Descriptive Classification Theory and Separable Banach Spaces

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Three Notions of Classification

Consider your favorite class of mathematical structures, be it groups, modules, measure-preserving transformations, C^*-algebras, Lie groups, smooth manifolds, or something completely different. With some probability, the classification problem for these objects, that is, the problem of determining the structures up to some relevant notion of isomorphism, is, or has been, one of the central problems of the corresponding field of study.

Of course, inasmuch as mathematical theories stem from attempts to model or organize physical or other phenomena, the classification problem might not be the primordial challenge. But once the basic theorems of a theory have been worked out, there is often an internal motivation to categorize its different models.

For example, the definition of a Banach space as a complete normed vector space is motivated by the study of function spaces as the potential solution sets to various differential equations modeling physical phenomena. But, as is known to all of us, the common aspects of the individual problems often simplify through abstraction, whence the concept of an abstract Banach space. And therefore having an isomorphic classification of Banach spaces would certainly be helpful when dealing with more concrete problems involving these spaces.

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Without doubt, one the most gratifying examples of classification is that of finite simple groups. In this case, we have a catalogue or explicit listing of all isomorphism types of finite simple groups, something we certainly cannot hope for in all other classes of mathematical objects. So with finite simple groups as our paragon, an optimal classification of some class $\mathcal{A}$ of mathematical structures up to a corresponding notion of isomorphism would seem to be an explicit listing of all isomorphism types of $\mathcal{A}$-structures, plus perhaps some reasonable algorithm or procedure for deciding the isomorphism type of each $\mathcal{A}$-structure.

In order to better understand what we mean by an isomorphic classification, we must first make the distinction between the isomorphism types and the concrete instances or realizations of these. It is the latter that we wish to classify. For example, by a *classification* of finitely generated groups, we understand some abstract procedure that given two presentations of finitely generated groups decides whether these are presentations of the same group up to isomorphism, i.e., if they are instances of the same isomorphism type. From our perspective, the isomorphism types themselves are abstract platonic objects that can be grasped and concretely manipulated only in special instances, namely, when the isomorphism relation is "smooth" and thus the types correspond to points in a sufficiently nice topological space (we shall come back to this later).

With this in mind, we can, with some amount of simplification, distinguish at least three notions of classification:

- An explicit listing of all isomorphism types of elements of $\mathcal{A}$.

– A classification of the objects of $\mathcal{A}$ via an assignment of complete invariants.
– A determination of the "irreducible" or "prime" $\mathcal{A}$-objects.

Assigning to every object $X \in \mathcal{A}$ a complete invariant from another class of mathematical objects $\mathcal{B}$ would ideally mean a function $\phi\colon \mathcal{A} \to \mathcal{B}$ such that two $\mathcal{A}$-objects X and Y are isomorphic if and only if $\phi(X) = \phi(Y)$. However, this holds only if we consider the $\mathcal{B}$-objects as *isomorphism types*. Thus, to avoid presupposing that we already understand the isomorphism types of $\mathcal{B}$-objects, we see that the right formulation is rather that $\phi\colon \mathcal{A} \to \mathcal{B}$ should be *isomorphism invariant*, i.e.,

$$X \cong Y \;\Rightarrow\; \phi(X) \cong \phi(Y),$$

and *complete*, i.e.,

$$\phi(X) \cong \phi(Y) \;\Rightarrow\; X \cong Y.$$

Prominent examples of classification by complete invariants include the Elliot classification of approximately finite-dimensional C^*-algebras via their dimension groups and the Ornstein classification of Bernoulli shifts by their entropy.

The third type of classification sometimes can lead to complete classification results, provided that any $\mathcal{A}$-object can be uniquely represented in terms of its irreducible or prime components. But, oftentimes, one can hope only to isolate the irreducible parts without an actual reconstruction of the full object from these. We shall return to this later in the article in connection with Banach spaces.

Parametrizations and Standard Borel Spaces

When parametrizing a class of mathematical objects, we choose a particular method of presenting these and then regard the totality of such presentations. For example, the finitely generated groups can be parametrized by infinite tuples

$$G = \langle a_1, a_2, \dots, a_n \mid w_1 = 1, w_2 = 1, \dots \rangle,$$

where $a_1, \dots, a_n$ is a distinguished set of generators and $w_i = w_i(a_1, \dots, a_n)$ are group words in these generators listed in some canonical way. An only slightly different way of presenting such G would be as quotients of the free group $\mathbb{F}_\infty$ on the alphabet $a_1, a_2, \dots$ by normal subgroups $N \trianglelefteq \mathbb{F}_\infty$ such that $a_n \in N$ for all but finitely many n. So, the set $\mathcal{G}$ of these special normal subgroups of $\mathbb{F}_\infty$ can be seen as a parametrization of the class of finitely generated groups. The space $\mathcal{G}$ is called the *space of finitely generated groups* and has some very interesting properties making it a particularly well-behaved model. In particular, if $\mathrm{Aut}_f(\mathbb{F}_\infty)$ denotes the group of automorphisms ϕ of $\mathbb{F}_\infty$ such that $\phi(a_n) = a_n$ for all but finitely many n, then $\mathrm{Aut}_f(\mathbb{F}_\infty)$ is countable. Also, if $N, M \in \mathcal{G}$, then the two groups $\mathbb{F}_\infty/N$ and $\mathbb{F}_\infty/M$ are isomorphic if and only if for some $\phi \in \mathrm{Aut}_f(\mathbb{F}_\infty)$ we have $\phi[N] = M$. In other words, the relation of isomorphism between quotient groups $\mathbb{F}_\infty/N$ for $N \in \mathcal{G}$ is induced by the countable group $\mathrm{Aut}_f(\mathbb{F}_\infty)$ acting on $\mathcal{G}$ (C. Champetier (2000)).

A convenient framework for dealing with classification problems is that of Polish and standard Borel spaces.

Definition 1. A topological space $(\mathcal{X}, \tau)$ is said to be *Polish* if it is separable and its topology can be given by a complete metric on $\mathcal{X}$.

A measurable space $(\mathcal{X}, \mathfrak{B})$, i.e., a set $\mathcal{X}$ equipped with a σ-algebra of subsets $\mathfrak{B}$, is said to be *standard Borel* if there is a Polish topology τ on $\mathcal{X}$ with respect to which $\mathfrak{B}$ is the class of Borel sets.

In the latter case, the sets in $\mathfrak{B}$ are called the Borel sets of $\mathcal{X}$.

For example, the space $\mathcal{G}$ of finitely generated groups can be made into a standard Borel space by equipping it with the σ-algebra generated by sets of the form

$$\{N \in \mathcal{G} \mid g \in N\},$$

where g varies over elements of $\mathbb{F}_\infty$.

