

FINITELY APPROXIMABLE GROUPS AND ACTIONS
PART I: THE RIBES–ZALESSKIĀ PROPERTY

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Abstract. We investigate extensions of S. Solecki’s theorem on closing off finite partial isometries of metric spaces [11] and obtain the following exact equivalence: any action of a discrete group Γ by isometries of a metric space is finitely approximable if and only if any product of finitely generated subgroups of Γ is closed in the profinite topology on Γ .

§1. Introduction. Suppose a discrete group G acts by isometries on a metric space (X, d_X) . The main question we shall consider here is the conditions on the group G that ensure that such an action can always be finitely approximated. So, before we go any further, we need to precisely state our notion of approximation.

Let G be a group acting on sets X and Y and let $A \subseteq X$ and $F \subseteq G$ be arbitrary subsets. An F -map from A to Y is a function $\pi: A \rightarrow Y$ such that whenever $g \in F$ and $x, gx \in A$, then $\pi(gx) = g\pi(x)$. Moreover, if X and Y are relational structures of the same type, an F -embedding is simply an injective F -map that is an isomorphism of A with its image $F[A]$.

DEFINITION 1. Let G be a group acting by isometries on a metric space (X, d_X) . We say that the action is finitely approximable if for any finite $A \subseteq X$ and $F \subseteq G$ there is a finite metric space (Y, d_Y) , on which G acts by isometries, and an isometric F -embedding $\pi: A \rightarrow Y$.

We note that this is a very strong notion of finite approximation in the sense that we require π to be an isometric embedding and not just a map with small distortion. However, as we shall see later, this is not a real restriction.

The goal here is to isolate the properties of a group G that ensure that every isometric action of G is finitely approximable and, in particular, show this is so for finitely generated Abelian groups. These results form the basis of our investigation in our companion paper [10]. The first result along these lines is due to S. Solecki [11], who proved that this is verified when G is a finitely generated free group. Solecki’s proof was based on earlier work of B. Herwig and D. Lascar [6] on equations in

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free groups and ultimately relied on the solution to Rhodes' Type II Conjecture proved independently by C. J. Ash [1] and L. Ribes and P. A. Zalesskiĭ [9]. The proof by Ribes and Zalesskiĭ used a connection between the Type II Conjecture and the profinite topology on free groups established by J.-E. Pin and C. Reutenauer [8], and, indeed, the main result of Ribes and Zalesskiĭ is that products of finitely generated subgroups of free groups are closed in the profinite topology. While part of the result presented here is not entirely novel and can be proved with the methods of the literature, it is not clear from Solecki's and Herwig and Lascar's papers that this topological property of G is exactly what is needed for the approximation result. Our main goal here is to establish the following exact correspondence between the two properties.

THEOREM 2. *The following two properties are equivalent for a countable discrete group G .*

- (1) *If H_1, \dots, H_n are finitely generated subgroups of G , their product $H_1 H_2 \cdots H_n$ is a closed subset in the profinite topology on G ,*
- (2) *any action of G by isometries on a metric space is finitely approximable.*

1.1. The profinite topology and finitely approximable groups. We recall that if G is a discrete group, the *profinite topology* on G is the group topology on G generated by the basic open sets

$$gK,$$

where $g \in G$ and K is a finite index subgroup of G . Since any finite index subgroup $K \leq G$ contains a further subgroup $N \leq K$, which is both of finite index and normal in G , in the definition of the basis, one can always assume that K is moreover normal in G . Thus, a subset $S \subseteq G$ is closed in the profinite topology on G if for any $g \in G \setminus S$, there is a finite index (normal) subgroup $K \leq G$ such that $gK \cap S = \emptyset$ or, equivalently, such that $g \notin SK$. Since this is a group topology, i.e., the group operations are continuous, G is Hausdorff if and only if $\{1\}$ is closed, i.e., if for any $g \neq 1$ there is a finite index subgroup K not containing g . In other words, G is Hausdorff if and only if it is residually finite.

A stronger notion than residual finiteness is subgroup separability or being LERF (locally extended residually finite). Here a group G is *subgroup separable*, or *LERF*, if any finitely generated subgroup $H \leq G$ is closed in the profinite topology on G . Since $\{1\}$ is finitely generated, subgroup separability implies residual finiteness. M. Hall [4, 5] originally proved that free groups are subgroup separable.

