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TURBULENCE, AMALGAMATION, AND GENERIC AUTOMORPHISMS OF HOMOGENEOUS STRUCTURES

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Abstract

We study topological properties of conjugacy classes in Polish groups, with emphasis on automorphism groups of homogeneous countable structures. We first consider the existence of dense conjugacy classes (the topological Rokhlin property). We then characterize when an automorphism group admits a comeager conjugacy class (answering a question of Truss) and apply this to show that the homeomorphism group of the Cantor space has a comeager conjugacy class (answering a question of Akin, Hurley and Kennedy). Finally, we study Polish groups that admit comeager conjugacy classes in any dimension (in which case the groups are said to admit ample generics). We show that Polish groups with ample generics have the small index property (generalizing results of Hodges, Hodkinson, Lascar and Shelah) and arbitrary homomorphisms from such groups into separable groups are automatically continuous. Moreover, in the case of oligomorphic permutation groups, they have uncountable cofinality and the Bergman property. These results in particular apply to automorphism groups of many ω-stable, $\aleph_0$-categorical structures and of the random graph. In this connection, we also show that the infinite symmetric group S_∞ has a unique non-trivial separable group topology. For several interesting groups we also establish Serre's properties (FH) and (FA).

1. *Introduction*

1.1. *Polish groups*

In this paper we study topological properties of conjugacy classes in Polish groups. There are two questions in which we are particularly interested. First, does a Polish group G have a dense conjugacy class? This is equivalent (see, for example, Kechris [**31**, (8.47)]) to the following generic ergodicity property of G: every conjugacy invariant subset $A \subseteq G$ with the Baire property (for example, a Borel set) is either meager or comeager. There is an extensive list of Polish groups that have dense conjugacy classes, like, for example, the automorphism group $\mathrm{Aut}(X,\mu)$ of a standard measure space (X,μ), that is, a standard Borel space X with a non-atomic Borel probability measure μ (see, for example, Halmos [**22**]), the unitary group $U(H)$ of separable infinite-dimensional Hilbert space H (see, for example, Choksi and Nadkarni [**10**]) and the homeomorphism groups $H(X)$ of various compact metric spaces X with the uniform convergence topology, including $X = [0,1]^{\mathbb{N}}$ (the Hilbert cube), $X = 2^{\mathbb{N}}$ (the Cantor space), $X = S^{2d}$ (even-dimensional spheres), etc. (see Glasner and Weiss [**19**] and Akin, Hurley and Kennedy [**3**]). In [**19**] groups that have dense conjugacy classes are said to have the *topological Rokhlin property*, motivated by the existence of dense conjugacy classes in $\mathrm{Aut}(X,\mu)$, which is usually seen as a consequence of the well-known Rokhlin Lemma in ergodic theory. Recently, Akin, Glasner and Weiss [**2**] have also found an example of a locally compact Polish group with dense conjugacy class. Any such group must be non-compact and it appears likely that it must also be totally disconnected. (Karl Hofmann has shown that if a locally compact group not equal to $\{1\}$ has a dense conjugacy class, then it is not pro-Lie and in particular, its quotient by the connected component of the identity cannot be compact.)

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The second question we consider is whether G has a (necessarily unique) dense G_δ conjugacy class (which is well known to be equivalent to whether it has a dense non-meager class, see, for example, Becker and Kechris [**5**]). Following Truss [**47**], we call any element of G whose conjugacy class is dense G_δ a *generic element* of G. This is a much stronger property which fails in very 'big' groups such as $\mathrm{Aut}(X,\mu)$ or $U(H)$ but can often occur in automorphism groups $\mathrm{Aut}(\mathbf{K})$ of countable structures $\mathbf{K}$. It was first studied in this context by Lascar [**36**] and Truss [**47**]. For example, $\mathrm{Aut}(\mathbb{N},=)$ (that is, *the infinite symmetric group* S_∞), $\mathrm{Aut}(\mathbb{Q},<)$, $\mathrm{Aut}(\mathbf{R})$, where $\mathbf{R}$ is the random graph, and $\mathrm{Aut}(\mathbf{P})$, where $\mathbf{P}$ is the random poset, all have a dense G_δ conjugacy class (see Truss [**47**], and Kuske and Truss [**35**]). For more on automorphism groups having dense G_δ classes, see also the recent paper by Macpherson and Thomas [**38**]. However, the question of whether certain groups have a dense G_δ conjugacy class arose as well in topological dynamics, where Akin, Hurley and Kennedy [**3**, p. 104] posed the problem of the existence of a dense G_δ conjugacy class in $H(2^{\mathbb{N}})$, that is, the existence of a generic homeomorphism of the Cantor space.

1.2. *Model theory*

Our goal here is to study these questions in the context of automorphism groups, $\mathrm{Aut}(\mathbf{K})$, of countable structures $\mathbf{K}$. It is well known that these groups are (up to topological group isomorphism) exactly the closed subgroups of S_∞. However, quite often such groups can be densely embedded in other Polish groups G (that is, there is an injective continuous homomorphism from $\mathrm{Aut}(\mathbf{K})$ into G with dense image) and therefore establishing the existence of a dense conjugacy class of $\mathrm{Aut}(\mathbf{K})$ implies the same for G. So we can use this method to give simple proofs that such conjugacy classes exist in several interesting Polish groups. This can be viewed as another instance of the idea of reducing questions about the structure of certain Polish groups to those of automorphism groups $\mathrm{Aut}(\mathbf{K})$ of countable structures $\mathbf{K}$, where one can employ methods of model theory and combinatorics. An earlier use of this methodology is found in [**33**] by Kechris, Pestov and Todorcevic in connection with the study of extreme amenability and its relation with Ramsey theory.

It is also well known that to every closed subgroup $G \leqslant S_\infty$ one can associate the structure $\mathbf{K}_G = (\mathbb{N}, \{R_{i,n}\})$, where $R_{i,n} \subseteq \mathbb{N}^n$ is the ith orbit in some fixed enumeration of the orbits of G on $\mathbb{N}^n$, which is such that $G = \mathrm{Aut}(\mathbf{K}_G)$ and moreover $\mathbf{K}_G$ is *ultrahomogeneous*, that is, every isomorphism between finite substructures of $\mathbf{K}_G$ extends to an automorphism of $\mathbf{K}_G$. Fraïssé has analyzed such structures in terms of their finite 'approximations'.

To be more precise, let $\mathcal{K}$ be a class of finite structures in a fixed (countable) signature L which has the following properties:

(i) (HP) $\mathcal{K}$ is *hereditary* (that is, $\mathbf{A} \leqslant \mathbf{B} \in \mathcal{K}$ implies $\mathbf{A} \in \mathcal{K}$, where $\mathbf{A} \leqslant \mathbf{B}$ means that $\mathbf{A}$ can be embedded into $\mathbf{B}$);

(ii) (JEP) $\mathcal{K}$ satisfies the *joint embedding property* (that is, if $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is $\mathbf{C} \in \mathcal{K}$ with $\mathbf{A}, \mathbf{B} \leqslant \mathbf{C}$);

(iii) (AP) $\mathcal{K}$ satisfies the *amalgamation property* (that is, if $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, are embeddings, there exist $\mathbf{D} \in \mathcal{K}$ and embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ with $r \circ f = s \circ g$);

(iv) $\mathcal{K}$ contains only countably many structures, up to isomorphism, and contains structures of arbitrarily large (finite) cardinality.

We call any $\mathcal{K}$ that satisfies (i)–(iv) a *Fraïssé class*. Examples include the class of trivial structures ($L = \emptyset$), graphs, linear orderings, Boolean algebras, metric spaces with rational distances, etc. For any Fraïssé class $\mathcal{K}$ one can define, following Fraïssé [**16**] (see also Hodges [**25**]), its so-called *Fraïssé limit* $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, which is the unique countably infinite structure such that:

(a) $\mathbf{K}$ is locally finite (that is, finitely generated substructures of $\mathbf{K}$ are finite);

(b) $\mathbf{K}$ is *ultrahomogeneous* (that is, any isomorphism between finite substructures of $\mathbf{K}$ extends to an automorphism of $\mathbf{K}$);
(c) $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$, where $\mathrm{Age}(\mathbf{K})$ is the class of all finite structures that can be embedded in $\mathbf{K}$.

A countably infinite structure $\mathbf{K}$ satisfying (a) and (b) is called a *Fraïssé structure*. If $\mathbf{K}$ is a Fraïssé structure, then $\mathrm{Age}(\mathbf{K})$ is a Fraïssé class; therefore the maps $\mathcal{K} \mapsto \mathrm{Flim}(\mathcal{K})$ and $\mathbf{K} \mapsto \mathrm{Age}(\mathbf{K})$ provide a canonical bijection between Fraïssé classes and structures. Examples of Fraïssé structures include the trivial structure $(\mathbb{N}, =)$, the random graph $\mathbf{R}$, $(\mathbb{Q}, <)$, the countable atomless Boolean algebra $\mathbf{B}_\infty$, the rational Urysohn space $\mathbf{U}_0$ (which is the Fraïssé limit of the class of finite metric spaces with rational distances), etc.