Another example, which will figure prominently here, is that of separable Banach spaces. By the Banach–Mazur theorem, any separable Banach space embeds linearly isometrically into the space $C([0,1])$ of continuous functions on $[0,1]$. Thus one way of parametrizing separable Banach spaces is as the set $\mathcal{SB}$ of closed linear subspaces of $C([0,1])$. Of course, there are other equally natural parametrizations, for example, any separable Banach space X is linearly isometric to a quotient of ℓ_1 by a closed linear subspace $Y \subseteq \ell_1$, and hence we can use the set $\mathcal{SB}(\ell_1)$ of closed linear subspaces Y of ℓ_1 as another parametrization. In order to make these sets into standard Borel spaces, we equip them with the *Effros Borel structure*, which is the σ-algebra generated by the sets of the form

$$\{X \in \mathcal{SB} \mid X \cap U \neq \emptyset\},$$

respectively

$$\{X \in \mathcal{SB}(\ell_1) \mid X \cap U \neq \emptyset\},$$

where U runs over the open subsets of $C([0,1])$, respectively of ℓ_1. However, it is a fact that these two models are equivalent in the following precise sense: There is a Borel isomorphism $\phi\colon \mathcal{SB} \to \mathcal{SB}(\ell_1)$, i.e., an isomorphism of measurable spaces, such that any $X \in \mathcal{SB}$ is linearly isometric to $\ell_1/\phi(X)$. This is just one instance of the empirical fact that different, but natural, parametrizations of the same class of mathematical objects are equivalent. So it is not too important which specific parametrization one is working with, though, of course, certain computations might be more easily performed in one

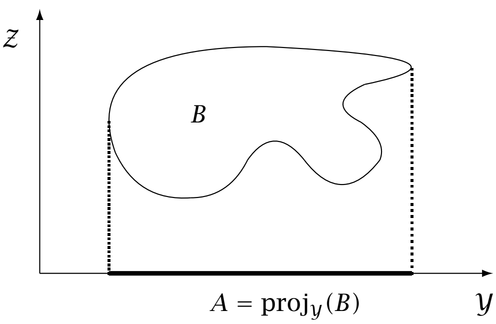

Figure 1. Projections.

model than in another. Therefore, we can think of $\mathcal{G}$ and $\mathcal{SB}$ as the standard Borel spaces of finitely generated groups, respectively separable Banach spaces, even though $\mathcal{G}$ and $\mathcal{SB}$ themselves are of a very different nature from groups or Banach spaces.

There are admittedly limitations to the classes of mathematical objects that admit standard Borel parametrizations. For, as the underlying Polish topology of a standard Borel space is Hausdorff and admits a countable basis for its topology, any standard Borel space has cardinality at most $2^{\aleph_0}$. In fact, by a result of K. Kuratowski, all uncountable standard Borel spaces are isomorphic to $\mathbb{R}$ equipped with its algebra of Borel sets. So we cannot model classes having too many objects. The ones that do admit standard Borel parametrizations are mostly either classes of countable algebraic and combinatorial objects (groups, rings, graphs, ...) or separable topological and analytical objects (algebraic subvarieties of $\mathbb{C}^n$, compact smooth manifolds, measure-preserving transformations, complete separable metric spaces, ...).

Once we have constructed our standard Borel space $(\mathcal{X}, \mathfrak{B})$, we can go on to the isomorphism relation itself. For example, in the case of $\mathcal{G}$, we see that the relevant notion of isomorphism is not that of isomorphism $\cong$ between $N, M \in \mathcal{G}$, but rather isomorphism between the quotients $\mathbb{F}_\infty/N$ and $\mathbb{F}_\infty/M$. So let

$$N \mathsf{E} M \iff \mathbb{F}_\infty/N \cong \mathbb{F}_\infty/M.$$

Then E is an equivalence relation on $\mathcal{G}$ induced by an action of the countable group $\mathrm{Aut}_f(\mathbb{F}_\infty)$ and hence has countable classes. Moreover, E is a Borel subset of the standard Borel space $\mathcal{G} \times \mathcal{G}$. However, in general, the corresponding isomorphism relation might not be Borel, though most often it is *analytic*.

Definition 2. Let $(\mathcal{Y}, \mathfrak{B})$ be a standard Borel space. A subset $A \subseteq \mathcal{Y}$ is said to be *analytic* if there is a standard Borel space $(\mathcal{Z}, \mathfrak{C})$ and a Borel subset B of $(\mathcal{Y} \times \mathcal{Z}, \mathfrak{B} \otimes \mathfrak{C})$ such that

$$y \in A \iff \exists z \in \mathcal{Z}\ (y, z) \in B.$$

In other words, a set is analytic if it is the projection of a Borel set (see Figure 1).

A very useful way of thinking of Borel and analytic sets, which is now known as the Kuratowski–Tarski algorithm, is in terms of the quantifier complexity of their definitions. Thus Borel sets are those that can be inductively defined from open sets in the underlying Polish topology by using only quantifiers over countable sets, while analytic sets are those that can be defined using quantifiers over countable sets and a single positive instance of an existential quantifier over a standard Borel space.

As an example, recall that two Banach spaces X and Y are said to be *isomorphic* if there is a bounded, bijective, linear operator $T \colon X \to Y$ (whereby T is a linear homeomorphism). Let F denote the relation of isomorphism between elements of $\mathcal{SB}$. We can construct a standard Borel space $\mathcal{Z}$ of isomorphisms $T : X \to Y$ between closed linear subspaces of $C([0,1])$ such that the set

$$\begin{aligned} B = \{&(T, X, Y) \in \mathcal{Z} \times \mathcal{SB} \times \mathcal{SB} \mid \\ &T \text{ is an isomorphism between } X \text{ and } Y\} \end{aligned}$$

is Borel. So as

$$X \mathsf{F} Y \iff \exists T \in \mathcal{Z}\ (T, X, Y) \in B,$$

we see that F is analytic as a subset of $\mathcal{SB} \times \mathcal{SB}$.

While it is easy to see that any Borel set is analytic, not every analytic set is Borel. For example, the relation F above is not Borel (B. Bossard (1993)).

We now have the necessary framework to formulate the abstract concept of classification by complete invariants.

Definition 3. Let E and F be analytic equivalence relations on standard Borel spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. We say that E is *Borel reducible* to F, in symbols $\mathsf{E} \leqslant_B \mathsf{F}$, if there is a Borel measurable function $\phi \colon \mathcal{X} \to \mathcal{Y}$ such that for all $x, x' \in \mathcal{X}$,

$$x \mathsf{E} x' \iff \phi(x) \mathsf{F} \phi(x').$$

When

$$\mathsf{E} \leqslant_B \mathsf{F} \leqslant_B \mathsf{E},$$

E and F are said to be *Borel bireducible*, $\mathsf{E} \sim_B \mathsf{F}$, and when

$$\mathsf{F} \not\leqslant_B \mathsf{E} \leqslant_B \mathsf{F},$$

we write $\mathsf{E} <_B \mathsf{F}$.

The partial preorder $\leqslant_B$ of Borel reducibility between analytic equivalence relations should be understood as a relative measure of complexity, so that if $\mathsf{E} \leqslant_B \mathsf{F}$, E is simpler than F.