However, the even stronger notion that concerns us here is what we shall call the Ribes–Zalesskiĭ property, or property (RZ) for brevity. Namely, a group G is said to have the *Ribes–Zalesskiĭ property* if any product $H_1 H_2 \cdots H_n$ of finitely generated subgroups $H_i \leq G$ is closed in the profinite topology on G . This property was originally proven for free groups in [9] and T. Coulbois [3] showed that if both G and F have property (RZ), then so does $G * F$.

1.2. Finitely approximable actions. The requirement in the definition of finitely approximable actions that the map π be an isometry may seem very strong, but can actually be weakened somewhat without effect.

DEFINITION 3. *Suppose (X, d_X) and (Y, d_Y) are metric spaces and $K \geq 1$. A function $f: X \rightarrow Y$ is said to be a map with constant K if for some constant $c > 0$*

and all $x, y \in X$,

$$cd_X(x, y) \leq d_Y(f(x), f(y)) \leq Kcd_X(x, y).$$

The minimum C of the numbers K for which f is a map with constant K is said to be the distortion of f . In this case, we say that (X, d_X) embeds into (Y, d_Y) with distortion C .

Note that if we rescale the metric d_Y by a factor $\frac{1}{Kc}$, the function f will be 1-Lipschitz and f^{-1} will be K -Lipschitz on its domain $f[X]$.

A finite set D of non-negative real numbers is said to be a *good value set* if $0 \in D$ and for all $s, t \in D$, if $s + t \leq \max D$, then also $s + t \in D$. With this definition in hand, the following lemma is trivial to verify.

LEMMA 4. Suppose (X, d_X) is a metric space and D is a good value set. Then,

$$\partial_X(x, y) = \begin{cases} \min(s \in D \mid d_X(x, y) \leq s) & \text{if } d_X(x, y) \leq \max D, \\ \max D & \text{otherwise.} \end{cases}$$

defines a metric on X . Moreover, if G is a group acting by isometries of (X, d_X) , then the same action is also an isometric action on (X, ∂_X) .

DEFINITION 5. If (X, d_X) is a metric space and $A \subseteq X$ is a non-empty finite subset, the expanded distance set of A is defined by

$$\text{Ex}(A) = \{r_1 + \cdots + r_n \mid r_i \in d_X[A \times A] \ \& \ r_1 + \cdots + r_n \leq \text{diam}(A)\}.$$

Note that $\text{Ex}(A)$ is a good value set in the above sense. So with these preliminaries off our hands, we can now show the following equivalence.

PROPOSITION 6. Suppose G is a group acting by isometries on a metric space (X, d_X) and that $A \subseteq X$ and $F \subseteq G$ are finite subsets. Then the following are equivalent,

- (1) there is a finite $\text{Ex}(A)$ -valued metric space (Y, d_Y) , on which G acts by isometries, and an isometric F -embedding $\pi: A \rightarrow Y$,
- (2) there is a finite metric space (Y, d_Y) , on which G acts by isometries, and an isometric F -embedding $\pi: A \rightarrow Y$,
- (3) for every $K > 1$, there is a finite metric space (Y, d_Y) , on which G acts by isometries, and an F -map $\pi: A \rightarrow Y$ of distortion at most K .

PROOF. The only non-trivial part is that (3) implies (1). So pick $K > 1$ small enough such that for any $s < t$ in $\text{Ex}(A)$, $Ks < t$. Assume (Y, d_Y) is a finite metric space on which G acts by isometries and that $\pi: A \rightarrow Y$ is an F -map with constant K . Also, by rescaling the metric d_Y , we can suppose that π is 1-Lipschitz and π^{-1} is K -Lipschitz on its domain. By Lemma 4, we can define a G -invariant, $\text{Ex}(A)$ -valued metric ∂_Y on Y by

$$\partial_Y(x, y) = \begin{cases} \min(s \in \text{Ex}(A) \mid d_Y(x, y) \leq s), & \text{if } d_Y(x, y) \leq \max \text{Ex}(A); \\ \max \text{Ex}(A), & \text{otherwise.} \end{cases}$$

Then, for all $x, y \in A$,

$$\frac{1}{K}d_X(x, y) \leq d_Y(\pi(x), \pi(y)) \leq d_X(x, y),$$

and so, by assumption on K , we have for all $x, y \in A$, $\partial_Y(\pi(x), \pi(y)) = d_X(x, y)$. It follows that π is an isometric F -embedding of (A, d_X) into (Y, ∂_Y) . \dashv

Thus, approximating with arbitrarily small distortion or isometrically are equivalent and we shall therefore stick to the stronger notion throughout the rest of the paper.

