MCS-360 Project-4 Solutions

Problem 1.

0 7 2 3 4 8 5 9 ⟶ selection sort!

Step 1:

min=0 min=0 min=0 min=0

0 7 2 3 4 8 5 9

We first find the min of the array. Starting with min=0, we compare it with 7, 2, 3, ..., 9 and finally conclude that 0 is the minimum.

Comparisons: 7   Swaps: 0

Step 2:

min=7 min=2 min=2 . . . min=2

$\boxed{0}$ | 7 2 3 4 8 5 9

sorted part

We find out that min in the new subarray is 2, and hence swap 2 with 7.

Comparisons: 6   swaps: 1

step 3:

min=7 min=3 min=3 . . . min=3

$\boxed{0 \quad 2}$ | 7 3 4 8 5 9

Sorted part

min =3, gets swapped with 7

Comparisons: 5   swaps: 1

step 4:

min=7 min=4 . . . min=4

$\boxed{0 \quad 2 \quad 3}$ | 7 4 8 5 9

sorted part

min=4, gets swapped with 7

Comparisons: 4   swaps: 1

step 5:

min=7 min=7 min=5 min=5

| 0 2 3 4 | 7 8 5 9

sorted part

Comp: 3
Swap: 1

min = 5, gets swapped with 7.

Step 6:

min=8 min=7 min=7

| 0 2 3 4 5 | 8 7 9

comp: 2
swap: 1

min = 7, gets swapped with 8

Step 7:

min=8 min=8

| 0 2 3 4 5 7 | 8 9

Comp: 1
swap: 0

min = 8, no swaps occur.

step 8:

| 0 2 3 4 5 7 8 | 9

Comp: 0
swap: 0

step 9:

| 0 2 3 4 5 7 8 9 | is the sorted array.

So, total # comparisons = $7+6+\dots+1 = 24$

total # swaps = 5

Generalizing, let us start with any arbitrary array

$a[0] \quad a[1] \quad \dots \quad a[n-1]$.

Then, in step 1 there will be $(n-1)$ comparisons

in step 2 . . . . $(n-2)$ . .

⋮

in step $(n-1)$ there will be 1 comparison

$\therefore$ Total # of comparisons $= 1+2+\dots+(n-1)$

We know $1+2+\dots+k = \dfrac{k(k+1)}{2}$

$\therefore$ Set $k = n-1 \Rightarrow 1+2+\dots+(n-1) = \boxed{\dfrac{(n-1)n}{2}}$ these many comparisons.

Problem 2.

$$n!,\ \binom{n}{5},\ n\log n,\ n^{\ln n},\ \left(\frac{n}{3}\right)^{\frac{n}{3}},\ n^n,\ (\log 3)^n,\ (\ln n)^{100}.$$

The correct order of these functions is:

$$\underbrace{(\ln n)^{100}}_{\text{logarithmic}} \ll \underbrace{n\log n \ll \binom{n}{5}^{*}}_{\text{polynomial region}} \ll \underbrace{n^{\ln n}}_{\text{superpolynomial}} \ll \underbrace{(\log 3)^n}_{\text{exponential}}$$

$$\ll \left(\frac{n}{3}\right)^{n/3} \ll n! \ll n^n.$$

Let us check only the difficult limits:

- $\lim\limits_{n\to\infty} \dfrac{\binom{n}{5}}{n^{\log n}} = \lim\limits_{n\to\infty} \dfrac{n^5}{5!\cdot n^{\log n}} = \lim\limits_{n\to\infty} \dfrac{1}{5!\,n^{\log n - 5}} = 0$ (✓)

- $n^{\ln n} \ll (\log 3)^n$: take ln of both sides

  $\Leftrightarrow \underbrace{\ln n \cdot \ln n}_{\text{logarithmic}} \ll \underbrace{n \ln \log 3}_{\text{linear}}$ (✓)

- $(\log 3)^n \ll \left(\frac{n}{3}\right)^{\frac{n}{3}}$: take ln of both sides

  $\Leftrightarrow n\cdot \ln\log 3 \ll \frac{n}{3}\cdot \ln\left(\frac{n}{3}\right) \Leftrightarrow \underbrace{3\ln\log 3}_{\text{constant}} \ll \underbrace{\ln\frac{n}{3}}_{\text{logarithmic}}$ (✓)

- $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sim n^{1/2}\frac{n^n}{e^n}.$

  $\lim\limits_{n\to\infty} \dfrac{\left(\frac{n}{3}\right)^{n/3}}{n^{1/2}\frac{n^n}{e^n}} = \lim\limits_{n\to\infty} \dfrac{e^n}{3^{n/3}\cdot n^{2n/3}\cdot n^{1/2}} = 0$ as $e^n \ll n^{2n/3}$. (✓)

- $\sqrt{n}\,\frac{n^n}{e^n} \ll n^n$. So, $n! \ll n^n$. (✓)

---

* Asymptotically, $\binom{n}{5} \sim \frac{n^5}{5!}$. This is because $\binom{n}{5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}$

and when n is large, $n-1, n-2, \dots, n-5$ are all close to n.