In particular, a Borel reduction ϕ of the relation E of isomorphism defined on $\mathcal{G}$ to the relation F of isomorphism on $\mathcal{SB}$ can be seen as a Borel assignment of separable Banach spaces to finitely generated groups as complete isomorphism invariants. While, as we shall see, there is such a reduction, there is no reduction the other way, i.e., $\mathsf{E} <_B \mathsf{F}$. So in terms of complexity, the isomorphism relation between separable Banach spaces is strictly more complex than that of isomorphism between finitely generated groups. And there is no way of using the latter as complete invariants (at least in a Borel manner) for isomorphism of separable Banach spaces.

The requirement that the reduction be Borel corresponds to a requirement that the assignment of invariants be somehow explicit. In particular, one sees that there are both $2^{\aleph_0}$ finitely generated groups and $2^{\aleph_0}$ separable Banach spaces up to isomorphism, so, by simple cardinality considerations, one easily gets non-Borel reductions in both directions. However, as assignments of invariants these are of little practical use, and indeed only one of the reductions can be made Borel.

Analytic Equivalence Relations

The notion of Borel reducibility was introduced independently by H. Friedman–L. Stanley (1989) and L. A. Harrington–A. S. Kechris–A. Louveau (1990) as a distillation of ideas originating in model theory and operator algebras. Since its inception, many prominent classification problems have been considered in this light in order to determine their relative complexity with respect to the Borel reducibility ordering. For example, in ergodic theory, two of the grand successes have been the classification by P. R. Halmos and J. von Neumann (1942) of the measure-preserving automorphisms with discrete spectrum and D. S. Ornstein's classification of Bernoulli shifts by Kolmogorov and Sinai's notion of entropy (1970).

Though certainly the deeper of the two, Ornstein's theorem is actually easier to understand, so let us consider this first. Suppose $p_1, \dots, p_n$ are positive real numbers such that $\sum_{i=1}^{n} p_i = 1$ and give the set $\{1, \dots, n\}$ the distribution $\vec{p} = (p_1, \dots, p_n)$, which makes it into a probability space. We then equip $\Omega = \{1, \dots, n\}^{\mathbb{Z}}$ with the product probability measure and define a measure-preserving automorphism S of Ω as the bilateral shift

$$S(\dots, x_{-2}, x_{-1}, \dot{x}_0, x_1, x_2, \dots) = (\dots, x_{-1}, x_0, \dot{x}_1, x_2, x_3, \dots).$$

The system (Ω, S) is called a *Bernoulli shift*, and its *entropy* is defined to be the real number

$$h(\Omega, S) = -\sum_{i=1}^{n} p_i \log p_i.$$

We say that two Bernoulli shifts (Ω, S) and (Ω', S') given by distributions $\vec{p} = (p_1, \dots, p_n)$ and $\vec{q} = (q_1, \dots, q_m)$ are *measurably conjugate* if there is an isomorphism of measure spaces, $T\colon \Omega \to \Omega'$, such that

$$S = T^{-1} \circ S' \circ T.$$

A. N. Kolmogorov and Ya. G. Sinai showed that entropy is an invariant for Bernoulli shifts, i.e., that measurably conjugate Bernoulli shifts have the same entropy. On the other hand, that entropy is a complete invariant was discovered by Ornstein, who showed that two Bernoulli shifts (Ω, S) and (Ω', S') with the same entropy, $h(\Omega, S) = h(\Omega', S')$, are measurably conjugate. Moreover, it is easy to see that the entropy function ϕ defined on the standard Borel space of Bernoulli shifts is Borel and so ϕ is a Borel reduction of conjugacy of Bernoulli shifts to the equivalence relation $=_{\mathbb{R}}$ of equality on the set of real numbers.

We say that an analytic equivalence relation E on a standard Borel space $\mathcal{X}$ is *smooth* if E is Borel reducible to the relation $=_{\mathbb{R}}$, i.e., if there is a Borel function $\phi\colon \mathcal{X} \to \mathbb{R}$ such that

$$x \mathsf{E} y \Leftrightarrow \phi(x) = \phi(y).$$

What makes the smooth equivalence relations particularly tangible is that, in this case, the quotient space, $\mathcal{X}/\mathsf{E}$, is Borel isomorphic with an analytic set, and hence the E-classes can be identified with points in a standard Borel space. For nonsmooth E, on the other hand, the space $\mathcal{X}/\mathsf{E}$ has no reasonable measurable structure. So the smooth equivalence relations, such as conjugacy of Bernoulli shifts, are especially simple and, in the literature on operator algebras, are usually singled out as those that are *classifiable*. However, we shall avoid that terminology here, as some archetypical examples of classification are in fact of nonsmooth isomorphism relations.

An example of this is the Halmos–von Neumann theorem. A measure-preserving automorphism T of $[0,1]$ has *discrete spectrum* if there is an orthonormal basis of eigenvectors for the associated unitary operator U_T on $L_2([0,1])$ defined by $U_T(f) = f \circ T$. The Halmos–von Neumann theorem states that two discrete spectrum transformations are measurably conjugate if they have the same spectrum. So, in other words, the spectrum, which is a countable subset[1] of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, is a complete invariant for conjugacy of discrete spectrum measure-preserving automorphisms. However, there is no *canonical* way of enumerating a countable subset of S^1 (the reader is invited to try to construct such an enumeration of an arbitrary countable subset of $\mathbb{R}$ and see why this fails). In fact, being able to uniformly choose an enumeration of each countable subset of S^1 relies essentially on a weak version of the Axiom of Choice. So it is impossible for this choice to

[1] *In fact, a countable* subgroup.

be Borel. The best we can do is let $\mathsf{E}_{\mathrm{set}}$ be the equivalence relation on $(S^1)^{\mathbb{N}}$ defined by

$$(x_n)\mathsf{E}_{\mathrm{set}}(y_n) \Leftrightarrow \{x_1, x_2, x_3, \ldots\} = \{y_1, y_2, y_3, \ldots\},$$

i.e., (x_n) and (y_n) are $\mathsf{E}_{\mathrm{set}}$-equivalent if they enumerate the same *set*. Then $\mathsf{E}_{\mathrm{set}}$ is a Borel equivalence relation and is actually Borel bireducible with conjugacy of discrete spectrum measure-preserving automorphisms. However, $=_{\mathbb{R}} <_B \mathsf{E}_{\mathrm{set}}$ and so $\mathsf{E}_{\mathrm{set}}$ is nonsmooth.