For further reference, we note that condition (b) in the definition of a Fraïssé structure can be replaced by the following equivalent condition, called the *extension property*: if $\mathbf{A}$ and $\mathbf{B}$ are finite, $\mathbf{A}, \mathbf{B} \leqslant \mathbf{K}$, and $f : \mathbf{A} \to \mathbf{K}$ and $g : \mathbf{A} \to \mathbf{B}$ are embeddings, then there is an embedding $h : \mathbf{B} \to \mathbf{K}$ with $h \circ g = f$.

Thus every closed subgroup $G \leqslant S_\infty$ is of the form $G = \mathrm{Aut}(\mathbf{K})$ for a Fraïssé structure $\mathbf{K}$. We study in this paper the question of the existence of dense or comeager conjugacy classes in the Polish groups $\mathrm{Aut}(\mathbf{K})$, equipped with the pointwise convergence topology, for Fraïssé structures $\mathbf{K}$, in terms of properties of $\mathbf{K}$, and also we derive consequences for other groups.

1.3. *Dense conjugacy classes*

Following Truss [**47**], we associate to each Fraïssé class $\mathcal{K}$ the class $\mathcal{K}_p$ of all systems, $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\mathbf{B}, \mathbf{C} \subseteq \mathbf{A}$ (that is, $\mathbf{B}$ and $\mathbf{C}$ are *substructures* of $\mathbf{A}$), and ψ is an isomorphism of $\mathbf{B}$ and $\mathbf{C}$. For such systems, we can define the notion of embedding as follows: an *embedding* of $\mathcal{S}$ into $\mathcal{T} = \langle \mathbf{D}, \varphi : \mathbf{E} \to \mathbf{F} \rangle$ is an embedding $f : \mathbf{A} \to \mathbf{D}$ such that f embeds $\mathbf{B}$ into $\mathbf{E}$ and $\mathbf{C}$ into $\mathbf{F}$ and $f \circ \psi \subseteq \phi \circ f$. Using this concept of embedding we then see clearly what it means to say that $\mathcal{K}_p$ satisfies JEP or AP.

We now have the following result.

THEOREM 1.1. *Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$. Then the following are equivalent:*
(i) *there is a dense conjugacy class in* $\mathrm{Aut}(\mathbf{K})$*;*
(ii) $\mathcal{K}_p$ *satisfies the* JEP.

For example, it is easy to verify the JEP of $\mathcal{K}_p$ for the following classes $\mathcal{K}$:
(i) finite metric spaces with rational distances,
(ii) finite Boolean algebras,
(iii) finite measure Boolean algebras with rational measure,

and this immediately implies that $\mathrm{Aut}(\mathbf{U}_0) = \mathrm{Iso}(\mathbf{U}_0)$ (the isometry group of the rational Urysohn space $\mathbf{U}_0$), $\mathrm{Aut}(\mathbf{B}_\infty)$ and $\mathrm{Aut}(\mathbf{F}, \lambda)$ (the automorphism group of the Boolean algebra generated by the rational intervals of $[0, 1]$ with Lebesgue measure) all have dense conjugacy classes. However, $\mathrm{Aut}(\mathbf{U}_0)$ can be densely embedded into $\mathrm{Iso}(\mathbf{U})$, the isometry group of the Urysohn space (see [**33**] by Kechris, Pestov and Todorcevic), $\mathrm{Aut}(\mathbf{B}_\infty)$ is isomorphic to $H(2^{\mathbb{N}})$ by Stone duality, and $\mathrm{Aut}(\mathbf{F}, \lambda)$ can be densely embedded in $\mathrm{Aut}(X, \mu)$, so we have simple proofs of the following result.

COROLLARY 1.2. *The following Polish groups have dense conjugacy classes:*
(i) $\mathrm{Iso}(\mathbf{U})$,
(ii) $H(2^{\mathbb{N}})$ (*Glasner and Weiss* [**19**], *Akin, Hurley and Kennedy* [**3**]),
(iii) $\mathrm{Aut}(X, \mu)$ (*Rokhlin*).

Glasner and Pestov have also proved part (i). We also obtain a similar result for the (diagonal) conjugacy action of G on $G^{\mathbb{N}}$ for all the above groups G.

We say that a Polish group G has a *cyclically dense conjugacy class* if there are $g, h \in G$ such that $\{g^n h g^{-n}\}_{n\in\mathbb{Z}}$ is dense in G. In this case G is topologically 2-generated, that is, has a dense 2-generated subgroup. For example, S_∞ has this property and so does the automorphism group of the random graph (see Macpherson [**37**]). Using a version of Theorem 1, we can give simple proofs that the following groups admit cyclically dense conjugacy classes and therefore are topologically 2-generated: $H(2^{\mathbb{N}})$, $H(2^{\mathbb{N}}, \sigma)$ (the group of the measure-preserving homeomorphisms of $2^{\mathbb{N}}$ with the usual product measure), $\mathrm{Aut}(X, \mu)$, and $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ (the automorphism group of the infinitely splitting rooted tree).

1.4. *Comeager conjugacy classes*

We now turn to the existence of a dense G_δ conjugacy class in $\mathrm{Aut}(\mathbf{K})$, with $\mathbf{K}$ a Fraïssé structure, that is, the existence of a *generic automorphism* of $\mathbf{K}$. Truss [**47**] showed that the existence of a subclass $\mathcal{L} \subseteq \mathcal{K}_p$ such that $\mathcal{L}$ is cofinal under embeddability and satisfies the AP, a property which we will refer to as the *cofinal amalgamation property* (CAP), together with the JEP for $\mathcal{K}_p$, is sufficient for the existence of a generic automorphism. He also raised the question of whether the existence of a generic automorphism is equivalent to some combination of amalgamation and joint embedding properties for $\mathcal{K}_p$.

Motivated by this problem, we have realized that the question of generic automorphisms is closely related to Hjorth's concept of turbulence (see Hjorth [**24**] or Kechris [**32**]) for the conjugacy action of the automorphism group $\mathrm{Aut}(\mathbf{K})$, a connection that is surprising at first sight since turbulence is a phenomenon usually thought of as incompatible with actions of closed subgroups of the infinite symmetric group, like $\mathrm{Aut}(\mathbf{K})$. Once however this connection is realized, it leads naturally to the formulation of an appropriate amalgamation property for $\mathcal{K}_p$ which in combination with JEP is equivalent to the existence of a generic automorphism.

We say that $\mathcal{K}_p$ satisfies the *weak amalgamation property* (WAP) if for any

$$u\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \quad \text{in } \mathcal{K}_p,$$

there exist

$$\mathcal{T} = \langle \mathbf{D}, \varphi : \mathbf{E} \to \mathbf{F} \rangle$$

and an embedding

$$e : \mathcal{S} \to \mathcal{T}$$

such that for any embeddings $f : \mathcal{T} \to \mathcal{T}_0$ and $g : \mathcal{T} \to \mathcal{T}_1$, where $\mathcal{T}_0, \mathcal{T}_1 \in \mathcal{K}_p$, there are $\mathcal{U} \in \mathcal{K}_p$ and embeddings $r : \mathcal{T}_0 \to \mathcal{U}$ and $s : \mathcal{T}_1 \to \mathcal{U}$ with $r \circ f \circ e = s \circ g \circ e$.

We now have the following.

THEOREM 1.3. *Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ its Fraïssé limit. Then the following are equivalent:*
(i) $\mathbf{K}$ *has a generic automorphism;*
(ii) $\mathcal{K}_p$ *satisfies* JEP *and* WAP.

After obtaining this result, we found out that Ivanov [**29**] had already proved a similar theorem, in response to Truss' question, in a somewhat different context, that of $\aleph_0$-categorical structures (he also calls WAP the *almost amalgamation property*). Our approach however, through the idea of turbulence, is different and we proceed to explain it in more detail.

Suppose that a Polish group G acts continuously on a Polish space X. Given $x \in X$, an open neighborhood U of x and an open symmetric neighborhood V of the identity of G, the

(U, V)-*local orbit* of x, $\mathcal{O}(x, U, V)$, is the set of all $y \in U$ for which there is a finite sequence $g_0, g_1, \ldots, g_k \in V$ with

$$x_0 = x, \quad g_i \cdot x_i = x_{i+1}, \quad x_{k+1} = y \quad \text{and} \quad x_i \in U \quad \text{for all } i.$$

A point x is *turbulent* if for every U and V as above $\mathrm{Int}(\overline{\mathcal{O}(x, U, V)}) \neq \emptyset$. This turns out to be equivalent to saying that $x \in \mathrm{Int}(\overline{\mathcal{O}(x, U, V)})$; see Kechris [**32**]. It is easy to see that this only depends on the orbit $G \cdot x$ of x, so we can refer to *turbulent orbits*. This action is called (*generically*) *turbulent* if:

(i) every orbit is meager,
(ii) there is $x \in X$ with dense, turbulent orbit.

(This is not quite the original definition of turbulence, as in [**24**], but it is equivalent to it; see, for example, [**32**].)

Examples of turbulent actions, relevant to our context, include the conjugacy actions of $U(H)$ (see [**34**] by Kechris and Sofronidis) and $\mathrm{Aut}(X, \mu)$ (see [**15**] by Foreman and Weiss).

Now Hjorth [**24**] has shown that no closed subgroup of S_∞ has a turbulent action and from this one has the following corollary, which can also be easily proved directly.