§2. Groups with the Ribes–Zalesskii property. We are now ready for our main equivalence.

THEOREM 7. *The following properties are equivalent for a countable group G .*

- (a) G has property (RZ),
- (b) any action of G by isometries on a metric space is finitely approximable.

We note also that it follows from Theorem 7 together with Proposition 6, that if G has property (RZ) and acts by isometries on a rational valued metric space, then any finite approximation can be done by rational valued metric spaces. This will be important for our later applications.

PROOF. Suppose first G is a group with property (RZ) acting by isometries on a metric space X and $A \subseteq X$ and $F \subseteq G$ are finite subsets. Replacing d_X with the metric ∂_X defined from the good value set $\text{Ex}(A)$, as in Lemma 4, we can, without changing distances between points of A , assume that d_X only takes values in the finite set $\text{Ex}(A)$.

Now let Ga_1, \dots, Ga_m list the orbits of G on X that intersect A . Without loss of generality, we can suppose that $X = Ga_1 \cup \dots \cup Ga_m$. Let also $N_i = \{g \in G \mid ga_i = a_i\}$ and let G act by left-translation on the space of left cosets

$$G/N_1 \sqcup \dots \sqcup G/N_m.$$

Then the map

$$\begin{aligned} \pi: X &\rightarrow G/N_1 \sqcup \dots \sqcup G/N_m \\ ga_i \in Ga_i &\mapsto gN_i \in G/N_i \end{aligned}$$

is a conjugacy of G -actions and we may therefore assume that $X = G/N_1 \sqcup \dots \sqcup G/N_m$, i.e., that G acts by left-translation on $G/N_1 \sqcup \dots \sqcup G/N_m$ preserving a metric d taking values in the finite set $\text{Ex}(A)$.

Now, since $A \subseteq G/N_1 \sqcup \dots \sqcup G/N_m$ is finite, there is some finite set $C \subseteq G$ such that $A \subseteq \{gN_i \mid i \leq m \ \& \ g \in C\}$. By enlarging F , we can also assume that $1 \in F$.

Define finitely generated subgroups $M_i \leq N_i$ by

$$M_i = \langle g^{-1}fh \mid f \in F, g, h \in C \ \& \ g^{-1}fh \in N_i \rangle$$

and subsets $E_r^{ij} \subseteq G$ for $i, j \leq m$ and $r \in \text{Ex}(A) \setminus \{0\}$ by

$$E_r^{ij} = M_i \{g^{-1}h \mid g, h \in C \ \& \ d(gN_i, hN_j) = r\} M_j.$$

Note that $E_r^{ij} = (E_r^{ji})^{-1}$ and clearly also $E_r^{ij} = M_i E_r^{ij} M_j$.

We claim that whenever $i_0, \dots, i_n \leq m$, $r_j, r \in \text{Ex}(A) \setminus \{0\}$ and $r_1 + \dots + r_n < r$, we have

$$1 \notin E_{r_1}^{i_0 i_1} E_{r_2}^{i_1 i_2} \dots E_{r_n}^{i_{n-1} i_n} E_r^{i_n i_0}.$$

To see this, suppose toward a contradiction that $h_1 \in E_{r_1}^{i_0 i_1}$ and $h \in E_r^{i_n i_0}$ are such that $1 = h_1 h_2 \dots h_n h$. Then

$$d(N_{i_0}, h_1 N_{i_1}) = r_1, \ d(N_{i_1}, h_2 N_{i_2}) = r_2, \dots, \ d(N_{i_{n-1}}, h_n N_{i_n}) = r_n$$