As we see from the above examples, it is useful to have a catalogue of various more combinatorial examples of analytic equivalence relations with which we can compare concrete classification problems. For example, $\mathsf{E}_{\mathrm{set}}$ is much easier to handle directly than conjugacy of discrete spectrum measure-preserving automorphisms. One such combinatorial example that has long been considered in the literature on operator algebras is the relation E_0 of eventual agreement of infinite binary sequences, i.e., for $(x_n), (y_n) \in \{0,1\}^{\mathbb{N}}$, we set

$$(x_n)\mathsf{E}_0(y_n) \quad\Leftrightarrow\quad \exists N\ \forall n \geqslant N\ x_n = y_n.$$

While it is easy to Borel reduce $=_{\mathbb{R}}$ to E_0, a simple measure-theoretic argument shows that there is no reduction in the other direction: First let μ be the product probability measure on $\{0,1\}^{\mathbb{N}}$ given by the distribution $(\frac{1}{2}, \frac{1}{2})$ on $\{0,1\}$. By Kolmogorov's zero-one law, any Borel subset of $\{0,1\}^{\mathbb{N}}$ that depends only on the tail of its elements, i.e., which is E_0-invariant, must either have μ-measure 0 or 1. So suppose toward a contradiction that

$$\phi\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}$$

is a Borel function such that

$$(x_n)\mathsf{E}_0(y_n) \quad\Leftrightarrow\quad \phi\big((x_n)\big) = \phi\big((y_n)\big).$$

Then if $\{I_n\}$ is a listing of all open intervals of $\mathbb{R}$ with rational endpoints, we see that each $\phi^{-1}(I_n) \subseteq \{0,1\}^{\mathbb{N}}$ only depends on the tail of its elements and hence has μ-measure 0 or 1. Letting A be the intersection of all $\phi^{-1}(I_n)$ and $\{0,1\}^{\mathbb{N}} \setminus \phi^{-1}(I_n)$ of measure 1,

$$A = \Big[\bigcap_{\mu(\phi^{-1}(I_n))=1} \phi^{-1}(I_n)\Big] \setminus \Big[\bigcup_{\mu(\phi^{-1}(I_n))=0} \phi^{-1}(I_n)\Big],$$

we have $\mu(A) = 1$, while ϕ must be constant on A. But, as any E_0-class is countable and hence of measure 0, there are $(x_n), (y_n) \in A$ that are E_0-inequivalent, while $\phi\big((x_n)\big) = \phi\big((y_n)\big)$. This contradicts the assumptions on ϕ, and so E_0 is nonsmooth.

Other examples are the relation E_1 defined on $\mathbb{R}^{\mathbb{N}}$ in a similar fashion by

$$(x_n)\mathsf{E}_1(y_n) \quad\Leftrightarrow\quad \exists N\ \forall n \geqslant N\ x_n = y_n,$$

and the relation E_{ℓ_∞} also defined on $\mathbb{R}^{\mathbb{N}}$ by

$$(x_n)\mathsf{E}_{\ell_\infty}(y_n) \quad\Leftrightarrow\quad \exists K\ \forall n\ |x_n - y_n| \leqslant K.$$

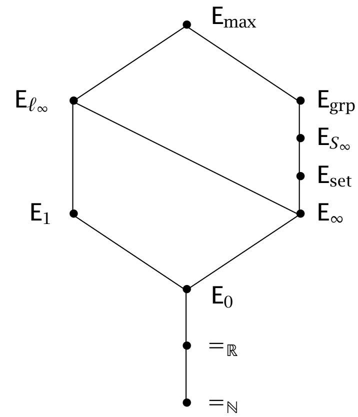

Figure 2. Significant examples of analytic equivalence relations under Borel reducibility.

As $=_{\mathbb{R}}$, we can of course also define $=_{\mathbb{N}}$, which is the equality relation on $\mathbb{N}$. The relationship between these and other equivalence relations is most easily visualized in a diagram with the simplest relations at the bottom (see Figure 2).

The remainder of the relations in Figure 2 have more complex descriptions, namely, E_∞ is the orbit equivalence relation induced by the shift action of $\mathbb{F}_2$ on the space $\{0,1\}^{\mathbb{F}_2}$ of subsets of $\mathbb{F}_2$, and E_{S_∞} is the $\leqslant_B$-maximum isomorphism relation between countable algebraic structures, which can, e.g., be realized as the isomorphism relation between countable groups. Similarly, $\mathsf{E}_{\mathrm{grp}}$ is the equivalence relation $\leqslant_B$-maximum among all orbit equivalence relations induced by an action of a Polish topological group, while $\mathsf{E}_{\max}$ is the analytic equivalence relation maximum in the Borel reducibility ordering.

Other examples of classification problems, or more precisely isomorphism relations, Borel bireducible with the relations of Figure 2 are the isomorphism relation between torsion-free Abelian groups of rank 1, which is bireducible with E_0; Lipschitz isomorphism of compact metric spaces, which is bireducible with E_{ℓ_∞}; and isometry of separable, complete metric spaces, which is bireducible with $\mathsf{E}_{\mathrm{grp}}$. If one instead considers finitely generated algebraic structures, the complexity of the corresponding isomorphism relation decreases and will be Borel reducible to E_∞. For example, isomorphism of finitely generated groups is bireducible with E_∞.

Dichotomy Theorems for Borel Reducibility

The continuum hypothesis, as proposed by G. Cantor, is the statement that every infinite subset of the real line is in bijective correspondence with either $\mathbb{N}$ or $\mathbb{R}$. In the first attempt at proving the hypothesis, Cantor showed that any *closed* subset of $\mathbb{R}$ is either countable or contains a homeomorphic copy of the Cantor space $2^{\mathbb{N}}$, which is itself of cardinality equal to that of the continuum. Somewhat later, P. S. Alexandrov and F. Hausdorff (1915) independently extended this result to include all Borel subsets of $\mathbb{R}$, and again this was extended by M. Y. Souslin (1917) to the class of analytic subsets of $\mathbb{R}$. However, after this, further development stalled for some time, due to the failure to prove a similar result even for sets that are complements of analytic sets, called *coanalytic sets*. Similarly, other regularity properties that had been established for analytic sets, such as Lebesgue measurability, could not be verified for more complex *projective* sets, i.e., those defined from the Borel sets by iterated complementations and projections.

N. Lusin (1925) made serious, but fruitless, attempts at deciding whether any uncountable coanalytic set is of the power of the continuum and showed enormous prescience by declaring

> Les efforts que j'ai faits pour résoudre cette question m'ont conduit à ce résultat tout inattendu: *il existe une famille admettant une application sur le continu d'ensembles effectifs telle qu'on ne sait pas* et l'on ne saura jamais *si un ensemble quelconque de cette famille (supposé non dénombrable) a la puissance du continu, s'il est ou non de troisième catégorie, ni même s'il est mesurable.*

That is, "one does not know and one will never know [of the uncountable projective sets] whether or not they are all of the power of the continuum."

His bold contention was corroborated by K. Gödel's development of the *constructible universe L* (1938), a model of set theory in which the continuum hypothesis holds true, but in which there are non-Lebesgue measurable projective sets and so-called *thin* coanalytic sets, i.e., uncountable coanalytic sets not containing a homeomorphic copy of $2^{\mathbb{N}}$. With P. J. Cohen's introduction of the method of forcing (1963), it was realized that one can construct thin coanalytic sets of size $\aleph_1$ in models of set theory where $|\mathbb{R}| > \aleph_1$ and so even coanalytic sets can be counterexamples to the continuum hypothesis.[2]

In line with the feeling that the continuum hypothesis should hold for explicitly defined sets in analysis, R. L. Vaught conjectured (1961) that any first-order theory T in a countable language has either at most countably many countable nonisomorphic models or, alternatively, a continuum of such. The conjecture remains open now after close to fifty years, but the strongest partial result was proved by M. Morley (1970), who showed that any first-order theory T in a countable language has either at most $\aleph_1$ countable nonisomorphic models or a continuum of these. Morley's theorem can be formulated in a stronger way once we notice that the set of models of T with underlying domain $\mathbb{N}$ can be made into a standard Borel space $\mathfrak{M}_T$ on which the relation of isomorphism $\cong$ is an analytic equivalence relation. The theorem then states that either there are only $\aleph_1$ $\cong$-classes on $\mathfrak{M}_T$ or there is an uncountable Borel set $A \subseteq \mathfrak{M}_T$ of pairwise nonisomorphic models (and hence continuum many by the theorem of Alexandrov and Hausdorff).