Proposition 1.4. *Let G be a closed subgroup of S_∞ and suppose G acts continuously on the Polish space X. Then the following are equivalent for any $x \in X$:*

(i) *the orbit $G \cdot x$ is dense G_δ;*
(ii) *$G \cdot x$ is dense and turbulent.*

Thus a dense G_δ orbit exists if and only if a dense turbulent orbit exists.

We can now apply this to the conjugacy action of $\mathrm{Aut}(\mathbf{K})$, with $\mathbf{K}$ a Fraïssé structure, on itself by conjugacy and this leads to the formulation of the WAP and our approach to the proof of Theorem 3. In view of that result, it is interesting that, in the context of this action, the existence of dense, turbulent orbits is equivalently manifested as a combination of joint embedding and amalgamation properties.

1.5. *Homeomorphisms of the Cantor space*

We next use these ideas to answer the question of Akin, Hurley and Kennedy [**3**] about the existence of a generic homeomorphism of the Cantor space $2^{\mathbb{N}}$, that is, the existence of a dense G_δ conjugacy class in $H(2^{\mathbb{N}})$. Since $H(2^{\mathbb{N}})$ is isomorphic (as a topological group) to $\mathrm{Aut}(\mathbf{B}_\infty)$, where $\mathbf{B}_\infty$ is the countable atomless Boolean algebra (that is, the Fraïssé limit of the class of finite Boolean algebras), this follows from the following result.

Theorem 1.5. *Let $\mathcal{BA}$ be the class of finite Boolean algebras. Then $\mathcal{BA}_p$ has the* CAP. *So $\mathbf{B}_\infty$ has a generic automorphism and there is a generic homeomorphism of the Cantor space.*

Using well-known results, we can also see that there is a generic element of the group of order-preserving homeomorphisms of the interval $[0, 1]$.

1.6. *Ample generics*

Finally, we discuss the concept of ample generics in Polish groups, which is a tool that has been used before (see, for example, Hodges *et al.* [**26**]) in the structure theory of automorphism groups. We say that a Polish group G has *ample generic* elements if for each finite n there is a comeager orbit for the (diagonal) conjugacy action of G on

$$G^n : g \cdot (g_1, \ldots, g_n) = (g g_1 g^{-1}, \ldots, g g_n g^{-1}).$$

Obviously this is a stronger property than just having a comeager conjugacy class and for example $\mathrm{Aut}(\mathbb{Q}, <)$ has the latter, but not the former (see Kuske and Truss [**35**] for a discussion of this).

There is now an extensive list of permutation groups known to have ample generics. These include the automorphism groups of the following structures: many ω-stable, $\aleph_0$-categorical structures (see [**26**]), the random graph (Hrushovski [**28**], see also [**26**]), the free group on countably many generators (Bryant and Evans [**9**]) and arithmetically saturated models of true arithmetic (Schmerl [**43**]). Moreover, Herwig and Lascar [**23**] have extended the result of Hrushovski to a much larger class of structures in finite relational languages and the isometry group of the rational Urysohn space $\mathbf{U}_0$ is now also known to have ample generics (this follows from recent results of Solecki [**45**] and Vershik).

We will add another two groups to this list, which incidentally are automorphism groups of structures that are not $\aleph_0$-categorical, namely, the group of (Haar) measure-preserving homeomorphisms of the Cantor space, $H(2^{\mathbb{N}}, \sigma)$, and the group of Lipschitz homeomorphisms of the Baire space. This latter group is canonically isomorphic to $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$, where $\mathbb{N}^{<\mathbb{N}}$ is seen as the infinitely splitting regular rooted tree.

Let us first notice that in the same manner as for the existence of a comeager conjugacy class, we are able to determine an equivalent model-theoretic condition on a Fraïssé class $\mathcal{K}$ for when the automorphism group of its Fraïssé limit $\mathbf{K}$ has ample generic elements. This criterion is in fact a trivial generalization of the one-dimensional case; but we shall be more interested in the consequences of the existence of ample generics.

Recall that a second countable topological group G is said to have the *small index property* if any subgroup of index less than $2^{\aleph_0}$ is open. Then we can show the following, which generalizes the case of automorphism groups of ω-stable, $\aleph_0$-categorical structures due to Hodges *et al.* [**26**].

THEOREM 1.6. *Let G be a Polish group with ample generic elements. Then G has the small index property.*

In the case of G being a closed subgroup of S_∞, that is, G having a neighborhood basis at the identity consisting of clopen subgroups, this essentially says that the topological structure of the group is completely determined by its algebraic structure.

We subsequently study the cofinality of Polish groups. Recall that the *cofinality* of a group G is the least cardinality of a well-ordered chain of proper subgroups whose union is G. Again generalizing results of Hodges *et al.* [**26**], we prove the following.

THEOREM 1.7. *Let G be a Polish group with ample generic elements. Then G is not the union of countably many non-open subgroups* (*or even cosets of subgroups*).

Note that if G, a closed subgroup of S_∞, is oligomorphic, that is, has only finitely many orbits on each $\mathbb{N}^n$, then, by a result of Cameron, any open subgroup of G is contained in only finitely many subgroups of G; thus, if G has ample generics, it has uncountable cofinality. The same holds for connected Polish groups, and Polish groups with a finite number of topological generators.

It turns out that the existence of generic elements of a Polish group has implications for its actions on trees. So let us recall some basic notions of the theory of group actions on trees (see Serre [**44**]).

A group G is said to act *without inversion* on a tree T if G acts on T by automorphisms such that for no $g \in G$ are there two adjacent vertices $a, b \in T$ such that

$$g \cdot a = b \quad \text{and} \quad g \cdot b = a.$$

The action is said to have a *fixed point* if there is an $a \in T$ such that

$$g \cdot a = a, \quad \text{for all } g \in G.$$

We say that a group G has *property* (FA) if whenever G acts without inversion on a tree, there is a fixed point. When G is not countable this is known to be equivalent to the conjunction of the following three properties [**44**]:

(i) G is not a non-trivial free product with amalgamation,
(ii) $\mathbb{Z}$ is not a homomorphic image of G,
(iii) G has uncountable cofinality.

Macpherson and Thomas [**38**] recently showed that (i) follows if G has a comeager conjugacy class and, as (ii) trivially also holds in this case, we are left with verifying (iii).

Another way of proving property (FA) is through a slightly different study of the generation of Polish groups with ample generics. Obviously, any generating set for an uncountable group must be uncountable, but ideally we would still like to understand the structure of the group by studying a set of generators. Let us say that a group G has the *Bergman property* if for each exhaustive sequence of subsets

$$W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G,$$

there are n and k such that $W_n^k = G$. Bergman [**7**] showed this property for S_∞ by methods very different from those employed here and we will extend his result to a fairly large class of automorphism groups.

Theorem 1.8. *Let G be a closed oligomorphic subgroup of S_∞ with ample generic elements. Then G has the Bergman property.*

In particular, this result applies to many automorphism groups of ω-stable, $\aleph_0$-categorical structures and the automorphism group of the random graph.

It is not hard to see, and has indeed been noticed independently by other authors (for example, Cornulier [**11**]) that the Bergman property also implies that any action of the group by isometries on a metric space has bounded orbits. In fact, this is actually an equivalent formulation of the Bergman property. However, well-known results of geometric group theory (see B. Bekka, P. de la Harpe and A. Valette [**6**]) state that if a group action by isometries on a real Hilbert space has a bounded orbit, then it has a fixed point. A similar result holds for an action by automorphisms without inversion on a tree. Thus Bergman groups automatically have property (FH) and (FA), where property (FH) is the statement that any isometric action on a real Hilbert space has a fixed point.

Corollary 1.9. *Let G be a closed oligomorphic subgroup of S_∞ with ample generic elements. Then G has properties* (FA) *and* (FH).

The phenomenon of automatic continuity is well known and has been extensively studied, in particular in the context of Banach algebras. In this category morphisms of course preserve much more structure than homomorphisms of the underlying groups and therefore automatic continuity is easier to obtain. However, there are also plenty of examples of this phenomenon for topological groups, provided we add some definability constraints on the homomorphisms. An example of this is the classical result of Pettis, saying that any Baire measurable homomorphism from a Polish group into a separable group is continuous. Surprisingly though, when one assumes ample generics one can completely eliminate any definability assumption and still obtain the same result. In fact, one does not even need as much as separability for the target group, but essentially need only rule out that its topology is discrete. Let us recall that the

Souslin number of a topological space is the least cardinal κ such that there is no family of κ many disjoint open sets. In analogy with this, if H is a topological group, we let the *uniform Souslin number* of H be the least cardinal κ such that there is no non-empty open set having κ many disjoint translates. Then we can prove the following.

THEOREM 1.10. *Suppose G is a Polish group with ample generic elements and*

$$\pi : G \to H$$

is a homomorphism into a topological group with uniform Souslin number at most $2^{\aleph_0}$ (in particular, if H is separable). Then π is continuous.

This shows, in particular, that any action of a Polish group with ample generics by isometries on a Polish space or by homeomorphisms on a compact metric space is actually a continuous action. For such an action is essentially just a homomorphism into the isometry group, respectively into the homeomorphism group.

Moreover, one also sees that in this case there is a unique Polish group topology, and coupled with a result of Gaughan [**18**], one can in the case of S_∞ prove the following stronger fact.