and

$$d(N_{i_n}, hN_{i_0}) = r$$

whereby

$$\begin{aligned} r &= d(N_{i_0}, h^{-1}N_{i_n}) \\ &= d(N_{i_0}, h_1h_2 \cdots h_nN_{i_n}) \\ &\leq d(N_{i_0}, h_1N_{i_1}) + d(h_1N_{i_1}, h_1h_2N_{i_2}) \\ &\quad + \cdots + d(h_1 \cdots h_{n-1}N_{i_{n-1}}, h_1 \cdots h_{n-1}h_nN_{i_n}) \\ &= d(N_{i_0}, h_1N_{i_1}) + d(N_{i_1}, h_2N_{i_2}) + \cdots + d(N_{i_{n-1}}, h_nN_{i_n}) \\ &= r_1 + r_2 + \cdots + r_n < r, \end{aligned}$$

which is a contradiction. So the claim holds.

We remark that each of the sets E_r^{ij} can be written as a finite union of sets, $M_i g^{-1} h M_j$ where $g, h \in C$, and these sets are themselves products of left-cosets of finitely generated subgroups of G . Thus, any set

$$E_{r_1}^{i_0 i_1} E_{r_2}^{i_1 i_2} \cdots E_{r_n}^{i_{n-1} i_n} E_r^{i_n i_0}$$

is again a finite union of products of left-cosets of finitely generated subgroups of G . Using that for any subgroups $H_i \leq G$ and any $g_i \in G$

$$\begin{aligned} g_1 H_1 g_2 H_2 \cdots g_k H_k &= g_1 \cdots g_k \cdot (g_2 \cdots g_k)^{-1} H_1 (g_2 \cdots g_k) \\ &\quad \cdot (g_3 \cdots g_k)^{-1} H_2 (g_3 \cdots g_k) \cdots g_k^{-1} H_{k-1} g_k \cdot H_k, \end{aligned}$$

we see that such sets are actually finite unions of left-translates of products of finitely generated subgroups of G . Since G has property (RZ), it follows that the sets

$$E_{r_1}^{i_0 i_1} E_{r_2}^{i_1 i_2} \cdots E_{r_n}^{i_{n-1} i_n} E_r^{i_n i_0}$$

are closed in the profinite topology on G .

Using that the sets are closed and do not contain 1, we can find a finite index, normal subgroup $K \trianglelefteq G$ such that whenever $i_0, \dots, i_n \leq m$ and $r_j, r \in \text{Ex}(A) \setminus \{0\}$ satisfy $r_1 + \cdots + r_n < r$, we have

$$1K \cap E_{r_1}^{i_0 i_1} E_{r_2}^{i_1 i_2} \cdots E_{r_n}^{i_{n-1} i_n} E_r^{i_n i_0} = \emptyset,$$

i.e., as K is normal,

$$1 \notin E_{r_1}^{i_0 i_1} E_{r_2}^{i_1 i_2} \cdots E_{r_n}^{i_{n-1} i_n} E_r^{i_n i_0} K = (E_{r_1}^{i_0 i_1} K)(E_{r_2}^{i_1 i_2} K) \cdots (E_{r_n}^{i_{n-1} i_n} K)(E_r^{i_n i_0} K).$$

We set $H_i = M_i K$ and $B_r^{ij} = E_r^{ij} K$. Then, as K is normal in G ,

$$B_r^{ij} = E_r^{ij} K = K E_r^{ij} = K^{-1} (E_r^{ji})^{-1} = (E_r^{ji} K)^{-1} = (B_r^{ji})^{-1}$$

and

$$H_i B_r^{ij} H_j = (M_i K)(E_r^{ij} K)(M_j K) = M_i E_r^{ij} M_j K = E_r^{ij} K = B_r^{ij}.$$

Finally, whenever $i_0, \dots, i_n \leq m$ and $r_j, r \in \text{Ex}(A) \setminus \{0\}$ satisfy $r_1 + \cdots + r_n < r$, we have

$$1 \notin B_{r_1}^{i_0 i_1} B_{r_2}^{i_1 i_2} \cdots B_{r_n}^{i_{n-1} i_n} B_r^{i_n i_0}.$$

Using the sets B_r^{ij} , we can define an $\text{Ex}(A)$ -valued metric ρ on the left-coset space

$$G/H_1 \sqcup G/H_2 \sqcup \cdots \sqcup G/H_m$$

by setting for distinct gH_i, fH_j ,

$$\rho(gH_i, fH_j) = \min \left(\max \text{Ex}(A), \inf \{r_1 + \dots + r_n \mid g^{-1}f \in B_{r_1}^{i_1} B_{r_2}^{i_2} \dots B_{r_n}^{i_n-1j} \text{ for some } i_1, \dots, i_{n-1} \leq m\} \right). \quad (1)$$