In view of Vaught's conjecture and Souslin's result for analytic sets, it is natural to ask simply if for any analytic equivalence relation E on a standard Borel space $\mathcal{X}$ there are either countably many E-classes or an uncountable Borel set $A \subseteq \mathcal{X}$ of E-inequivalent points. However, this is easily seen to be false by considering the following example: Let $\mathcal{X} = \{0,1\}^{\mathbb{Q}}$ be the space of all subsets of $\mathbb{Q}$ and define an analytic equivalence relation E on $\mathcal{X}$ by setting

$$x\mathsf{E}y \iff <\restriction_x \text{ and } <\restriction_y \text{ are isomorphic well orderings, or neither } <\restriction_x \text{ nor } <\restriction_y \text{ is well ordered.}$$

Then E has $\aleph_1$ equivalence classes: a class corresponding to the subsets of $\mathbb{Q}$ of order type ξ for every countable ordinal $\xi < \aleph_1$, and a single class corresponding to the non-well-ordered subsets of $\mathbb{Q}$. Thus, if the continuum is larger than $\aleph_1$, E cannot satisfy the stated conjecture, but, in fact, it can be shown that independently from any extra set theoretical assumptions, there cannot be an uncountable Borel set $A \subseteq \mathcal{X}$ of E-inequivalent points.

Instead, in a technical tour de force involving a large arsenal of modern set theory, J. H. Silver proved the optimal conclusion for coanalytic equivalence relations.

Theorem 4 (J. H. Silver (1980))**.** *Let E be a coanalytic equivalence relation on a standard Borel space $\mathcal{X}$. Then exactly one of the following holds:*

- *there are at most countably many E-classes,*
- *there is an uncountable Borel set $A \subseteq \mathcal{X}$ of E-inequivalent points.*

Using Silver's dichotomy one can recuperate the result of Souslin: if $A \subseteq \mathbb{R}$ is an infinite analytic set, define a coanalytic equivalence relation E on $\mathbb{R}$ by setting

$$x\mathsf{E}y \Leftrightarrow x, y \notin A \text{ or } x = y.$$

Then $|\mathcal{X}/\mathsf{E}| = |A|$, and if there is an uncountable Borel set of E-inequivalent points, A contains an uncountable Borel set.

[2] *For a thorough discussion of the current research on this, one can consult W. H. Woodin's article in this journal* [12].

Unfortunately, coanalytic equivalence relations very rarely come up, unless they are actually Borel, while analytic equivalence relations abound and hence are the more important object of study. Deducing from Silver's dichotomy, J. P. Burgess showed that the above example is as bad as it can get.

Theorem 5 (J. P. Burgess (1978))**.** *Let* E *be an analytic equivalence relation on a standard Borel space* $\mathcal{X}$*. Then exactly one of the following holds:*

- *there are at most* $\aleph_1$ E*-classes,*
- *there is an uncountable Borel set* $A \subseteq \mathcal{X}$ *of* E*-inequivalent points.*

Silver's proof, which used substantial parts of modern set theory, was soon simplified by L. A. Harrington, who found a proof based on effective considerations, i.e., ultimately computability theory on the integers. Subsequently a number of dichotomies for various structures were proved using variations of this technique, but the theory would soon advance with deepening connections to other branches of mathematics.

Theorem 6 (L. A. Harrington, A. S. Kechris, A. Louveau (1990))**.** *Let* E *be a Borel equivalence relation on a standard Borel space* $\mathcal{X}$*. Then exactly one of the following holds:*

- E *is smooth, i.e.,* $\mathsf{E} \leqslant_B =_{\mathbb{R}}$,
- $\mathsf{E}_0 \leqslant_B \mathsf{E}$.

This dichotomy extends previous work by J. G. Glimm (1961) and E. G. Effros (1965) on the representation theory of C^*-algebras. In his work on a conjecture of G. W. Mackey, Glimm originally proved the above dichotomy for orbit equivalence relations induced by the action of a second countable, locally compact group, and Effros subsequently generalized this to F_σ equivalence relations induced by actions of Polish topological groups.

The dichotomies of Silver and Harrington, Kechris and Louveau explain the linearity of the lower part of Figure 2. Indeed, for *Borel* equivalence relations E, either E is Borel reducible to $=_{\mathbb{N}}$, E is Borel bireducible with $=_{\mathbb{R}}$, or E_0 Borel reduces to E. For general *analytic* equivalence relations, this picture breaks down.

A number of other dichotomies for Borel equivalence relations are now known, most notably a dichotomy stating that E_1 is minimal (but not minimum) above E_0 among Borel equivalence relations (Kechris and Louveau (1997)). Even though the statements of these results involve nothing more than Borel sets and Borel measurable functions on standard Borel spaces, until recently the only known proofs of these results involved computability theory on the integers. However, recent proofs due to B. D. Miller (2009) have completely removed the need for such effective considerations.

Banach Spaces and Gowers's Classification Program

The two main classification problems for Banach spaces are arguably those of isomorphism and isometry. While the problem of isometrically distinguishing two Banach spaces is somewhat tangible, mainly due to a number of quantitative invariants at hand, e.g., moduli of smoothness and convexity, there are fewer general techniques for distinguishing the isomorphic structure of Banach spaces. One successful subcase is the isomorphic classification by A. A. Miljutin (1966) and C. Bessega and A. Pełczyński (1960) of the $C(K)$ spaces, for K compact metrizable, in terms of the cardinality and Cantor–Bendixon rank of K. However, with respect to isomorphic classification by complete invariants, there has otherwise only been limited progress. Though it has long been suspected that a complete classification of separable Banach spaces up to isomorphism would be too complicated to be of any practical value, the evidence was mostly circumstantial. However, based on work by S. A. Argyros and P. Dodos (2006) on amalgamations of analytic sets of Banach spaces, we now know the exact complexity of isomorphism.

Theorem 7 (V. Ferenczi, A. Louveau, C. Rosendal (2009))**.** *The relation of isomorphism between separable Banach spaces is maximum in the Borel reducibility ordering, i.e., is Borel bireducible with* $\mathsf{E}_{\max}$.

By contrast, isometry is of substantially lower complexity, namely, J. Melleray (2007) showed that it is bireducible with the equivalence relation $\mathsf{E}_{\mathrm{grp}}$ from Figure 2.