THEOREM 1.11. *There is exactly one non-trivial separable group topology on S_∞.*

In the literature one can find several results on automatic continuity of homomorphisms between topological groups when one puts restrictions on the target groups. For example, the small index property of a group G can be seen to imply that any homomorphism from G into S_∞ is continuous. Also a classical theorem due to Van der Waerden (see Hofmann and Morris [**27**]) states that if $\pi\colon G \to H$ is a group homomorphism from an n-dimensional Lie group G, whose Lie algebra agrees with its commutator algebra, into a compact group H, then π is continuous. The final result we should mention is due to Dudley [**14**], which says that any homomorphism from a complete metric group into a 'normed' group with the discrete topology is automatically continuous. We shall not go into his definition of a normed group, other than saying that these include the additive group of the integers and more general free groups. The novelty of Theorem 1.10 lies in the fact that it places no restrictions on the target group (other than essentially ruling out the trivial case of the topology being discrete).

Adding $H(2^{\mathbb{N}}, \sigma)$ and $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ to the list of Polish groups with ample generics, we finally show the following, where a closed subgroup of S_∞ has the *strong small index property* if any subgroup of index less than $2^{\aleph_0}$ is sandwiched between the pointwise and setwise stabilizer of a finite set.

THEOREM 1.12. *Let G be either $H(2^{\mathbb{N}}, \sigma)$, the group of measure-preserving homeomorphisms of the Cantor space, or $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$, the group of Lipschitz homeomorphisms of the Baire space. Then*

(i) *G has ample generic elements;*
(ii) *G has the strong small index property;*
(iii) *G has uncountable cofinality;*
(iv) *G has the Bergman property and thus properties* (FH) *and* (FA).

The strong small index property for $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ was previously proved by Rögnvaldur Möller in [**39**].

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2. *Automorphisms with dense conjugacy classes*

Let $\mathcal{K}$ be a Fraïssé class. We let $\mathcal{K}_p$ be the class of all systems of the form

$$\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle,$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\mathbf{B}, \mathbf{C} \subseteq \mathbf{A}$ and ψ is an isomorphism of $\mathbf{B}$ and $\mathbf{C}$. An embedding of one system $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle$ into another $\mathcal{T} = \langle \mathbf{D}, \phi : \mathbf{E} \to \mathbf{F} \rangle$ is an embedding $f : \mathbf{A} \to \mathbf{D}$ such that f embeds $\mathbf{B}$ into $\mathbf{E}$ and $\mathbf{C}$ into $\mathbf{F}$, and moreover $f \circ \psi \subseteq \phi \circ f$.

Thus we can define JEP and AP for $\mathcal{K}_p$ as well.

THEOREM 2.1. *Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbf{K}$. Then the following are equivalent:*

(i) *there is a dense conjugacy class in* $\mathrm{Aut}(\mathbf{K})$*;*
(ii) $\mathcal{K}_p$ *satisfies the* JEP.

Proof. (i) $\Rightarrow$ (ii) Fix some element $f \in \mathrm{Aut}(\mathbf{K})$ having a dense conjugacy class in $\mathrm{Aut}(\mathbf{K})$ and suppose

$$\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \quad \text{and} \quad \mathcal{T} = \langle \mathbf{D}, \phi : \mathbf{E} \to \mathbf{F} \rangle$$

are two systems in $\mathcal{K}_p$.

Replacing $\mathcal{S}$ and $\mathcal{T}$ by isomorphic copies we can of course assume that $\mathbf{A}, \mathbf{D} \subseteq \mathbf{K}$. Now by ultrahomogeneity of $\mathbf{K}$ we know that both ψ and ϕ have extensions in $\mathrm{Aut}(\mathbf{K})$, so, by the density of the conjugacy class of f, there are $g_1, g_2 \in \mathrm{Aut}(\mathbf{K})$ such that $\psi \subseteq g_1^{-1} f g_1$ and $\phi \subseteq g_2^{-1} f g_2$. Let

$$\mathbf{H} = \langle g_1'' \mathbf{A} \cup g_2'' \mathbf{D} \rangle, \quad \mathbf{M} = \langle g_1'' \mathbf{B} \cup g_2'' \mathbf{E} \rangle, \quad \mathbf{N} = \langle g_1'' \mathbf{C} \cup g_2'' \mathbf{F} \rangle, \quad \text{and} \quad \chi = f \restriction \mathbf{M}.$$

Then it is easily seen that $g_1 \restriction_{\mathbf{A}}$ and $g_2 \restriction_{\mathbf{D}}$ embed $\mathcal{S}$ and $\mathcal{T}$ into $\langle \mathbf{H}, \chi : \mathbf{M} \to \mathbf{N} \rangle$.

(ii) $\Rightarrow$ (i) A basis for the open subsets of $\mathrm{Aut}(\mathbf{K})$ consists of sets of the form

$$[\psi] = [\psi : \mathbf{B} \to \mathbf{C}] = \{ f \in \mathrm{Aut}(\mathbf{K}) : f \supseteq \psi \},$$

where $\psi : \mathbf{B} \to \mathbf{C}$ is an isomorphism of finite substructures of $\mathbf{K}$. We refer to such $\psi : \mathbf{B} \to \mathbf{C}$ as *conditions*. We wish to construct an element $f \in \mathrm{Aut}(\mathbf{K})$ such that for any condition $\psi : \mathbf{B} \to \mathbf{C}$ there is a $g \in \mathrm{Aut}(\mathbf{K})$ such that $g \psi g^{-1} \subseteq f$. So let

$$D(\psi : \mathbf{B} \to \mathbf{C}) = \{ f \in \mathrm{Aut}(\mathbf{K}) : \exists g \in \mathrm{Aut}(\mathbf{K}) (g \psi g^{-1} \subseteq f) \}.$$

Now $D(\psi : \mathbf{B} \to \mathbf{C})$ is clearly open, but we shall see that it is also dense. If $[\phi : \mathbf{E} \to \mathbf{F}]$ is a basic open set, then there exist, by the JEP of $\mathcal{K}_p$ and the extension property of $\mathbf{K}$, some isomorphism $\chi : \mathbf{H} \to \mathbf{L}$ of finite substructures of $\mathbf{K}$ such that $\phi \subseteq \chi$, and some $g \in \mathrm{Aut}(\mathbf{K})$ embedding $\psi : \mathbf{B} \to \mathbf{C}$ into $\chi : \mathbf{H} \to \mathbf{L}$. Then any $f \in \mathrm{Aut}(\mathbf{K})$ extending χ witnesses that $D(\psi : \mathbf{B} \to \mathbf{C}) \cap [\phi : \mathbf{E} \to \mathbf{D}] \neq \emptyset$. Therefore any $f \in \mathrm{Aut}(\mathbf{K})$ that is in the intersection of all $D(\psi : \mathbf{B} \to \mathbf{C})$ has dense conjugacy class. $\square$

REMARK. One can also give a more direct proof of Theorem 1.1 by using the following standard fact: if G is a Polish group which acts continuously on a Polish space X and $\mathcal{B}$ is a countable basis of non-empty open sets for the topology of X, then there is a dense orbit for this action if and only if $G \cdot U \cap V \neq \emptyset$ for all $U, V \in \mathcal{B}$. (The direction $\Rightarrow$ is obvious. For the direction $\Leftarrow$, let $D_V = \{ x \in X : G \cdot x \cap V \neq \emptyset \}$ for $V \in \mathcal{B}$. This is clearly open, dense, and so $\bigcap_{V \in \mathcal{B}} D_V \neq \emptyset$. Any $x \in \bigcap_{V \in \mathcal{B}} D_V$ has dense orbit.)

We can simply apply this to the conjugacy action of $\mathrm{Aut}(\mathbf{K})$ on itself and the basis consisting of the sets $[\psi]$ as above.

We will proceed to some applications of Theorem 2.1.

THEOREM 2.2. *Let* $\mathbf{U}_0$ *be the rational Urysohn space. Then* $\mathrm{Aut}(\mathbf{U}_0)$ *has a dense conjugacy class. Thus, as* $\mathrm{Aut}(\mathbf{U}_0)$ *can be continuously, densely embedded into* $\mathrm{Iso}(\mathbf{U})$*, the isometry group of the Urysohn space,* $\mathrm{Iso}(\mathbf{U})$*, also has a dense conjugacy class.*

Proof. Let $\mathcal{K}$ be the class of finite metric spaces with rational distances; we will see that $\mathcal{K}_p$ has the JEP.

Given $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle$ and $\mathcal{T} = \langle \mathbf{D}, \phi : \mathbf{E} \to \mathbf{F} \rangle \in \mathcal{K}_p$, we let $\mathbf{H} = \mathbf{A} \sqcup \mathbf{B}$ be the disjoint union of the two metric spaces $\mathbf{A}$ and $\mathbf{B}$, where we have put some distance $k > \mathrm{diam}(\mathbf{A}) + \mathrm{diam}(\mathbf{B})$ between any two points $x \in \mathbf{A}$ and $y \in \mathbf{B}$. The triangle inequality is clearly satisfied. So now we can let $\mathbf{M}$ and $\mathbf{N}$ be the corresponding unions of $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{C}$, $\mathbf{F}$, and finally we let $\chi = \psi \cup \varphi$. Then clearly both $\mathcal{S}$ and $\mathcal{T}$ embed into $\langle \mathbf{H}, \chi : \mathbf{M} \to \mathbf{N} \rangle$. $\square$

Glasner and Pestov have informed us that they have also proved that $\mathrm{Iso}(\mathbf{U})$ has a dense conjugacy class.