Note first that since $H_i B_r^{il} = B_r^{il}$ and $B_r^{lj} H_j = B_r^{lj}$ for any $r \in \text{Ex}(A) \setminus \{0\}$ and $l \leq m$, this definition does not depend on the choice of representatives g and f from the cosets gH_i and fH_j . Also, ρ is easily seen to satisfy the triangle inequality and the positivity condition, so ρ is indeed a metric on

$$G/H_1 \sqcup G/H_2 \sqcup \dots \sqcup G/H_m.$$

Moreover, for all $g, f, h \in G$ and i, j , we have

$$\rho(gH_i, fH_j) = \rho(hgH_i, hfH_j),$$

so ρ is invariant under the action of G by left-translation on $G/H_1 \sqcup \dots \sqcup G/H_m$.

We claim that if $g^{-1}h \in B_r^{ij}$, then $\rho(gH_i, hH_j) = r$. For if not, we would have $g^{-1}h \in B_{r_1}^{i_1} B_{r_2}^{i_2} \dots B_{r_n}^{i_n-1j}$ for some $i_1, \dots, i_{n-1} \leq m$ and $r_1 + \dots + r_n < r$. But, as $h^{-1}g \in B_r^{ji}$, this implies

$$1 = g^{-1}hh^{-1}g \in B_{r_1}^{i_1} B_{r_2}^{i_2} \dots B_{r_n}^{i_n-1j} B_r^{ji},$$

contradicting the assumptions on the sets B_r^{ij} .

Now define

$$\pi: A \rightarrow G/H_1 \sqcup G/H_2 \sqcup \dots \sqcup G/H_m$$

by $\pi(gN_i) = gH_i$ for all $g \in C$ such that $gN_i \in A$. We claim that π is an isometric F -embedding of A into $G/H_1 \sqcup \dots \sqcup G/H_m$.

First, to see that π is well-defined, i.e., that $\pi(gN_i)$ does not depend on the representative g of gN_i , suppose that $g, h \in C$ and $gN_i = hN_i \in A$. Then $g^{-1}h \in N_i$, so, as $1 \in F$, also $g^{-1}h \in M_i \leq M_i K = H_i$, whereby

$$\pi(gN_i) = gH_i = hH_i = \pi(hN_i).$$

Secondly, to see that π is an isometric embedding of A into Y , suppose $gN_i, hN_j \in A$ are distinct, where $g, h \in C$. Let $r = d(gN_i, hN_j) > 0$, whereby $g^{-1}h \in E_r^{ij} \subseteq B_r^{ij}$. It therefore follows that $\rho(gH_i, hH_j) = r$. So π is an isometry.

Finally, to see that π is a partial conjugation of the action of G , suppose $gN_i \in A$ and $f \cdot gN_i = hN_i \in A$ for some $g, h \in C$ and $f \in F$. Then $h^{-1}fg \in N_i$ and so $h^{-1}fg \in M_i \leq H_i$. Therefore, $fgH_i = hH_i$ and so

$$f\pi(gN_i) = fgH_i = hH_i = \pi(hN_i) = \pi(fgN_i).$$

This shows that if G has property (RZ), then any action of G by isometries is finitely approximable.

Suppose now conversely that any isometric action of G is finitely approximable and let H_1, \dots, H_n be finitely generated subgroups of G . We must show that for any $g \notin H_1 \cdots H_n$ there is a finite index subgroup $K \leq G$ such that $g \notin H_1 \cdots H_n K$. So let H_1, \dots, H_n and g be given. We set

$$X = G/H_1 \sqcup G/H_2 \sqcup \dots \sqcup G/H_n$$

and define a graph $\Gamma = (X, E)$ with vertex set X and edges $\{fH_i, kH_{i+1}\}$, where $fH_i \cap kH_{i+1} \neq \emptyset$. Now set

$$d_X(fH_i, kH_j) = \min(n, \text{number of edges in the shortest path from } fH_i \text{ to } kH_j \text{ in } \Gamma).$$

So (X, d_X) is an integer valued metric space. Also, the left shift action of G on X clearly preserves the distance d_X .