Partially motivated by the inherent complexity of the isomorphic structure of Banach spaces, W. T. Gowers (2002) proposed an alternative classification program by instead determining the irreducible components from which other spaces are built up. Now, as the finite-dimensional Banach spaces are fully determined up to isomorphism by their dimension, we should concentrate on the infinite-dimensional building blocks. So to avoid endless repetition, henceforth all Banach spaces will be assumed to be separable and infinite-dimensional. Concretely, Gowers proposed that one should determine a list $(C_i)_{i\in I}$ of isomorphism invariant classes of Banach spaces such that

(1) if a space X belongs to a class C_i, then so do all of its subspaces,
(2) the classes are disjoint for obvious reasons,
(3) any space X has a subspace $Y \subseteq X$ belonging to some class C_i,

(4) knowing that a space X belongs to some class C_i should yield substantial knowledge about its structure and about the operators that may be defined on it.

Gowers's classification program can be thought of as a prototypical example of the third type of classification, but it has one distinguishing feature, namely, it makes no promise that we should be able to reconstruct a space X from those of its subspaces belonging to the various classes C_i. This is in contrast with, for example, the theory of unitary representations, where one seeks to write representations as direct integrals of their irreducible subrepresentations.

Another feature that sets it apart from the first two types of classification is that *embedding*, i.e., isomorphism with a subspace, rather than isomorphism, is the focal point (we shall write $X \sqsubseteq Y$ to denote that X embeds into Y). Thus, for example, properties (1) and (2) together imply that spaces X and Y belonging to different classes are totally incomparable. Here X and Y are *incomparable* if neither $X \sqsubseteq Y$ nor $Y \sqsubseteq X$ and are *totally incomparable* if there is no other space Z such that both $Z \sqsubseteq X$ and $Z \sqsubseteq Y$.

For many years (long before Gowers formulated his program), it was conjectured that any Banach space contains a copy of some ℓ_p or c_0, whereby if C_p and C_0 denote the set of subspaces of ℓ_p, respectively of c_0, then $(C_p)_{p\in\{0\}\cup[1,\infty[}$ would provide a list, as above. This was refuted by B. S. Tsirelson (1974), who constructed a space, now known as the *Tsirelson space*, not containing any ℓ_p or c_0. Nevertheless, Tsirelson's space in many ways resembled the classical sequence spaces c_0 and ℓ_p, and it was not clear just how far the structure of a space could diverge from these.

Schauder Bases

In order to manipulate the subspaces of a Banach space, we need a way of representing these efficiently, namely via bases. A sequence $(e_n)_{n=1}^{\infty}$ of vectors in a Banach space X is said to be a *Schauder basis* for X if any vector $x \in X$ can be uniquely represented as a norm convergent series

$$x = \sum_{n=1}^{\infty} a_n e_n,$$

where the a_n are scalars. So, for example, if e_n denotes the sequence

$$(0, 0, \dots, 0, 1, 0, 0 \dots),$$

where the 1 occurs in the nth place, then $(e_n)_{n=1}^{\infty}$ is a Schauder basis for the space

$$\ell_1 = \{(a_1, a_2, a_3, \dots) \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty} |a_i| < \infty\}$$

of absolutely summable sequences (and similarly for c_0 and ℓ_p, $1 \leqslant p < \infty$). For the combinatorial analysis of bases, however, another equivalent formulation is more useful: a sequence $(e_n)_{n=1}^{\infty}$ of nonzero vectors is a Schauder basis for X if its linear span is dense in X and there is a constant K such that

$$(*) \qquad \Big\| \sum_{n=1}^{k} a_n e_n \Big\| \leqslant K \Big\| \sum_{n=1}^{\infty} a_n e_n \Big\|,$$

whenever a_n are scalars and $k \in \mathbb{N}$.

Obviously, as in linear algebra, it can be extremely useful to represent a Banach space via a basis, as, for example, operators then can be written as matrices with respect to this basis. However, P. Enflo (1973) showed that not every Banach space has a Schauder basis. Actually, Enflo did more by constructing a space X without Grothendieck's approximation property, i.e., such that the identity operator on X cannot be approximated uniformly on compact sets by finite rank operators. Nevertheless, a classical result due to S. Banach states that any space has a subspace with a Schauder basis.

A distinguishing feature of the bases of c_0 and ℓ_p is that not only do we have uniformly bounded projections onto initial segments of the basis as in $(*)$, but we also have uniformly bounded projections onto any subset. That is, for some constant K (actually $K = 1$ works for c_0 and ℓ_p) and all sets $A \subseteq \mathbb{N}$, we have

$$\Big\| \sum_{n\in A} a_n e_n \Big\| \leqslant K \Big\| \sum_{n\in\mathbb{N}} a_n e_n \Big\|,$$

for all scalars a_n. Bases satisfying this stronger property are said to be *unconditional*. Now if X is a Banach space with an unconditional basis (e_n), then X has a multitude of complemented subspaces. For if $A \subseteq \mathbb{N}$, then the closed linear span $[e_n]_{n\in A}$ of the subsequence $(e_n)_{n\in A}$ will be the image of the bounded projection P_A on X defined by

$$P_A\Big(\sum_{n\in\mathbb{N}} a_n e_n\Big) = \sum_{n\in A} a_n e_n,$$

and hence is complemented by the subspace $[e_n]_{n\notin A}$.

The existence of a Schauder basis allows us to largely replace the analytical theory of Banach spaces with combinatorics in vector spaces. To set this up, suppose X is a Banach space with a Schauder basis (e_n) and let E denote the set of finitely supported vectors

$$E = \{\sum_{n=1}^{k} a_n e_n \mid a_n \in \mathbb{R} \,\&\, k \in \mathbb{N}\}.$$

Then E is a countable-dimensional normed vector space with basis (e_n). A *block sequence* of (e_i) is an infinite sequence (y_i) of nonzero vectors in E such that

$$\max \operatorname{support}(y_n) < \min \operatorname{support}(y_{n+1})$$

for all $n \in \mathbb{N}$ (see Figure 3).

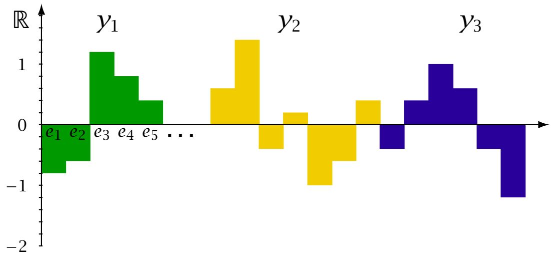

Figure 3. A block basis (y_n) of (e_n).

Any such block sequence will be a Schauder basis for its closed linear span $[y_i]$, which is an infinite-dimensional subspace of X called a *block subspace*. Also, a classical result implies that, modulo a small isomorphic perturbation, any subspace of X contains a block sequence and hence, up to embeddability, the block sequences of (e_n) parametrize the closed subspaces of X.