We next consider the class $\mathcal{K} = \mathcal{MBA}_{\mathbb{Q}}$ of all finite boolean algebras with an additive probability measure taking positive rational values on each non-zero element. So an element of $\mathcal{K}$ is of the form $(\mathbf{A}, \mu : \mathbf{A} \to [0,1] \cap \mathbb{Q})$, but, in order to get into first order model theory, we will view μ as a collection of unary predicates on $\mathbf{A}$, $\{M_r\}_{r \in [0,1] \cap \mathbb{Q}}$, where $M_r(x) \Leftrightarrow \mu(x) = r$.

PROPOSITION 2.3. *The class* $\mathcal{K}$ *is a Fraïssé class.*

Proof. Obviously $\mathcal{K}$ has the HP. Now as all structures in $\mathcal{K}$ have a common substructure (the trivial boolean algebra {0,1}), it is enough to verify AP, from which JEP will follow.

So suppose $f : (\mathbf{A}, \mu) \to (\mathbf{B}, \nu)$ and $g : (\mathbf{A}, \mu) \to (\mathbf{C}, \rho)$ are embeddings. Let $a_1, \ldots, a_n$ be the atoms of $\mathbf{A}$, $b_1, \ldots, b_m$ be the atoms of $\mathbf{B}$, and $c_1, \ldots, c_k$ be the atoms of $\mathbf{C}$, and let $\Gamma_1 \sqcup \ldots \sqcup \Gamma_n$ partition $\{1, \ldots, m\}$ and $\Lambda_1 \sqcup \ldots \sqcup \Lambda_n$ partition $\{1, \ldots, k\}$ such that

$$f(a_l) = \bigvee_{i \in \Gamma_l} b_i \quad \text{and} \quad g(a_l) = \bigvee_{j \in \Lambda_l} c_j.$$

Let $\mathbf{D}$ be the boolean algebra with formal atoms $b_i \otimes c_j$ for $(i,j) \in \Gamma_l \times \Lambda_l$ with $l \leqslant n$, and let the embeddings $e : \mathbf{B} \to \mathbf{D}$ and $h : \mathbf{C} \to \mathbf{D}$ be defined by

$$e(b_i) = \bigvee_{j \in \Lambda_l} b_i \otimes c_j, \quad \text{where } i \in \Gamma_l,$$

$$h(c_j) = \bigvee_{i \in \Gamma_l} b_i \otimes c_j, \quad \text{where } j \in \Lambda_l.$$

This is the usual amalgamation of boolean algebras and we now only have to check that we can define an appropriate measure σ on $\mathbf{D}$ to finish the proof.

Notice first that $\mu(a_l) = \sum_{i \in \Gamma_l} \nu(b_i) = \sum_{j \in \Lambda_l} \rho(c_j)$, so the appropriate measure is

$$\sigma(b_i \otimes c_j) = \frac{\nu(b_i)\rho(c_j)}{\mu(a_l)}, \quad \text{for } (i,\, j) \in \Gamma_l \times \Lambda_l.$$

Then e and h are indeed embeddings, as, for example,

$$\sigma(e(b_i)) = \sigma\Big(\bigvee_{j \in \Gamma_l} b_i \otimes c_j\Big) = \sum_{j \in \Gamma_l} \frac{\nu(b_i)\rho(c_j)}{\mu(a_l)} = \nu(b_i). \tag{1}$$

$\square$

We can now identify the Fraïssé limit of the class $\mathcal{K}$. It follows from Fraïssé's construction that the limit must be the countable atomless boolean algebra equipped with some finitely additive probability measure. Let $\mathbf{F}$ be the boolean algebra generated by the rational intervals in the measure algebra of $([0,1],\lambda)$, where λ is Lebesgue measure, together with the restriction of Lebesgue measure.

PROPOSITION 2.4. *The measure algebra* $(\mathbf{F},\lambda)$ *is the Fraïssé limit of* $\mathcal{K}$.

Proof. The age of $(\mathbf{F},\lambda)$ is clearly $\mathcal{K}$, so we need only check that $(\mathbf{F},\lambda)$ has the extension property. So suppose $(\mathbf{A},\mu)\subseteq(\mathbf{B},\nu)\in\mathcal{K}$ and $f:\mathbf{A}\to\mathbf{F}$ is an embedding of $(\mathbf{A},\mu)$ into $(\mathbf{F},\lambda)$. Then we can extend f to an embedding $\tilde{f}$ of $(\mathbf{B},\nu)$ into $(\mathbf{F},\lambda)$ as follows.

An atom a of $\mathbf{A}$ corresponds by f to a finite disjoint union of rational intervals in [0,1] and is also the join of atoms $b_1,\dots,b_k$ in $\mathbf{B}$ with rational measure. So by appropriately decomposing these rational intervals into finitely many pieces we can find images for each of $b_1,\dots,b_k$ of the same measure. □

If we let $\mathcal{K}=\mathcal{MBA}_{\mathbb{Q}_2}$ be the class of finite boolean algebras with a measure that takes positive dyadic rational values on each non-zero element, then one can show (though it is more complicated than the preceding proof) that $\mathcal{K}$ is a Fraïssé class with limit $(\mathrm{clop}(2^{\mathbb{N}}),\sigma)$, where σ is the usual product measure on $2^{\mathbb{N}}$. Moreover, $\mathrm{Aut}(\mathrm{clop}(2^{\mathbb{N}}),\sigma)$ is (isomorphic to) the group of measure-preserving homeomorphisms of $2^{\mathbb{N}}$ (with the uniform convergence topology), $H(2^{\mathbb{N}},\sigma)$.

We now have the following result.

THEOREM 2.5. *There is a dense conjugacy class in* $\mathrm{Aut}(\mathbf{F},\lambda)$.

Proof. We verify that $\mathcal{K}_p$ has the JEP. Therefore suppose that $\langle\mathbf{A},\mu,\psi:\mathbf{B}\to\mathbf{C}\rangle$ and $\langle\mathbf{D},\nu,\phi:\mathbf{E}\to\mathbf{H}\rangle$ are given, where $\mathbf{A}\supseteq\mathbf{B},\mathbf{C}$ and $\mathbf{D}\supseteq\mathbf{E},\mathbf{H}$ are finite boolean algebras with rational-valued probability measures μ and ν, and ψ and ϕ are isomorphisms preserving the measure. We amalgamate $\langle\mathbf{A},\mu\rangle$ and $\langle\mathbf{D},\nu\rangle$ over the trivial boolean algebra as in the proof of Proposition 2.3, so our atoms in the new algebra $\mathbf{J}$ are $a_i\otimes d_j$, where $a_1,\dots,a_n$ and $d_1,\dots,d_m$ are the atoms at $\mathbf{A}$ and $\mathbf{D}$, respectively, and $i\leqslant n$ and $j\leqslant m$. Find partitions

$$\Gamma_1\sqcup\ldots\sqcup\Gamma_k=\Lambda_1\sqcup\ldots\sqcup\Lambda_k=\{1,\dots,n\}$$

and

$$\Delta_1\sqcup\ldots\sqcup\Delta_l=\Theta_1\sqcup\ldots\sqcup\Theta_l=\{1,\dots,m\}$$

such that $\bigvee_{\Gamma_e}a_i$, $\bigvee_{\Lambda_e}a_i$, $\bigvee_{\Delta_e}d_j$ and $\bigvee_{\Theta_e}d_j$ are the atoms of $\mathbf{B}$, $\mathbf{C}$, $\mathbf{E}$ and $\mathbf{H}$ respectively and

$$\psi\Big(\bigvee_{\Gamma_e}a_i\Big)=\bigvee_{\Lambda_e}a_i,\qquad\phi\Big(\bigvee_{\Delta_e}d_j\Big)=\bigvee_{\Theta_e}d_j.$$

Then we let $\mathbf{M}$ be the subalgebra of $\mathbf{J}$ generated by the atoms $\bigvee_{\Gamma_e\times\Delta_f}a_i\otimes d_j$, $\mathbf{N}$ be the subalgebra generated by the atoms $\bigvee_{\Lambda_e\times\Theta_f}a_i\otimes d_j$, and $\chi:\mathbf{M}\to\mathbf{N}$ be the isomorphism given by

$$\chi\Big(\bigvee_{\Gamma_e\times\Delta_f}a_i\otimes d_j\Big)=\bigvee_{\Lambda_e\times\Theta_f}a_i\otimes d_j.$$

Then $\langle\mathbf{J},\chi:\mathbf{M}\to\mathbf{N}\rangle$ clearly amalgamates $\langle\mathbf{A},\psi:\mathbf{B}\to\mathbf{C}\rangle$ and $\langle\mathbf{D},\phi:\mathbf{E}\to\mathbf{H}\rangle$ over the trivial boolean algebra, so the only thing we need to check is that χ preserves the measure σ

on $\mathbf{J}$ given by $\sigma(a_i \otimes d_j) = \mu(a_i) \cdot \nu(d_j)$:

$$\begin{aligned}
\sigma\Big(\bigvee_{\Gamma_e \times \Delta_f} a_i \otimes d_j\Big) &= \sum_{i \in \Gamma_e} \sum_{j \in \Delta_f} \mu(a_i)\nu(d_j) \\
&= \Big(\sum_{i \in \Gamma_e} \mu(a_i)\Big) \cdot \Big(\sum_{j \in \Delta_f} \nu(d_j)\Big) \\
&= \Big(\sum_{i \in \Lambda_e} \mu(a_i)\Big) \cdot \Big(\sum_{j \in \Theta_f} \nu(d_j)\Big) \\
&= \sum_{i \in \Lambda_e} \sum_{j \in \Theta_f} \mu(a_i)\nu(d_j) \\
&= \sigma\Big(\bigvee_{\Lambda_e \times \Theta_f} a_i \otimes d_j\Big). \qquad \square
\end{aligned}$$

By using $\mathcal{MBA}_{\mathbb{Q}_2}$ instead of $\mathcal{MBA}_{\mathbb{Q}}$, we obtain the same result for $H(2^{\mathbb{N}}, \sigma)$.

