Worksheet-8 (181)

$d(uv) = u\,dv + v\,du$

$\int u\,dv = uv - \int v\,du$

1. $\int x^2 \sin(2x)\,dx$

$u = x^2 \qquad dv = \sin(2x)\,dx$

$du = 2x\,dx \qquad v = -\frac{\cos(2x)}{2}$

$$= x^2 \cdot \left(-\frac{\cos(2x)}{2}\right) - \int -\frac{\cos(2x)}{2} \cdot 2x\,dx$$

$$= -\frac{x^2\cos(2x)}{2} + \int x\cos(2x)\,dx$$

$u = x \qquad dv = \cos(2x)\,dx$

$du = dx \qquad v = \frac{\sin(2x)}{2}$

$$= -\frac{x^2\cos(2x)}{2} + x \cdot \frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx$$

$$= -\frac{x^2\cos(2x)}{2} + \frac{x\sin(2x)}{2} - \frac{1}{2}\int \sin(2x)\,dx$$

$$= -\frac{x^2\cos(2x)}{2} + \frac{x\sin(2x)}{2} - \frac{1}{2}\cdot\left(\frac{-\cos(2x)}{2}\right)$$

$$= -\frac{x^2\cos(2x)}{2} + \frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4} + C$$

2. $\int x^n \sin(x)\,dx$ would require n iterations of by-parts.

$\int x^n \sin x\,dx = -x^n \cos x + \int n x^{n-1} \cos x\,dx \Rightarrow$ one iteration

kills one power of x.

3. (a) $\displaystyle\int_1^e \ln(2x)\,dx$

$u = \ln(2x) \quad dv = dx$

$du = \frac{1}{2x} \cdot 2\,dx \quad v = x$

$= \frac{1}{x}\,dx.$

$$\int \ln(2x)\,dx = \int u\,dv$$

$$= x\ln(2x) - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln(2x) - \int 1\,dx$$

$$= x\ln(2x) - x + C.$$

$$\Rightarrow \int_1^e \ln(2x)\,dx = x\ln(2x) - x \Big|_{x=1}^{x=e}$$

$$= e\ln(2e) - e - \ln 2 + 1$$

$$= e(\ln 2 + \ln e) - e - \ln 2 + 1$$

$$= e\ln 2 - \ln 2 + 1.$$

(b) $\displaystyle\int_0^{\frac{1}{2}} \arccos(x)\,dx$

$u = \arccos(x), \quad dv = dx$

$du = -\frac{1}{\sqrt{1-x^2}}\,dx \quad v = x$

$$\int \arccos(x)\,dx = x\arccos(x) - \int -\frac{x}{\sqrt{1-x^2}}\,dx.$$

$$= x\arccos(x) + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = 1-x^2$

$\Rightarrow du = -2x\,dx \quad \Rightarrow \quad x\,dx = -\frac{du}{2}$

$$= x\arccos(x) + \int \frac{-du}{2\sqrt{u}}$$

$$= x\arccos(x) - \frac{1}{2}\int u^{-1/2}\,du$$

$$= x\arccos(x) - \frac{1}{2} \cdot u^{1/2} \cdot 2$$

$$= x\arccos(x) - \sqrt{1-x^2}.$$

$$\int_0^{\frac{1}{2}} \arccos(x)\,dx = x\arccos(x) - \sqrt{1-x^2}\Big|_{x=0}^{x=\frac{1}{2}} = \left(\frac{1}{2}\arccos\frac{1}{2} - \sqrt{1-\frac{1}{4}}\right) - (0-1)$$

$$= \frac{1}{2} \cdot \frac{\pi}{3} - \frac{\sqrt{3}}{2} + 1$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1.$$