These considerations allow us to replace the notions of Banach space theory with combinatorial properties of block sequences in the normed vector space E. The goal of Gowers's classification program is then to isolate mutually exclusive and hereditary classes $(C_i)_{i\in I}$ of block subspaces, such that any Schauder basis (e_n) has a block subspace $Y = [y_n]$ belonging to some C_i. Of course, the import of such a classification largely depends on how informative membership in the classes C_i is and on the "logical gap" or conceptual distance between them. For example, no information is gained by splitting into the classes

$$C_1 = \{\text{reflexive spaces}\}$$

and

$$C_2 = \{\text{spaces without reflexive subspaces}\}.$$

Ramsey Theory for Block Sequences

If one attempts to apply Ramsey theoretical methods to Banach spaces, it is natural to first review the classical Ramsey theory for integers to look for principles that might transfer to the new context. So for any $n = 1, 2, \ldots, \infty$ and infinite subset $A \subseteq \mathbb{N}$, we let $[A]^n$ denote the set of all strictly increasing n-tuples of numbers in A. Then whenever

$$c\colon [\mathbb{N}]^n \to \{\text{green}, \text{blue}\}$$

is a coloring with two colors, we can ask if there is an infinite subset $A \subseteq \mathbb{N}$ such that $[A]^n$ is monochromatic. In other words, is there an infinite set $A \subseteq \mathbb{N}$ such that all increasing n-tuples from A get the same color?

In the case $n = 1$, c is really just a coloring of the natural numbers themselves, so of course there is such a monochromatic subset A (this is just the Dirichlet or pigeonhole principle). For $1 < n < \infty$, though any such coloring c still admits a monochromatic set $[A]^n$, this is certainly less trivial and is just the statement of the infinite version of Ramsey's theorem (from which the usual finite version follows by a simple compactness argument).

On the other hand, when coloring infinite increasing sequences of natural numbers new phenomena occur, and one has to restrict the allowable colorings. However, a result due to F. Galvin and K. Prikry (1973) states that if the coloring c is a Borel function, where $[\mathbb{N}]^\infty$ has the topology induced from $\mathbb{N}^\mathbb{N}$, then one can still find a monochromatic set $[A]^\infty$.

Now, when transferring these concepts to Banach spaces, our base set $\mathbb{N}$ should be replaced with the unit sphere of a Banach space X (recall that all spaces are assumed infinite-dimensional), the infinite subsets of $\mathbb{N}$ with unit spheres of subspaces $Y \subseteq X$ and the natural numbers with vectors of norm 1 in X. So if

$$c\colon S_X \to [0,1]$$

is a Lipschitz function defined on the unit sphere of a Banach space X and $\epsilon > 0$, does there exist a subspace $Y \subseteq X$ such that

$$\operatorname{diam}(c[S_Y]) < \epsilon\ ?$$

Well, at least for one space we do have such a result.

Theorem 8 (W. T. Gowers (1992))**.** *Suppose*

$$c\colon S_{c_0} \to [0,1]$$

is a Lipschitz function and $\epsilon > 0$. Then there is a subspace $Y \subseteq c_0$ such that

$$\operatorname{diam}(c[S_Y]) < \epsilon.$$

Unfortunately, for the classification of Banach spaces, c_0 is essentially the only space for which this holds.

Theorem 9 (E. Odell, Th. Schlumprecht (1994))**.** *Suppose X is a Banach space not containing an isomorphic copy of c_0 as a subspace. Then there is a subspace $Y \subseteq X$ and a Lipschitz function*

$$c\colon S_Y \to [0,1]$$

such that for any subspace $Z \subseteq Y$, we have

$$c[S_Z] = [0,1].$$

So even the simplest of the Ramsey principles for $\mathbb{N}$, i.e., the Dirichlet principle, has no direct analogue for Banach spaces. However, Gowers overcame this predicament by taking a more dynamical perspective that led to a general Ramsey principle for Banach spaces. We now introduce such a principle in the context of pure vector spaces.

Suppose E is a countable-dimensional $\mathbb{Q}$-vector space with basis $(e_n)_{n=1}^\infty$. We let $\mathcal{B}$ be the set of all block sequences[3] of the basis (e_n) and equip $\mathcal{B}$ with a separable, complete metric d given by

$$d((x_n),(y_n)) = 2^{-\min(m \mid x_m \neq y_m)}.$$

So with the d-topology $\mathcal{B}$ is a Polish space.

We consider a pair of infinite games between two players I and II that will produce block sequences of (e_n). The *Gowers game* G_X below a block subspace $X \subseteq E$ is played by letting I and II alternate in choosing, respectively, block subspaces $Y_i \subseteq X$ and nonzero vectors $x_i \in Y_i$ subject to the constraint that

$$\max \operatorname{support}(x_n) < \min \operatorname{support}(x_{n+1}).$$

Diagrammatically:

$$\begin{array}{lccccccccc} \textbf{I} & Y_0 & & Y_1 & & Y_2 & & Y_3 & & \dots \\ \textbf{II} & & x_0 & & x_1 & & x_2 & & x_3 & \dots \end{array}$$

The *infinite asymptotic game* F_X below the block subspace $X \subseteq E$ is defined as the Gowers game, except that we now additionally demand that the block subspaces $Y_i \subseteq X$ played by I should have *finite codimension* in X. In both games, the *outcome* of an infinite run of the game is defined to be the infinite block sequence $(x_i)_{i=0}^{\infty}$ produced by player II.

In both G_X and F_X, II is thus the only player that is directly contributing to the outcome of the game, while player I only indirectly influences the play of II by determining in which block subspaces II has to choose her vectors. Clearly, I has greater powers in Gowers's game G_X than in the infinite asymptotic game F_X, since in G_X he is not restricted to choosing subspaces of finite codimension in X.

Definition 10. A subset $A \subseteq \mathcal{B}$ is said to be *strategically Ramsey*[4] if there is a block subspace $X \subseteq E$ such that one of the following two properties holds

- II has a strategy in G_X to play in A,
- I has a strategy in F_X to play in $\sim A$.

We note that this is stronger than requiring that the games G_X and F_X to play in A are determined. For, in general, it is much stronger for I to have a strategy in F_X to play in $\sim A$ than to have a strategy in G_X to play in $\sim A$. On the other hand, the notion of strategically Ramsey does involve passing to a block subspace $X \subseteq E$ and so is really both a Ramsey theoretical and a game theoretical concept.

The basic fact about strategically Ramsey sets is then

Theorem 11. *Analytic sets are strategically Ramsey.*

This principle can be further strengthened in the context of Banach spaces. For suppose X is a Banach space with a Schauder basis (e_n), $B \subseteq (S_X)^{\mathbb{N}}$, and let E be the $\mathbb{Q}$-vector subspace of X with basis (e_n). Assume that for some block subspace $Y \subseteq E$, player I has a strategy in the infinite asymptotic game F_Y to play in B. Then for any sequence $\Delta = (\delta_n)$ of positive real numbers there is a further block subspace $Z \subseteq Y$ such that any block sequence in S_Z belongs to

$$B_\Delta = \{(z_n) \mid \exists (v_n) \in B \;\; \forall n \;\; \|v_n - z_n\| < \delta_n\}.$$

From this follows

Theorem 12 (W. T. Gowers (2002))**.** *Suppose X is a Banach space with a basis (e_n) and A is an analytic set of block sequences. Assume moreover that for any Δ and any block subspace $Y \subseteq X$ there is a block sequence in S_Y not belonging to $(\sim A)_\Delta$. Then there is a block subspace $Y \subseteq X$ such that II has a strategy in G_Y to play in A.*

So, up to small perturbations, if an analytic set of normalized block sequences cannot be avoided by passing to a block subspace, then, for some block subspace Y, II has a strategy in G_Y to play into this set.