Notice that for any $f, k \in G$,

$$d_X(fH_1, kH_n) = \begin{cases} n-1 & \text{if there is a path } fH_1, f_2H_2, \dots, f_{n-1}H_{n-1}, kH_n \text{ in } \Gamma, \\ n & \text{otherwise.} \end{cases}$$

Note that if $f_1H_1, f_2H_2, \dots, f_nH_n$ is a path in Γ , then

$$f_1^{-1}f_2 \in H_1H_2 \ \& \ f_2^{-1}f_3 \in H_2H_3 \ \& \ \dots \ \& \ f_{n-1}^{-1}f_n \in H_{n-1}H_n,$$

whence

$$f_1^{-1}f_n = f_1^{-1}f_2 \cdot f_2^{-1}f_3 \cdots f_{n-1}^{-1}f_n \in H_1H_2 \cdot H_2H_3 \cdots H_{n-1}H_n = H_1H_2 \cdots H_n.$$

Since $g = 1^{-1}g \notin H_1H_2 \cdots H_n$, it follows that $d_X(H_1, gH_n) = n$.

Let now $A = \{H_1, H_2, \dots, H_n, gH_n\}$ and set

$$F = \{\text{generators of } H_1, H_2, \dots, H_n\} \cup \{1, g\}.$$

By the assumption on G , there is a finite metric space (Y, d_Y) upon which G acts by isometries and an isometric F -embedding $\pi: A \rightarrow Y$ of A into Y . Let now K_i be the stabiliser of $\pi(H_i)$ in G . Since Y is finite, K_i is a finite index subgroup of G . We claim that $H_i \leq K_i$. For if h is a generator of H_i , then, as $h \in F$ and $H_i, hH_i \in A$, we must have $\pi(H_i) = \pi(hH_i) = h\pi(H_i)$ and so $h \in K_i$. Thus, K_i contains the generators of H_i and hence $H_i \leq K_i$.

Now suppose $k_i \in K_i$. Since $H_1, \dots, H_n \in A$ and $d_X(H_i, H_{i+1}) = 1$, we have

$$\begin{aligned} d_Y(k_1 \cdots k_i \pi(H_i), k_1 \cdots k_i k_{i+1} \pi(H_{i+1})) &= d_Y(k_1 \cdots k_i \pi(H_i), k_1 \cdots k_i \pi(H_{i+1})) \\ &= d_Y(\pi(H_i), \pi(H_{i+1})) \\ &= 1. \end{aligned}$$

So, by the triangle inequality,

$$d_Y(\pi(H_1), k_1 k_2 \cdots k_n \pi(H_n)) = d_Y(k_1 \pi(H_1), k_1 k_2 \cdots k_n \pi(H_n)) \leq n-1.$$

In other words, for any $f \in K_1 \cdots K_n$, we have $d_Y(\pi(H_1), f\pi(H_n)) \leq n-1$.

Now, $g \in F$ and $H_n, gH_n \in A$, so $g\pi(H_n) = \pi(gH_n)$, whence

$$d_Y(\pi(H_1), g\pi(H_n)) = d_Y(\pi(H_1), \pi(gH_n)) = d_X(H_1, gH_n) = n.$$

So it follows that $g \notin K_1 K_2 \cdots K_n$, whence $g \notin H_1 H_2 \cdots H_n K_n \subseteq K_1 \cdots K_n$. \dashv

We should make a few comments on the above proof. Namely, one sees that in order for a product $H_1 \cdots H_n$ of finitely generated subgroups of G to be closed in the profinite topology, it suffices that any action of G by isometries on a metric space with distance set $\{0, 1, \dots, n\}$ is finitely approximable. Conversely, with somewhat more care, one can check that if any product $H_1 \cdots H_n$ of finitely generated subgroups of G is closed in the profinite topology, then any action of G by isometries on a metric space with distance set $\{0, 1, \dots, n\}$ is finitely approximable.