Let now $\mathcal{K} = \mathcal{BA}$ be the class of all finite Boolean algebras. It is well known that $\mathcal{K}$ is a Fraïssé class (the argument is essentially that of Proposition 2.3, forgetting about the measures), whose Fraïssé limit is the countable atomless Boolean algebra $\mathbf{B}_\infty$. As in the proof of Theorem 2.5, we can easily verify the JEP for $\mathcal{K}_p$, so we have the following.

THEOREM 2.6. *There is a dense conjugacy class in* $\mathrm{Aut}(\mathbf{B}_\infty)$.

From this we immediately deduce a corollary.

COROLLARY 2.7 (Glasner and Weiss [**19**], Akin, Hurley and Kennedy [**3**]). *In the uniform topology there is a dense conjugacy class on* $H(2^{\mathbb{N}})$ (*the group of homeomorphisms of the Cantor space*).

Note that Corollary 2.7 has as a consequence the known fact that the aperiodic homeomorphisms form a dense G_δ set in $H(2^{\mathbb{N}})$. This is because for each $n > 0$,

$$\{h \in H(2^{\mathbb{N}}) \colon \exists x\ h^n(x) = x\}$$

is closed and conjugacy invariant, so it has empty interior. Thus,

$$\mathrm{APER} = \{h \in H(2^{\mathbb{N}}) \colon \forall x\ \forall n > 0\ h^n(x) \neq x\}$$

is dense G_δ.

Consider now the group $\mathrm{Aut}(I, \lambda)$ of measure-preserving automorphisms of the unit interval I with Lebesgue measure λ. By a theorem of Sikorski (see Kechris [**31**]) this is the same as the group of automorphisms of the measure algebra of (I, λ).

Every element $\varphi \in \mathrm{Aut}(\mathbf{F}, \lambda)$ induces a unique automorphism φ^* of the measure algebra, as $\mathbf{F}$ is dense in the latter. And therefore φ^* can be seen as an element of $\mathrm{Aut}(I, \lambda)$.

It is not hard to see that the mapping $\varphi \mapsto \varphi^*$ is a continuous injective homomorphism from $\mathrm{Aut}(\mathbf{F}, \lambda)$ into $\mathrm{Aut}(I, \lambda)$. Moreover, the image of $\mathrm{Aut}(\mathbf{F}, \lambda)$ is dense in $\mathrm{Aut}(I, \lambda)$. To see this, use for example the following fact (see Halmos [**22**]): the set of $f \in \mathrm{Aut}(I, \lambda)$, such that for some n, f is just shuffling the dyadic intervals at length 2^n, $[i/2^n, (i+1)/2^n]$ (linearly on each interval), is dense in $\mathrm{Aut}(I, \lambda)$. Obviously any such f is the image of some element of $\mathrm{Aut}(\mathbf{F}, \lambda)$.

COROLLARY 2.8. *The group* $\mathrm{Aut}(I, \lambda)$ *has a dense conjugacy class.*

This is of course a weaker version of the conjugacy lemma of ergodic theory which asserts that the conjugacy class of any aperiodic transformation is dense in $\mathrm{Aut}(I, \lambda)$. However, the above proof may be of some interest as it avoids the use of Rokhlin's Lemma.

Finally, let $\mathcal{K}$ be the Fraïssé class of all finite linear orderings. Trivially $\mathcal{K}_p$ has the JEP, so the automorphism group of $(\mathbb{Q}, <)$ has a dense conjugacy class; but $\mathrm{Aut}(\mathbb{Q}, <)$ can be densely and continuously embedded into $H_+([0, 1])$, the group of order-preserving homeomorphisms of the unit interval with the uniform topology. So we have the following corollary.

Corollary 2.9 (Glasner and Weiss [**19**]). *There is a dense conjugacy class in* $H_+([0, 1])$.

Obviously there cannot be any dense conjugacy class in $H([0, 1])$, as it has a proper clopen normal subgroup, $H_+([0, 1])$.

In the same manner one can easily check that for $\mathcal{K}$ the class of finite graphs, finite hypergraphs, finite posets, etc., $\mathcal{K}_p$ has the JEP. So the automorphism group of the Fraïssé limit, for example, the random graph, has a dense conjugacy class.

Remark. For an example of a Fraïssé class $\mathcal{K}$ for which $\mathcal{K}_p$ does not have the JEP, consider the class of finite equivalence relations with at most two equivalence classes. Its Fraïssé limit is $(\mathbb{N}, E)$ where

$$nEm \iff \mathrm{parity}(n) = \mathrm{parity}(m).$$

So any $f \in \mathrm{Aut}(\mathbb{N}, E)$ either fixes each class setwise or permutes the two classes. Therefore the group of f setwise fixing the classes is a clopen normal subgroup (of index 2) and there cannot be any dense conjugacy class in $\mathrm{Aut}(\mathbb{N}, E)$. So the JEP does not hold for $\mathcal{K}_p$ and the counterexample is of course two equivalence relations with two classes each and two automorphisms, one of which fixes the two classes and the other switches them.

In certain situations one can obtain more precise information concerning dense conjugacy classes, which also has further interesting consequences.

Suppose that G is a Polish group. We say that G has *cyclically dense conjugacy classes* if there are $g, h \in G$ such that $\{g^n h g^{-n}\}_{n\in\mathbb{Z}}$ is dense in G.

Notice that the set

$$D = \{(g, h) \in G^2 \colon \{g^n h g^{-n}\}_{n\in\mathbb{Z}} \text{ is dense in } G\}$$

is G_δ. Also, if some section $D_g = \{h \in G \colon \{g^n h g^{-n}\}_{n\in\mathbb{Z}} \text{ is dense in } G\}$ is non-empty, then it is dense in G. Moreover, as $D_{fgf^{-1}} = fD_g f^{-1}$, the set of $g \in G$ for which $D_g \neq \emptyset$ is conjugacy invariant. So if there is some g with a dense conjugacy class such that $D_g \neq \emptyset$, then there is a dense set of $f \in G$ for which D_f is dense and hence D itself is dense and thus comeager in G^2.

For each Polish group G, we denote by $n(G)$ the smallest number of *topological generators* of G, that is, the smallest $1 \leqslant n \leqslant \infty$ such that there is an n-generated dense subgroup of G. Thus $n(G) = 1$ if and only if G is monothetic. It follows immediately that if G admits a cyclically dense conjugacy class, then $n(G) \leqslant 2$.

Consider now a Fraïssé structure $\mathbf{K}$ and suppose we can find $g \in \mathrm{Aut}(\mathbf{K})$ with the following property.

$(*)$ For any finite $\mathbf{A}, \mathbf{B} \subseteq \mathbf{K}$ and any isomorphisms $\varphi : \mathbf{C} \to \mathbf{D}$ and $\psi : \mathbf{E} \to \mathbf{F}$, where $\mathbf{C}, \mathbf{D} \subseteq \mathbf{A}$ and $\mathbf{E}, \mathbf{F} \subseteq \mathbf{B}$, there are $m, n \in \mathbb{Z}$ such that $g^m \varphi g^{-m}$ and $g^n \psi g^{-n}$ have a common extension.

Then, as in the proof of Theorem 1.1, we see that there is $h \in \mathrm{Aut}(\mathbf{K})$ with $\{g^n h g^{-n}\}$ dense in $\mathrm{Aut}(\mathbf{K})$, that is, $\mathbf{K}$ admits a cyclically dense conjugacy class, and moreover $\mathrm{Aut}(\mathbf{K})$ is topologically 2-generated.

In [**37**, 3.3] Macpherson has shown that the automorphism group of the random graph has cyclically dense conjugacy classes and so is topologically 2-generated. We note below that $\mathrm{Aut}(\mathbf{B}_\infty)$, and thus $H(2^{\mathbb{N}})$, as well as $H(2^{\mathbb{N}},\sigma)$, $\mathrm{Aut}(X,\mu)$ and $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$, admit cyclically dense conjugacy classes, so in particular, they are topologically 2-generated. That $\mathrm{Aut}(X,\mu)$ is topologically 2-generated was earlier proved by different means by Grzaslewicz [**20**] and Prasad [**41**].