4. $\int e^{3x}\cos(3x)\,dx$

$$\boxed{\begin{array}{ll} u=\cos(3x). & dv=e^{3x}\,dx \\ du=-3\sin(3x)\,dx\,, & v=\frac{e^{3x}}{3} \end{array}}$$

$$= \cos(3x).\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\cdot(-3\sin(3x))\,dx$$

$$= \frac{e^{3x}.\cos(3x)}{3} + \int e^{3x}.\sin(3x)\,dx \longrightarrow \boxed{\begin{array}{ll} u=\sin(3x)\,, & dv=e^{3x}dx \\ du=3\cos(3x)dx & v=\frac{e^{3x}}{3} \end{array}}$$

$$= \frac{e^{3x}.\cos(3x)}{3} + \frac{e^{3x}}{3}.\sin(3x) - \int\frac{e^{3x}}{3}\cdot 3\cos(3x)\,dx$$

$$= \frac{e^{3x}}{3}\left(\cos(3x)+\sin(3x)\right) - \int e^{3x}\cos(3x)\,dx$$

$$\Rightarrow\quad 2\int e^{3x}\cos(3x)\,dx = \frac{e^{3x}}{3}\left(\cos(3x)+\sin(3x)\right)$$

$$\Rightarrow\quad \int e^{3x}\cos(3x)\,dx = \frac{e^{3x}}{6}\left(\cos(3x)+\sin(3x)\right).$$

5. (a) $\displaystyle\int \frac{x}{\sqrt{x-2}}\,dx$

$u = x-2 \Rightarrow dx = du$ and $x = u+2$

$$\therefore \int\frac{x}{\sqrt{x-2}}\,dx = \int\frac{u+2}{\sqrt{u}}\,du = \int\left(\sqrt{u}+\frac{2}{\sqrt{u}}\right)du$$

$$= \int\left(u^{1/2}+2u^{-1/2}\right)du$$

$$= \frac{u^{3/2}}{3/2} + 2\cdot\frac{u^{1/2}}{1/2}$$

$$= \frac{2}{3}\sqrt{(x-2)^3} + 4\sqrt{x-2} + c$$

$$\frac{1}{\sqrt{x-2}} = (x-2)^{-1/2}$$

(b) $\displaystyle\int \frac{x}{\sqrt{x-2}}\,dx$

$$\left[\begin{array}{ll} u = \dfrac{1}{\sqrt{x-2}}, & dv = x\,dx \\ du = -\tfrac{1}{2}(x-2)^{-3/2}\,dx, & v = \dfrac{x^2}{2} \end{array}\right]$$

$\leftarrow$ didn't work out!

$$= \frac{x^2/2}{\sqrt{x-2}} - \int \frac{x^2}{2} \cdot -\frac{1}{2}(x-2)^{-3/2}\,dx$$

$$\int \frac{x}{\sqrt{x-2}}\,dx = \int x \cdot (x-2)^{-1/2}\,dx$$

$$u = x, \quad dv = (x-2)^{-1/2}\,dx$$

$$du = dx \qquad v = \frac{(x-2)^{1/2}}{1/2} = 2(x-2)^{1/2}$$

$$\Rightarrow \int \frac{x}{\sqrt{x-2}}\,dx = x \cdot 2(x-2)^{1/2} - \int 2(x-2)^{1/2} \cdot dx$$

$$= 2x\sqrt{x-2} - 2 \cdot \frac{(x-2)^{3/2}}{3/2}$$

$$= 2x\sqrt{x-2} - \frac{4}{3}\sqrt{(x-2)^3} \;.\; + c$$

These two answers are the same!

(a) gave $\frac{2}{3}\sqrt{(x-2)^3} + 4\sqrt{x-2}$

$$= \sqrt{x-2}\left(\frac{2}{3}\sqrt{(x-2)^2} + 4\right)$$

$$= \sqrt{x-2}\left(\frac{2}{3}(x-2) + 4\right)$$

$$= \sqrt{x-2} \cdot \frac{2x-4+12}{3}$$

$$= \sqrt{x-2} \cdot \left(\frac{2x+8}{3}\right)$$

(b) gave $2x\sqrt{x-2} - \frac{4}{3}\sqrt{(x-2)^3}$

$$= \sqrt{x-2}\left(2x - \frac{4}{3}(x-2)\right)$$

$$= \sqrt{x-2}\left(\frac{6x - 4x + 8}{3}\right)$$

$$= \sqrt{x-2} \cdot \left(\frac{2x+8}{3}\right).$$