Another type of actions of interest to us is actions by automorphisms on a graph. Here a *graph* is a pair (X, E_X) of a non-empty set X and a symmetric, irreflexive relation E_X on X . So an action $G \curvearrowright X$ of a discrete group by automorphisms of a graph (X, E_X) is said to be *finitely approximable* if for any finite $A \subseteq X$ and $F \subseteq G$ there is a finite graph (Y, E_Y) on which G acts by automorphisms and an F -embedding $\pi: A \rightarrow Y$ embedding (A, E_X) into (Y, E_Y) .

Similarly, an action $G \curvearrowright X$ of a discrete group by permutations on a set X is *finitely approximable* if for any finite $A \subseteq X$ and $F \subseteq G$ there is a finite set Y on which G acts by permutations and an injective F -map $\pi: A \rightarrow Y$. Then, identifying a discrete set with a metric space with distance set $\{0, 1\}$ and a graph with a metric space with distance set $\{0, 1, 2\}$, the proof above show the following probably well-known facts.

PROPOSITION 8. *Let G be a discrete group.*

(A) *The following are equivalent.*

- *Any action of G by automorphisms of a graph is finitely approximable,*
- *if H and K are finitely generated subgroups of G , then HK is closed in the profinite topology on G .*

(B) *The following are equivalent.*

- *Any action of G by permutations of a set is finitely approximable,*
- *G is subgroup separable.*

The *Rado* or *random graph* is the countable graph \mathbf{R} , i.e., a set with a symmetric irreflexive relation, defined up to isomorphism by the following extension property: if X is a finite graph and $\phi: Y \rightarrow \mathbf{R}$ is an embedding of an induced subgraph, then ϕ extends to an embedding of X into \mathbf{R} .

E. Hrushovski investigated the case of graphs in [7], and showed that any finite graph can be extended to a larger finite graph in such a way that any partial automorphism of the former extends to a full automorphism of the latter. Modulo the existence of an ultrahomogeneous universal graph, namely the random graph, this is equivalent to showing that any action of a free group by automorphisms of a graph is finitely approximable. So, by the above proposition, this implies the special case of the Ribes–Zalesskiĭ Theorem that a product of two finitely generated subgroups of a free group is closed in the profinite topology.

§3. Locally finite dense subgroups of automorphism groups. The *rational Urysohn metric space* \mathbb{QU} is the countable metric space all of whose distances are rational and satisfying the following extension property: if X is a finite metric space with rational distances and $\phi: Y \rightarrow \mathbb{QU}$ is an isometric embedding of a subset $Y \subseteq X$, then ϕ extends to an isometric embedding of X into \mathbb{QU} .

Using the extension property, it is easy to see that any countable metric space with rational distances embeds isometrically into \mathbb{QU} and also, by a standard back and forth argument, the above properties define \mathbb{QU} uniquely up to isometry.

The rational Urysohn space is useful for us in that it and its isometry group function as a universal framework for certain arguments about metric spaces. To see this, we need to introduce a few notions.

DEFINITION 9. *An isometry $f: A \rightarrow B$ between finite subsets A and B of the rational Urysohn space \mathbb{QU} is said to be a finite partial isometry of \mathbb{QU} .*

So the restriction of any full isometry of $\mathbb{Q}\mathbb{U}$ to a finite subset is a finite partial isometry, but more importantly, by a back and forth argument, any finite partial isometry of $\mathbb{Q}\mathbb{U}$ extends to a full isometry of $\mathbb{Q}\mathbb{U}$.

In fact, we also have the following interesting fact due to V. V. Uspenskiĭ [12]; namely, if G is a group acting by isometries on a finite subspace $A \subseteq \mathbb{Q}\mathbb{U}$, then the action of G extends to an action by isometries on all of $\mathbb{Q}\mathbb{U}$. To see this, we can without loss of generality suppose that G is finite. Also, modulo an inductive construction, it suffices to show that for any one-point extension $B \supseteq A$, there is a further finite extension $C \supseteq B$ and an action of G on C extending the action of G on A . We identify the unique point in $B \setminus A$ with $1 \in G$ and extend the metric d on $B = A \sqcup \{1\}$ to all of $C = A \sqcup G$, by letting $d(x, h) = d(h^{-1}x, 1)$ for $x \in A$ and $h \in G$ and setting

$$d(g, h) = \min (d(g, x) + d(x, h) \mid x \in A)$$

for all $g, h \in G$. This is easily seen to be a rational-valued metric extending the metric on B and the invariance under the left-shift action by G is trivial.

