Math 165 Final Exam
Spring 2003

1. $$f(x) = \begin{cases} \dfrac{x+1}{x-1} & x \le 0 \\[2ex] \dfrac{x-1}{x+1} & x > 0 \end{cases}$$

<u>Note:</u> Rational functions are continuous for all x except where the denominator is zero.

I. 1st ask is f continuous at $x=-1$ then ask the question asked.

$f(x) = \dfrac{x+1}{x-1}$ for $x \le 0 \Rightarrow f(x)$ is continuous for all $x \le 0$. $\Rightarrow f(x)$ is continuous at $x=-1$.

<u>Answer:</u> I. is false

II Is f(x) continuous at $x=0$?
1st observe that f(x) is continuous for all x to left and right of $x=0$. So f(x) is continuous at $x=0$ if both functions match at $x=0$. Check:

- from left $f(x) = \dfrac{x+1}{x-1} \Rightarrow f(0) = \dfrac{0+1}{0-1} = -1$
- from right $f(x) = \dfrac{x-1}{x+1} \Rightarrow f(0) = \dfrac{0-1}{0+1} = -1$

Same

Both sides of $f(x)$ match at $x=0 \Rightarrow$
$f(x)$ is Continuous at $x=0$.
Answer II. false

III. Is $f(x)$ continuous at $x=1$?
for all $x>0$ $f(x) = \frac{x-1}{x+1}$ and is continuous
for all $x>0$. $\Rightarrow$ Continuous at $x=1$

Answer III false

IV. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{x+1}{x-1} = \frac{0+1}{0-1} = -1.$

Note, to find the limit of a continuous function just "plug in".

$\Rightarrow$ Answer IV. True

V $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \frac{x-1}{x+1} = \frac{1-1}{1+1} = \frac{0}{2} = 0$

$\Rightarrow$ Answer V. True

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2. $\lim\limits_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{0+1}-1}{0} = \frac{0}{0}$ Indeterminant Form.

$\Rightarrow$ try some standard trick or rule.

$$= \lim_{x\to 0}\left[\left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)\right]$$

$$= \lim_{x\to 0}\frac{(\sqrt{x+1})^2-1^2}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{\overbrace{(x+1)-1}^{x}}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}$$

$$= \frac{1}{\sqrt{0+1}+1} = \boxed{\frac{1}{2}}$$

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3. $f(x,y) = x^3 + y^3 - 6xy$

$f_x = 3x^2 - 6y$, $f_{xx} = 6x$

$f_y = 3y^2 - 6x$, $f_{yy} = 6y$

$f_{xy} = \frac{d}{dy}(f_x) = \frac{d}{dy}(3x^2-6y) = -6$

$f_{yx} = \frac{d}{dx}(f_y) = \frac{d}{dx}(3y^2-6x) = -6$

same ✓

$D(x,y) = f_{xx}(x,y)\cdot f_{yy}(x,y) - f_{xy}(x,y)\cdot f_{yx}(x,y)$

$= (6x)(6y) - (-6)(-6) = 36xy - 36$

$$\boxed{D(x,y) = 36(xy-1)}$$

<u>Critical Points</u>:

Solve equations

$$f_x = \cancel{3}x^2 - \cancel{6}^{\,2}y = 0 \Rightarrow x^2 - 2y = 0 \quad (1)$$
$$f_y = \cancel{3}y^2 - \cancel{6}^{\,2}x = 0 \qquad\quad y^2 - 2x = 0 \quad (2)$$

(1) $\Rightarrow$ $y = \dfrac{x^2}{2}$ substitute into (2)

$$\Rightarrow \left(\frac{x^2}{2}\right)^2 - 2x = 0$$
$$\frac{x^4}{4} - 2x = 0$$
$$x^4 - 8x = 0$$
$$\underbrace{x}_{=0}\,(\underbrace{x^3 - 8}_{=0}) = 0$$

$\boxed{x = 0 \quad x^3 = 8 \quad x = 2}$

Repeat by eliminating x from (1)

(1) $x^2 - 2y = 0$

(2) $y^2 - 2x = 0 \Rightarrow x = \dfrac{y^2}{2}$

(2) $\to$ (1) $\Rightarrow \left(\dfrac{y^2}{2}\right)^2 - 2y = 0$
$$\frac{y^4}{4} - 2y = 0$$
$$y^4 - 8y = 0$$
$$y(y^3 - 8) = 0$$

$y = 0,\ y = 2$

Now find all possible (x,y)'s and then check to make sure they satisfy the equations (1) and (2)