THEOREM 2.10. *Each of the groups* $H(2^{\mathbb{N}})$, $H(2^{\mathbb{N}},\sigma)$, $\mathrm{Aut}(X,\mu)$, *and* $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ *has cyclically dense conjugacy classes and is topologically* 2*-generated.*

Proof. We will sketch the proof for $H(2^{\mathbb{N}})$ or rather its isomorphic copy $\mathrm{Aut}(\mathbf{B}_\infty)$. The proofs in the cases of $H(2^{\mathbb{N}},\sigma)$ and $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ are similar. Since $H(2^{\mathbb{N}},\sigma)$ can be densely embedded in $\mathrm{Aut}(X,\mu)$, the result follows for this group as well.

It is clear (see the proof of Theorem 2.5) that it is enough to show the following, in order to verify property $(*)$ for $\mathbf{K}=\mathbf{B}_\infty$: there is an automorphism g such that given any finite subalgebras $\mathbf{A}$ and $\mathbf{B}$, there is $n\in\mathbb{Z}$ with $g^n(\mathbf{A})$ and $\mathbf{B}$ independent (that is, any non-zero elements of $g^n(\mathbf{A})$ and $\mathbf{B}$ have non-zero join).

To find such a g view $\mathbf{B}_\infty$ as the algebra of clopen subsets of $2^{\mathbb{Z}}$ and take g to be the Bernoulli shift on this space. □

It is not hard to show that also $U(\ell_2)$ has a cyclically dense conjugacy class.

Solecki has recently shown that the group of isometries of the Urysohn space admits a cyclically dense conjugacy class and thus it is topologically 2-generated.

REMARK. Prasad [**41**] showed that there is a comeager set of pairs (g,h) in $\mathrm{Aut}(X,\mu)^2$ generating a dense subgroup of $\mathrm{Aut}(X,\mu)$ (moreover, using the results of § 6.9 one sees easily that generically the subgroup generated is free). This also follows from our earlier remarks, since the g that witnesses the cyclically dense conjugacy class is in this case the shift, whose conjugacy class is dense (by the conjugacy lemma of ergodic theory).

We can also consider the diagonal conjugacy action of $\mathrm{Aut}(\mathbf{K})$ on $\mathrm{Aut}(\mathbf{K})^n$, for $n=1,2,\ldots,\mathbb{N}$. Note that there is a dense diagonal conjugacy class in $\mathrm{Aut}(\mathbf{K})^{\mathbb{N}}$ if and only if for each $n=1,2,\ldots$, there is a dense diagonal conjugacy class in $\mathrm{Aut}(\mathbf{K})^n$. This follows from the fact that in the latter case, the set of elements $(g_m)\in\mathrm{Aut}(\mathbf{K})^{\mathbb{N}}$, whose diagonal conjugacy class is dense in $\mathrm{Aut}(\mathbf{K})^n$, is dense G_δ for all n, and hence we can pick a (g_m) such that it holds for all n and therefore also for $\mathbb{N}$.

Consider a Fraïssé class $\mathcal{K}$ with Fraïssé limit $\mathbf{K}$ and let $G=\mathrm{Aut}(\mathbf{K})$. Define for each $n\geqslant 1$ the following sets (which are all G_δ):

$$\begin{aligned}
F_n &= \{(f_1,\ldots,f_n)\in G^n\colon \text{ for all } x\in\mathbf{K} \text{ the orbit of } x \text{ under } \langle f_1,\ldots,f_n\rangle \text{ is finite}\}\\
&= \{(f_1,\ldots,f_n)\in G^n\colon \langle f_1,\ldots,f_n\rangle \text{ is relatively compact in } G\};\\
D_n &= \{(f_1,\ldots,f_n)\in G^n\colon \langle f_1,\ldots,f_n\rangle \text{ is dense in } G\};\\
E_n &= \{(f_1,\ldots,f_n)\in G^n\colon \langle f_1,\ldots,f_n\rangle \text{ is non-discrete in } G\};\\
H_n &= \{(f_1,\ldots,f_n)\in G^n\colon \langle f_1,\ldots,f_n\rangle \text{ freely generates a free subgroup of } G\}.
\end{aligned}$$

Since the sets F_n, D_n, E_n and H_n are G_δ sets invariant under the diagonal conjugacy action of G on G^n, it follows that if G^n has a dense diagonal conjugacy class then each of them is either comeager or nowhere dense in G^n. We also see that D_n is never dense unless $G=\{1\}$, for if $H<G$ is a proper open subgroup and $(g_1,\ldots,g_n)\in H^n$, then $\langle g_1,\ldots,g_n\rangle\subseteq H$ and hence is not dense. So $D_n\cap H^n=\emptyset$. On the other hand, if $\mathcal{K}$ satisfies the Hrushovski property (see Definition 6.3), then F_n is automatically comeager. Moreover, unless G itself is compact,

the sets F_n and D_n are of course disjoint and thus if also $\mathcal{K}$ has the Hrushovski property, then D_n is nowhere dense.

These results can also be combined with the comments in § 6.9, where we will study the sets H_n.

Suppose that $\mathcal{K}$ is a Fraïssé class, and $\mathbf{K}$ is its Fraïssé limit. As before, we would like to characterize when $\mathrm{Aut}(\mathbf{K})^n$ has a dense diagonal conjugacy class. For this we introduce the class of n-systems $\mathcal{K}_p^n$ for each $n \geqslant 1$. An *n-system* $\mathcal{S} = \langle \mathbf{A}, \psi_1 : \mathbf{B}_1 \to \mathbf{C}_1, \ \dots, \ \psi_n : \mathbf{B}_n \to \mathbf{C}_n \rangle$ consists of finite structures $\mathbf{A}, \mathbf{B}_i, \mathbf{C}_i \in \mathcal{K}$ with $\mathbf{B}_i, \mathbf{C}_i \subseteq \mathbf{A}$ and ψ_i an isomorphism of $\mathbf{B}_i$ and $\mathbf{C}_i$. As before, an embedding of one n-system $\mathcal{S} = \langle \mathbf{A}, \psi_1 : \mathbf{B}_1 \to \mathbf{C}_1, \ \dots, \ \psi_n : \mathbf{B}_n \to \mathbf{C}_n \rangle$ into another $\mathcal{T} = \langle \mathbf{D}, \phi_1 : \mathbf{E}_1 \to \mathbf{F}_1, \ \dots, \ \phi_n : \mathbf{E}_n \to \mathbf{F}_n \rangle$ is a function $f : \mathbf{A} \to \mathbf{D}$ embedding $\mathbf{A}$ into $\mathbf{D}$, $\mathbf{B}_i$ into $\mathbf{E}_i$, and $\mathbf{C}_i$ into $\mathbf{F}_i$ such that $f \circ \psi_i \subseteq \phi_i \circ f$ for $i = 1, \dots, n$. So we can talk about JEP, WAP, etc., for n-systems as well.

We can prove the following exactly as before.

THEOREM 2.11. *Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ its Fraïssé limit. Then the following are equivalent:*

(i) *there is a dense diagonal conjugacy class in* $\mathrm{Aut}(\mathbf{K})^n$*;*

(ii) $\mathcal{K}_p^n$ *has the* JEP.

From this we easily have the following.

THEOREM 2.12. *There is a dense diagonal conjugacy class in each of*

$$H(2^{\mathbb{N}})^{\mathbb{N}}, \ H(2^{\mathbb{N}}, \sigma)^{\mathbb{N}}, \ \mathrm{Aut}(X, \mu)^{\mathbb{N}}, \ \mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}, \ \mathrm{Aut}(\mathbf{U}_0)^{\mathbb{N}}, \ \mathrm{Iso}(\mathbf{U})^{\mathbb{N}}.$$

DEFINITION 2.13. Let $\mathcal{K}$ be a Fraïssé class and let $\mathcal{K}_p$ be the corresponding class of systems. We say that $\mathcal{K}_p$ satisfies the *cofinal joint embedding property* (CJEP) if for each $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{A} \leqslant \mathbf{B} \in \mathcal{K}$ such that for $\mathcal{J}_{\mathbf{B}} = \langle \mathbf{B}, \mathrm{id} \colon \mathbf{B} \to \mathbf{B} \rangle$, any systems $\mathcal{T}_0$ and $\mathcal{T}_1$ and embeddings $e \colon \mathcal{J}_{\mathbf{B}} \to \mathcal{T}_0$ and $f \colon \mathcal{J}_{\mathbf{B}} \to \mathcal{T}_1$, there exist a system $\mathcal{R}$ and embeddings $g \colon \mathcal{T}_0 \to \mathcal{R}$ and $h \colon \mathcal{T}_1 \to \mathcal{R}$ such that $g \circ e = h \circ f$.

This property is enough to ensure that if $\mathbf{K}$ is the Fraïssé limit of $\mathcal{K}$, then $\mathrm{Aut}(\mathbf{K})$ has a neighborhood basis at the identity consisting of clopen subgroups with a dense conjugacy class.