We equip the group $\text{Isom}(\mathbb{Q}\mathbb{U})$ of isometries of $\mathbb{Q}\mathbb{U}$ with the *permutation group topology*, that is, the basic open neighbourhoods of $f \in \text{Isom}(\mathbb{Q}\mathbb{U})$ are of the form

$$U(f, x_1, \dots, x_n) = \{g \in \text{Isom}(\mathbb{Q}\mathbb{U}) \mid g(x_i) = f(x_i), i \leq n\},$$

where $x_1, \dots, x_n \in \mathbb{Q}\mathbb{U}$. Since $\mathbb{Q}\mathbb{U}$ is countable, it is easy to see that $\text{Isom}(\mathbb{Q}\mathbb{U})$ is a *Polish group*, i.e., a separable and completely metrisable topological group.

We shall now use the correspondence developed above in conjunction with the main result of Coulbois [3] to give an alternative proof of an unpublished result of Solecki (we include the result here with Solecki's permission).

THEOREM 10 (S. Solecki). *The isometry group $\text{Isom}(\mathbb{Q}\mathbb{U})$ of the rational Urysohn metric space has a dense, locally finite subgroup.*

PROOF. Let f_0, f_1, f_2, \dots be a listing of a dense subset of $\text{Isom}(\mathbb{Q}\mathbb{U})$ in which every element is listed infinitely often. Let also $\emptyset = A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ be an increasing exhaustive sequence of finite subsets of $\mathbb{Q}\mathbb{U}$. We shall construct an increasing sequence of finite groups $G_0 \leq G_1 \leq G_2 \leq \dots$ and finite subsets $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots \subseteq \mathbb{Q}\mathbb{U}$, where each G_n acts faithfully by isometries on B_n in such a way that the action of G_{n+1} on B_{n+1} extends the action of G_n on B_n , $A_n \subseteq B_n$, and for every n there is some $g \in G_n$ such that $g|_{A_n} = f_n|_{A_n}$. In this case, $G = \bigcup_n G_n$ naturally acts by isometries on $\mathbb{Q}\mathbb{U} = \bigcup_n B_n$, and letting Γ be the image of G by the natural homomorphism into $\text{Isom}(\mathbb{Q}\mathbb{U})$, we see that Γ is locally finite and dense in $\text{Isom}(\mathbb{Q}\mathbb{U})$.

To see how this is done, suppose B_n and G_n are defined and consider f_{n+1} and A_{n+1} . Arbitrarily extend the action of G_n on the finite subset $B_n \subseteq \mathbb{Q}\mathbb{U}$ to an action by isometries on $\mathbb{Q}\mathbb{U}$. So letting the generator of the infinite cyclic group \mathbb{Z} act as f_{n+1} , we see that this defines an action of $G_n * \mathbb{Z}$ by isometries on $\mathbb{Q}\mathbb{U}$. As G_n is finite, G_n has property (RZ) and so does \mathbb{Z} . By the main result of Coulbois [3], also $G_n * \mathbb{Z}$ has property (RZ). It follows by Theorem 7 that there is a finite rational metric space Y on which $G_n * \mathbb{Z}$ acts by isometries and an isometric $G_n \cup \{f_{n+1}\}$ -embedding of $B_n \cup A_{n+1}$ into Y . By the extension property of $\mathbb{Q}\mathbb{U}$, we can suppose that actually $B_n \subseteq B_n \cup A_{n+1} \subseteq Y \subseteq \mathbb{Q}\mathbb{U}$. Now let $B_{n+1} = Y$ and let H be the image of $G_n * \mathbb{Z}$ in $\text{Isom}(Y)$. Since G_n acts faithfully on $B_n \subseteq Y$, the

canonical homomorphism is injective on G_n and so we can suppose that $G_n \leq H$. Set $G_{n+1} = H$. \dashv

By exactly the same argument as above, we see that when equipped with the permutation group topology, $\text{Aut}(\mathbf{R})$ has a locally finite dense subgroup. This gives an alternative proof of the main result of M. Bhattacharjee and D. Macpherson [2].

We see that the above method applies fairly generally to automorphism groups of many random relational structures, the main issue being whether they admit an analogue of Theorem 7.

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