Dichotomies for Banach Spaces

Up till the beginning of the 1990s, perhaps the major unsolved problem of Banach space theory was the *unconditional basic sequence problem*, i.e., the question of whether any Banach space contains a subspace with an unconditional basis and thus to some extent resembling the classical sequence spaces ℓ_p and c_0. This was solved negatively by Gowers and B. Maurey (1993) by the construction of a so-called *hereditarily indecomposable* space, i.e., a space in which no two infinite-dimensional subspaces form a direct sum and so admitting no nontrivial projections.

The construction of the Gowers-Maurey space spurred significant activity, most notably the recent construction by Argyros and R. G. Haydon (2009) of a Banach space on which every operator is of the form $\lambda I + K$, where K is a compact operator. But, surprisingly, despite the extreme intricacy of its construction, any counterexample to the unconditional basic sequence problem must resemble the Gowers-Maurey space. This was proved by Gowers as the first application of his Ramsey theorem of the preceding section.

Theorem 13 (Gowers's first dichotomy (1996))**.** *Any Banach space contains either a hereditarily indecomposable subspace or a subspace having an unconditional Schauder basis.*

Since being hereditarily indecomposable or having an unconditional basis are hereditary properties, i.e., are inherited by block subspaces, and are contradictory, this provides an initial step in

[3] *We import the relevant concepts from Schauder bases wholesale.*

[4] *The actual definition of strategically Ramsey that one needs is slightly more complicated and involves quantification over block spaces and finite block sequences.*

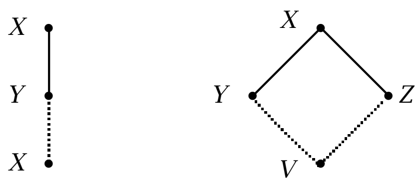

Figure 4. Minimality and quasi-minimality.

Gowers's classification program by splitting into two classes of spaces, namely the hereditarily indecomposable and those with unconditional basis. However, this division can be further refined by another application of the Ramsey theorem. First, a space X is said to be *minimal* in case it embeds into all of its subspaces, and X is *quasi-minimal* if whenever $Y, Z \sqsubseteq X$ there is some V such that $V \sqsubseteq Y, Z$, i.e., X has no pair of totally incomparable subspaces (see Figure 4).

Theorem 14 (Gowers's second dichotomy (1996))**.** *Any Banach space contains either a quasi-minimal subspace or a subspace with an unconditional basis such that disjointly supported block subspaces are totally incomparable.*

In other words, the hereditary failure to quasi-minimality can be explicitly witnessed on a subspace with an unconditional basis. Moreover, this latter property can be conveniently reformulated as follows. Suppose X is a space with a Schauder basis (e_n). Saying that disjointly supported block subspaces are totally incomparable is just requiring that whenever $Y = [y_n]$ is a block subspace, $I_n = \text{support}(y_n)$ and $T\colon Y \to X$ is an isomorphic embedding, then

$$\liminf_{n\to\infty} \|P_{I_n} T\| > 0$$

(recall that P_{I_n} denotes the projection of X onto the subspace $[e_i]_{i\in I_n}$). So any isomorphic copy of Y inside of X has essential support on the sequence of finite-dimensional subspaces $[e_i]_{i\in I_n}$.

Since Gowers's two dichotomies are logically dependent, putting them together only gives us a decomposition into three classes: hereditarily indecomposable, quasi-minimal with unconditional basis, and spaces with an unconditional basis such that disjointly supported subspaces are totally incomparable. However, it is natural to look for a dichotomy distinguishing the minimal spaces (a property characteristic of the classical sequence spaces ℓ_p and c_0) among the more general quasi-minimal spaces. For this, a Banach space X with a basis (e_n) is said to be *tight* if whenever $Y = [y_n]$ is a block subspace, there are finite sets

$$I_0 < I_1 < I_2 < \cdots \subseteq \mathbb{N}$$

such that any isomorphic embedding $T\colon Y \to X$ satisfies

$$\liminf_{n\to\infty} \|P_{I_n} T\| > 0.$$

Thus, for example, the spaces occurring above are seen to be tight by using the support of the vectors y_n. Tight spaces are easily seen to have no minimal subspaces.

Theorem 15 (V. Ferenczi, C. Rosendal (2009))**.** *Any Banach space contains either a tight or a minimal subspace.*

Of course, the bare existence of the finite sets I_n can be somewhat unsatisfactory, and in concrete cases one would like to know how these are computed from the block subspace $Y = [y_n]$. And, indeed, there are other dichotomies for weakened types of minimality in terms of the choice of sets I_n (involving constants of embeddability by the range of vectors, or, as in Gowers's second dichotomy, by their support).

The dichotomies obtained hitherto give a pretty detailed picture of the nonclassical Banach spaces, i.e., those not containing minimal subspaces, but our knowledge of the minimal spaces is still very far from being complete. In particular, one would like to have a truly informative dichotomy detecting the presence of c_0 or ℓ_p inside a Banach space. There are such dichotomies for ℓ_1 or c_0 due to H. P. Rosenthal (1974) and (1994), but none as informative exist for ℓ_p, $p \neq 1$.

On a somewhat different note, the following question still remains open.

Question 16. Let X be a Banach space. What is the number of nonisomorphic subspaces of X?

Since Hilbert space ℓ_2 is outright isometric to its subspaces, any space isomorphic to ℓ_2 is *homogeneous*, i.e., isomorphic to all of its subspaces. That the converse holds, i.e., that any homogeneous space is isomorphic to Hilbert space, is a consequence of Gowers's first dichotomy together with a result of R. Komorowski and N. Tomczak-Jaegermann (1995). But this leaves a huge gap of possible cardinalities for the number of nonisomorphic subspaces of a Banach space. Since it can be shown with the aid of tightness that any space without minimal subspaces contains a continuum of nonisomorphic subspaces, the main problem lies in understanding the minimal spaces. A perhaps optimistic guess is that any Banach space into which ℓ_2 does not embed contains a continuum of nonisomorphic subspaces.

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About the Cover

Old North Church, Boston

The cover shows a watercolor sketch of Old North Church in Boston, site of the annual Joint Mathematics Meetings in January 2012. It was done by Karl Hofmann. We don't mean to deceive you about Boston weather in January, so we show below a more typical winter scene, the Great Spirit statue next to the Boston Museum of Fine Arts, also by Karl.

This is the third cover we have had from Karl Hofmann. He is a member of the mathematics faculties of both Tulane University and Universität Darmstadt. He still works at mathematics, but his most unusual accomplishment is an extraordinarily long series of poster designs in tempera colors and felt pen calligraphy accompanying colloquium talks in Darmstadt, which you can see at

`http://www3.mathematik.tu-darmstadt.de/galerie-der-kolloquiumsposter.html`

He has also provided illustrations for mathematics books, most notably the very popular *Proofs from the Book* by Martin Aigner and Günter Ziegler.

The photograph below shows him at work in his "studio" at the university in Darmstadt.

—Bill Casselman
Graphics editor
(`notices-covers@ams.org`)

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