

Problem 1.

0 7 2 3 4 8 5 9 \longrightarrow selection sort!

Step 1: $\begin{matrix} \text{min}=0 & \text{min}=0 & \text{min}=0 & & & & \text{min}=0 \\ 0 & 7 & 2 & 3 & 4 & 8 & 5 & 9 \end{matrix}$

We first find the min of the array. Starting with $\text{min}=0$, we compare it with 7, 2, 3, ..., 9 and finally conclude that 0 is the minimum.

Comparisons: 7 Swaps: 0

Step 2: $\begin{matrix} & & \text{min}=7 & & \text{min}=2 & & \text{min}=2 & & \dots & & \text{min}=2 \\ \boxed{0} & | & \textcircled{7} & \longleftrightarrow & \textcircled{2} & & 3 & & 4 & & 8 & & 5 & & 9 \end{matrix}$
sorted part

We find out that min in the new subarray is 2, and hence swap 2 with 7.

Comparisons: 6 swaps: 1

Step 3: $\begin{matrix} & & \text{min}=7 & & \text{min}=3 & & \text{min}=3 & & \dots & & \text{min}=3 \\ \boxed{0 \quad 2} & | & \textcircled{7} & \longleftrightarrow & \textcircled{3} & & 4 & & 5 & & 5 & & 9 \end{matrix}$
sorted part

min = 3, gets swapped with 7

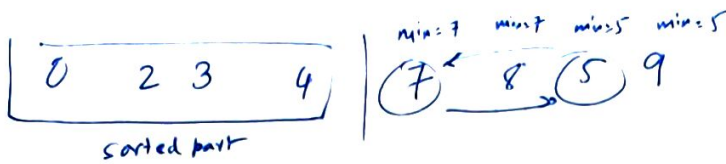
Comparisons: 5 swaps: 1

Step 4: $\begin{matrix} & & \text{min}=7 & & \text{min}=4 & & \dots & & \text{min}=4 \\ \boxed{0 \quad 2 \quad 3} & | & \textcircled{7} & \longleftrightarrow & \textcircled{4} & & 8 & & 5 & & 9 \end{matrix}$
sorted part

min = 4, gets swapped with 7

Comparisons: 4 swaps: 1

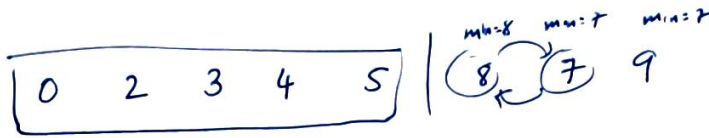
steps:



Comp: 3
Swap: 1

min = 5, gets swapped with 7.

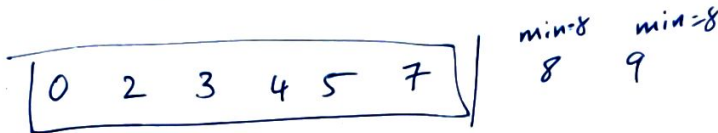
Step 6:



Comp: 2
Swap: 1

min = 7, gets swapped with 8

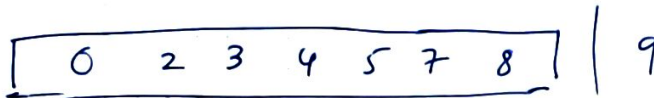
Step 7:



Comp: 1
Swap: 0

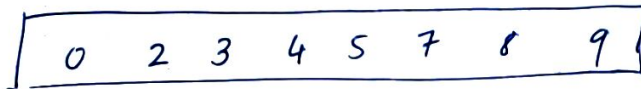
min = 8, no swaps occur.

Step 8:



Comp: 0
Swap: 0

Step 9:



is the sorted array.

So, total # comparisons = $7 + 6 + \dots + 1 = 28$

total # swaps = 5

Generalizing, let us start with any arbitrary array

$a[0] \quad a[1] \quad \dots \quad a[n-1]$.

Then, in step 1 there will be $(n-1)$ comparisons
in step 2 \dots $(n-2)$ \dots

\vdots
in step $(n-1)$ there will be 1 comparison

\therefore Total # of comparisons = $1 + 2 + \dots + (n-1)$

We know $1 + 2 + \dots + k = \frac{k(k+1)}{2}$

\therefore Set $k = n-1 \Rightarrow 1 + 2 + \dots + (n-1) =$

$$\boxed{\frac{(n-1)n}{2}}$$

these many comparisons.

Problem 2.

$$n!, \binom{n}{5}, n \log n, n^{\ln n}, \left(\frac{n}{3}\right)^{\frac{n}{3}}, n^n, (\log 3)^n, (\ln n)^{100}.$$

The correct order of these functions is:

$$\underbrace{(\ln n)^{100}}_{\text{logarithmic}} \ll \underbrace{n \log n}_{\text{polynomial region}} \ll \underbrace{\binom{n}{5}^*}_{\text{polynomial region}} \ll \underbrace{n^{\ln n}}_{\text{superpolynomial}} \ll \underbrace{(\log 3)^n}_{\text{exponential}} \ll \left(\frac{n}{3}\right)^{\frac{n}{3}} \ll n! \ll n^n.$$

Let us check only the difficult limits:

$$\circ \lim_{n \rightarrow \infty} \frac{\binom{n}{5}}{n \log n} = \lim_{n \rightarrow \infty} \frac{n^5}{5! \cdot n \log n} = \lim_{n \rightarrow \infty} \frac{1}{5! n^{\log n - 5}} = 0 \quad (\checkmark)$$

$$\circ n^{\ln n} \ll (\log 3)^n : \text{take ln of both sides}$$

$$\Leftrightarrow \underbrace{\ln n \cdot \ln n}_{\text{logarithmic}} \ll \underbrace{n \ln \log 3}_{\text{linear}} \quad (\checkmark)$$

$$\circ (\log 3)^n \ll \left(\frac{n}{3}\right)^{\frac{n}{3}} : \text{take ln of both sides}$$

$$\Leftrightarrow n \cdot \ln \log 3 \ll \frac{n}{3} \cdot \ln\left(\frac{n}{3}\right) \Leftrightarrow \underbrace{3 \ln \log 3}_{\text{constant}} \ll \underbrace{\ln \frac{n}{3}}_{\text{logarithmic}} \quad (\checkmark)$$

$$\circ n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \sim n^{\frac{1}{2}} \frac{n^n}{e^n}.$$

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{n}{3}\right)^{\frac{n}{3}}}{n^{\frac{1}{2}} \frac{n^n}{e^n}} = \lim_{n \rightarrow \infty} \frac{e^n}{3^{\frac{n}{3}} \cdot n^{\frac{2n}{3}} \cdot n^{\frac{1}{2}}} = 0 \text{ as } e^n \ll n^{2n/3}. \quad (\checkmark)$$

$$\circ \sqrt{n} \frac{n^n}{e^n} \ll n^n. \text{ So, } n! \ll n^n. \quad (\checkmark)$$

* Asymptotically, $\binom{n}{5} \sim \frac{n^5}{5!}$. This is because $\binom{n}{5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}$

and when n is large, $n-1, n-2, \dots, n-5$ are all close to n .