Worksheet - 9 (Math 181)

1. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

2. $\cos^2 x + \sin^2 x = 1$

$\sec^2 x - \tan^2 x = 1$

3. (a) $\int 4\cos^4 x \, dx$

$$= 4\int (\cos^2 x)^2 dx$$

$$= 4\int \left(\tfrac{1}{2}(1+\cos 2x)\right)^2 dx$$

$$= 4\cdot\frac{1}{4}\int (1+\cos(2x))^2 dx$$

$$= \int (1 + 2\cos(2x) + \cos^2(2x))\, dx$$

$$= \int 1\, dx + 2\int \cos(2x)\, dx + \int \cos^2(2x)\, dx$$

$$= x + 2\cdot\frac{\sin(2x)}{2} + \int \frac{1}{2}(1+\cos(4x))\, dx$$

$$= x + \sin(2x) + \frac{1}{2}x + \frac{1}{2}\int \cos(4x)\, dx$$

$$= x + \sin(2x) + \frac{1}{2}x + \frac{1}{2}\cdot\frac{\sin(4x)}{4}$$

$$= \frac{3x}{2} + \sin(2x) + \frac{\sin(4x)}{8} + C.$$

(b) $\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{4}(1+\cos 2x)(1-\cos 2x)\, dx$

$$= \int \frac{1}{4}(1-\cos^2 2x)\, dx$$

$$= \frac{1}{4}\int 1\, dx - \frac{1}{4}\int \cos^2 2x\, dx$$

$$= \frac{x}{4} - \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\, dx$$

$$= \frac{x}{4} - \frac{1}{8}\int (1 + \cos 4x)\, dx$$

$$= \frac{x}{4} - \frac{x}{8} - \frac{1}{8}\frac{\sin(4x)}{4} + c$$

$$= \frac{x}{8} - \frac{\sin(4x)}{\cancel{8}32} + c.$$

(c) $\int \tan^2(x)\, dx$

$$= \int (\sec^2(x) - 1)\, dx$$

$$= \int \sec^2 x\, dx - \int 1\, dx$$

$$= \tan x - x + c$$

4. (a) $\int \sin^3(x) \cos^2(x)\, dx$

$$= \int \cos^2 x \cdot \sin^2 x \cdot \underline{\sin x\, dx}$$

$(u = \cos x \Rightarrow du = \underline{-\sin x\, dx})$

$$= \int \cos^2 x\, (1 - \cos^2 x)\quad \sin x\, dx$$

$$= \int u^2(1-u^2)\,(-du)$$

$$= -\int (u^2 - u^4)\, du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5}$$

$$= -\frac{(\cos^3 x)}{3} + \frac{(\cos^5 x)}{5}. + c$$

(b) $\int 6\sin^2(x)\cos^5(x)\,dx$

$= \int 6\sin^2 x \cdot \cos^4 x \cdot \underline{\cos x\,dx}$

$(u = \sin x \Rightarrow du = \underline{\cos x\,dx})$

$= \int 6\sin^2 x\,(\cos^2 x)^2 \cdot \cos x\,dx$

$= \int 6\sin^2 x\,(1-\sin^2 x)^2 \cdot \cos x\,dx$

$= \int 6u^2(1-u^2)^2 \cdot du$

$= \int 6u^2(1-2u^2+u^4)\,du$

$= \int (6u^2 - 12u^4 + 6u^6)\,du$

$= \dfrac{6u^3}{3} - \dfrac{12u^5}{5} + \dfrac{6u^7}{7}$

$= 2(\sin^3 x) - \dfrac{12}{5}(\sin^5 x) + \dfrac{6}{7}(\sin^7 x) + c$

5. $\int \sec^4 x\,dx$

$= \int \sec^2 x \cdot \underline{\sec^2 x\,dx}$

$(u = \tan x \Rightarrow du = \underline{\sec^2 x\,dx})$

$= \int (1+\tan^2 x) \cdot \sec^2 x\,dx$

$= \int (1+u^2)\,du \quad = u + \dfrac{u^3}{3} = \tan x + \dfrac{\tan^3 x}{3} + c.$

6. (a) $$\frac{x^2-2x+3}{x^3-x^2-6x} = \frac{x^2-2x+3}{x(x^2-x-6)} = \frac{x^2-2x-3}{x(x-3)(x+2)}$$

$$= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}.$$

(b) $$\frac{x+1}{x^3+2x^2} = \frac{x+1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}.$$

(c) $$\frac{5}{(x^2-1)(x^2+1)} = \frac{5}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}.$$

7. (a) $$\frac{1}{(x-3)(x+1)^2} = \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow \quad 1 = A(x+1)^2 + B(x-3)(x+1) + C(x-3)$$

$x=3$: $\quad 1 = A(3+1)^2 \Rightarrow A = \frac{1}{16}$

$x=-1$: $\quad 1 = C(-1-3) \Rightarrow C = -\frac{1}{4}$

$x=0$: $\quad 1 = \frac{1}{16}(0+1)^2 + B(0-3)(0+1) + \cancel{c}\left(-\frac{1}{4}(0-3)\right)$

$$\frac{16}{16} = \frac{1}{16} - 3B + \frac{3}{4} = \frac{1}{16} - 3B + \frac{12}{16}$$

$$\Rightarrow 3B = -\frac{3}{16} \quad \therefore B = -\frac{1}{16}.$$

$$\therefore \int \frac{1}{(x-3)(x+1)^2}\,dx = \frac{1}{16}\int \frac{1}{x-3}\,dx - \frac{1}{16}\int \frac{1}{x+1}\,dx - \frac{1}{4}\int \frac{1}{(x+1)^2}\,dx$$

$$= \frac{1}{16}\ln|x-3| - \frac{1}{16}\ln|x+1| - \frac{1}{4}\left(\frac{(x+1)^{-1}}{-1}\right) + c$$

$$= \frac{1}{16}\ln|x-3| - \frac{1}{16}\ln|x+1| + \frac{1}{4(x+1)} + c.$$

7. (b) $\displaystyle\int \frac{6x^2+5x-2}{x^3+x^2-2x}\,dx$

$$\frac{6x^2+5x-2}{x^3+x^2-2x} = \frac{6x^2+5x-2}{x(x^2+x-2)} = \frac{6x^2+5x-2}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\Rightarrow 6x^2+5x-2 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

$x=1$: $B(3) = 6+5-2 \Rightarrow B = 3$

$x=0$: $A(-1)(2) = -2 \Rightarrow A = 1$

$x=-2$: $C(-2)(-3) = 24-10-2 \Rightarrow C = 2$

$$\int \frac{6x^2+5x-2}{x^3+x^2-2x}\,dx = 1\int \frac{1}{x}\,dx + 3\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x+2}\,dx$$

$$= \ln|x| + 3\ln|x-1| + 2\ln|x+2| + C.$$