0 < 1 < \infty$

$\sum b_n = \sum \frac{1}{n^3}$ converges by p-test

The terms a_n, b_n are positive.

$\therefore$ By LCT, $\sum a_n$ converges.

4. (a) $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n} \geq 0$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges } \overset{CT}{\Rightarrow} \sum_{n=1}^{\infty} \frac{n+1}{n^2} \text{ diverges.}$$

(b) For LCT, we use $b_n = \frac{n}{n^2} = \frac{1}{n}$

$\therefore \quad a_n, b_n \geq 0$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

$0 < L < \infty$.

So by LCT, $\sum a_n$ converges iff $\sum b_n$ converges

& $\sum a_n$ diverges if $\sum b_n$ diverges

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test ($p=1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges.

5. (a) LCT: use $b_n = \frac{n}{n^3} = \frac{1}{n^2}$

$\therefore \quad a_n, b_n \geq 0$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$0 < L < \infty$.

So by LCT, $\sum a_n$ converges if $\sum b_n$ converges

$\sum a_n$ diverges if $\sum b_n$ diverges

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test ($p=2$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{n+1}{n^3}$ converges.

(b) If we used $\frac{n+1}{n^3} > \frac{n}{n^3} = \frac{1}{n^2}$, then comparison test would not have given us anything, as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

- Comparison test: $a_n > b_n \geq 0$ and $\sum b_n$ diverges $\Rightarrow$ $\sum a_n$ diverges

  $\sum a_n$ converges $\Rightarrow$ $\sum b_n$ converges

- But, if we are a bit clever, we can still use the direct comparison test:

$$0 \leq \frac{n+1}{n^3} < \frac{n+n}{n^3} = \frac{2n}{n^3} = \frac{2}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$ also converges. $\square$