Math 170: Mock Exam (solution)

Sayan Mukherjee's discussion

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This exam does not count towards your final grade.
Compute the following indefinite integrals. Each problem is worth 10 points.

1. $\displaystyle\int x\sqrt{2x+1}\,dx$

   Solution. Let $u = 2x+1$, then $du = 2dx$ and $x = \frac{u-1}{2}$. Thus we have to integrate:

   $$\int \frac{u-1}{2}\sqrt{u}\,\frac{du}{2} = \frac{1}{4}\int (u-1)\cdot u^{1/2}\,du.$$

   The rest is a direct application of the power rule, and the integral evaluates to

   $$\begin{aligned}\frac{1}{4}\int u^{3/2} - u^{1/2}\,du &= \frac{1}{4}\left(u^{5/2}5/2 - \frac{u^{3/2}}{3/2}\right) + c\\ &= \frac{1}{4}\left(\frac{2}{5}(2x+1)^{5/2} - \frac{2}{3}(2x+1)^{3/2}\right) + c\\ &= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + c.\end{aligned}$$

   □

2. $\int x^2 \sin(x)\, dx$

Solution. We do integration by parts two times, since it's a poly × trig.

$$\begin{aligned}
\int x^2 \sin x\, dx &= \int x^2(-\cos x)'\, dx \\
&= x^2(-\cos x) - \int 2x(-\cos x)\, dx \\
&= -x^2 \cos x + 2\int x\cos x\, dx \\
&= -x^2 \cos x + 2\left(\int x(\sin x)'\, dx\right) \\
&= -x^2 \cos x + 2\left(x\sin x - \int \sin x\, dx\right) \\
&= -x^2 \cos x + 2x\sin x - 2\int \sin x\, dx \\
&= -x^2 \cos x + 2x\sin x + 2\cos x + c.
\end{aligned}$$

□

3. $\displaystyle\int e^{2x} \sin 3x \, dx$

Solution. Since it's exp × trig, it's the type of problem where we do integration by parts twice and solve a linear equation.

Let $I = \int e^{2x} \sin 3x \, dx$, then we have:

$$
\begin{aligned}
I &= \int \left(\frac{e^{2x}}{2}\right)' \sin 3x \, dx \\
&= \frac{e^{2x}}{2} \cdot \sin 3x - \int \frac{e^{2x}}{2} \cdot 3 \cos 3x \, dx \\
&= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx \\
&= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int \left(\frac{e^{2x}}{2}\right)' \cos 3x \, dx \\
&= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \left(\frac{1}{2} e^{2x} \cos 3x - \int \frac{e^{2x}}{2} \cdot (-3 \sin 3x)\right) dx \\
&= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x - \frac{9}{4} \int e^{2x} \sin 3x \, dx \\
&= \frac{1}{4} e^{2x} (2 \sin 3x - 3 \cos 3x) - \frac{9}{4} I,
\end{aligned}
$$

Therefore $\frac{13}{4} I = \frac{1}{4} e^{2x} (2 \sin 3x - 3 \cos 3x)$, implying

$$I = \frac{1}{13} e^{2x} (2 \sin 3x - 3 \cos 3x).$$

□

4. $\int x^9 \ln x \, dx$

Solution. Since it's poly $\times$ log, we gotta differentiate the log and integrate the poly (since the other option of integrating logs is hard)...

$$\begin{aligned}
\int x^9 \ln x \, dx &= \int \left(\frac{x^{10}}{10}\right)' \ln x \, dx \\
&= \frac{x^{10} \ln x}{10} - \int \frac{x^{10}}{10} \cdot \frac{1}{x} \, dx \\
&= \frac{x^{10} \ln x}{10} - \frac{1}{10} \int x^9 \, dx \\
&= \frac{x^{10} \ln x}{10} - \frac{x^{10}}{100} + c.
\end{aligned}$$

□

5. $\int \sin \sqrt{2x}\, dx$

Solution. Let $u = \sqrt{2x}$, then $2x = u^2$ implying $2dx = 2udu$, i.e. $dx = udu$. Therefore we see,

$$\begin{aligned}
\int \sin \sqrt{2x}\, dx &= \int \sin u \cdot udu \\
&= \int (-\cos u)' \cdot u\, du \\
&= -u \cos u - \int (-\cos u)\, du \\
&= -u \cos u + \int \cos u\, du \\
&= -u \cos u + \sin u + c \\
&= -\sqrt{2x} \cos \sqrt{2x} + \sin \sqrt{2x} + c.
\end{aligned}$$

□

All the best for the final exam! :)