THEOREM 2.14. *Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbf{K}$ and suppose that $\mathcal{K}_p$ satisfies the* CJEP*. Then for any finite substructure $\mathbf{A} \subseteq \mathbf{K}$, there is a finite substructure $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{K}$ such that* $\mathrm{Aut}(\mathbf{K})_{(\mathbf{B})} = \{g \in \mathrm{Aut}(\mathbf{K}) \colon g \restriction_{\mathbf{B}} = \mathrm{id}_{\mathbf{B}}\}$ *has a dense conjugacy class.*

Proof. Let $\mathbf{A} \subseteq \mathbf{K}$ be given and find by the CJEP some $\mathbf{A} \leqslant \mathbf{B} \in \mathcal{K}$ satisfying the conditions of the definition. By ultrahomogeneity of $\mathbf{K}$, we can suppose that $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{K}$. Let $\overline{b} = \langle b_1, \dots, b_n \rangle$ be new names for the elements of $\mathbf{B}$ and let $(\mathbf{K}, \overline{b})$ be the corresponding expansion of $\mathbf{K}$. Notice that $\mathrm{Aut}(\mathbf{K}, \overline{b}) = \mathrm{Aut}(\mathbf{K})_{(\mathbf{B})}$. Similarly, we let $(\mathcal{K}, \overline{b})$ be the class of expanded structures $(\mathbf{D}, \overline{b})$, where $\mathbf{B} \leqslant \mathbf{D} \in \mathcal{K}$ and $b_1, \dots, b_n$ are the names for some fixed copy of $\mathbf{B}$ in $\mathbf{D}$. We claim that

(i) $(\mathcal{K}, \overline{b})$ is a Fraïssé class, and

(ii) $(\mathcal{K}, \overline{b})_p$ has the JEP.

First we consider (i). The HP for $(\mathcal{K}, \overline{b})$ follows at once from the HP of $\mathcal{K}$. Now, if $(\mathbf{C}, \overline{b}), (\mathbf{D}, \overline{b}), (\mathbf{E}, \overline{b}) \in (\mathcal{K}, \overline{b})$ and e and f are embeddings of $(\mathbf{C}, \overline{b})$ into $(\mathbf{D}, \overline{b})$ and $(\mathbf{C}, \overline{b})$ into $(\mathbf{E}, \overline{b})$ respectively, then e and f embed $\mathbf{C}$ into $\mathbf{D}$, and $\mathbf{C}$ into $\mathbf{E}$ respectively. So by the AP for $\mathcal{K}$ there exist a structure $\mathbf{F} \in \mathcal{K}$ and embeddings $g \colon \mathbf{D} \to \mathbf{F}$ and $h \colon \mathbf{E} \to \mathbf{F}$ such that $g \circ e = h \circ f$. In particular, $g \circ e(b_i^{\mathbf{C}}) = h \circ f(b_i^{\mathbf{C}})$ for $i = 1, \dots, n$ and so we can expand $\mathbf{F}$ by setting $b_i^{\mathbf{F}} = g \circ e(b_i^{\mathbf{C}})$. Then g and h embed $(\mathbf{D}, \overline{b})$ and $(\mathbf{E}, \overline{b})$ into $(\mathbf{F}, \overline{b})$ and $g \circ e = h \circ f$. So $(\mathcal{K}, \overline{b})$ has the AP. Finally, the JEP follows for $(\mathcal{K}, \overline{b})$ from the AP, since all structures have a common substructure, namely the one generated by $b_1, \ldots, b_n$.

Now we prove (ii). Suppose that $\mathcal{S}, \mathcal{T} \in (\mathcal{K}, \overline{b})_p$. Then for some $\mathbf{D}, \mathbf{E} \subseteq \mathbf{C}$ and $\mathbf{G}, \mathbf{H} \subseteq \mathbf{F}$,

$$\mathcal{S} = \langle (\mathbf{C}, \overline{b}), \psi \colon (\mathbf{D}, \overline{b}) \to (\mathbf{E}, \overline{b}) \rangle$$

and

$$\mathcal{T} = \langle (\mathbf{F}, \overline{b}), \phi \colon (\mathbf{G}, \overline{b}) \to (\mathbf{H}, \overline{b}) \rangle.$$

However, because ψ and ϕ both preserve $b_1, \ldots, b_n$ this means that $\mathcal{J}_{\mathbf{B}}$ embeds into both $\tilde{\mathcal{S}} = \langle \mathbf{C}, \psi \colon \mathbf{D} \to \mathbf{E} \rangle$ and $\tilde{\mathcal{T}} = \langle \mathbf{F}, \phi \colon \mathbf{G} \to \mathbf{H} \rangle$. Therefore there is an amalgamation of $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ which can be expanded to a common super-system of $\mathcal{S}$ and $\mathcal{T}$.

We now only need to show that the Fraïssé limit of $(\mathcal{K}, \overline{b})$ is $(\mathbf{K}, \overline{b})$, but for this it is enough to notice that $(\mathbf{K}, \overline{b})$ is still ultrahomogeneous as $\mathbf{K}$ is, and $\mathrm{Age}(\mathbf{K}, \overline{b}) = (\mathcal{K}, \overline{b})$ as $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$ and $\mathbf{K}$ is ultrahomogeneous. So by Fraïssé's theorem on the uniqueness of the Fraïssé limit we have the required result. For now we can just apply Theorem 2.1 to $(\mathcal{K}, \overline{b})$ with limit $(\mathbf{K}, \overline{b})$. □

In any natural instance it is certainly easy to verify the CJEP; for example, it is true in any of the cases considered in this paper. However, we have no general theorem saying that it should follow from the existence of a comeager conjugacy class, and probably it does not.

3. *Generic automorphisms*

We will now consider the question of when $\mathrm{Aut}(\mathbf{K})$ has a comeager conjugacy class, when $\mathbf{K}$ is the Fraïssé limit of some Fraïssé class $\mathcal{K}$.

Now it is known, by a theorem due to Effros, Marker and Sami (see, for example, Becker and Kechris [**5**]), that any non-meager orbit of a Polish group acting continuously on a Polish space is in fact G_δ. So we are actually looking for criteria for when there is a dense G_δ orbit.

Now we recall some basic facts about Hjorth's notion of turbulence (see Hjorth [**24**] or Kechris [**32**]).

Suppose a Polish group G acts continuously on a Polish space X. A point $x \in X$ is said to be *turbulent* if for every open neighborhood U of x and every symmetric open neighborhood V of the identity $e \in G$, the *local orbit*, $\mathcal{O}(x, U, V)$, is somewhere dense, that is,

$$\mathrm{Int}(\overline{\mathcal{O}(x, U, V)}) \neq \emptyset. \tag{2}$$

Here

$$\mathcal{O}(x, U, V) = \{y \in X : \exists g_0, \ldots, g_k \in V \ \forall i \leqslant k \\ (g_i g_{i-1} \ldots g_0 \cdot x \in U \text{ and } g_k g_{k-1} \ldots g_0 \cdot x = y)\}. \tag{3}$$

Notice that the property of being turbulent is G-invariant (see Kechris [**32**, 8.3]), so we can talk about *turbulent orbits*. We also have the following fact (see [**32**, 8.4]).

Proposition 3.1. *Let a Polish group G act continuously on a Polish space X and $x \in X$. The following are equivalent:*

(i) *x is turbulent;*

(ii) *for every open neighborhood U of x and every symmetric open neighborhood V of $e \in G$, $x \in \mathrm{Int}(\overline{\mathcal{O}(x, U, V)})$.*

Notice that if V is an open subgroup of G, then $\mathcal{O}(x, U, V) = U \cap V \cdot x$. So if G is a closed subgroup of S_∞, then, as G has an open neighborhood basis at the identity consisting of open subgroups, we see that $x \in X$ is turbulent if and only if $x \in \mathrm{Int}(\overline{V \cdot x})$ for all open subgroups $V \leqslant G$. Therefore, the following proposition provides equivalent conditions for turbulence.

PROPOSITION 3.2. *Let G be a closed subgroup of S_∞ acting continuously on a Polish space X and let $x \in X$. Then the following are equivalent:*

(i) *the orbit $G \cdot x$ is non-meager;*
(ii) *for each open subgroup $V \leqslant G, V \cdot x$ is non-meager;*
(iii) *for each open subgroup $V \leqslant G, V \cdot x$ is somewhere dense;*
(iv) *for each open subgroup $V \leqslant G, x \in \mathrm{Int}(\overline{V \cdot x})$;*
(v) *the point x is turbulent.*

Proof. (i) $\Rightarrow$ (ii) Suppose $G \cdot x$ is non-meager and $V \leqslant G$ is an open subgroup of G. Then we can find $g_n \in G$ such that $G = \bigcup g_n V$, so some $g_n V \cdot x$ is non-meager and therefore $V \cdot x$ is non-meager.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (iv) Suppose V is an open subgroup of G such that $V \cdot x$ is dense in some open set $U \neq \emptyset$. Take $g \in V$ such that $g \cdot x \in U$. Then $g^{-1} V \cdot x = V \cdot x$ is dense in $g^{-1} \cdot U \ni x$ and $x \in \mathrm{Int}(\overline{V \cdot x})$.

(iv) $\Rightarrow$ (i) Suppose $F_n \subseteq X$ are closed nowhere dense such that $G \cdot x \subseteq \bigcup_n F_n$. Then $K_n = \{g \in G : g \cdot x \in F_n\}$ are closed and, as they cover G, some K_n has non-empty interior. So suppose $gV \subseteq K_n$ for some open subgroup $V \leqslant G$. Then $gV \cdot x \subseteq F_n$ and both $gV \cdot x$ and $V \cdot x$ are nowhere dense.

The equivalence of (iv) and (v) is clear from the remarks following Proposition 3.1. $\square$

It was asked by Truss in [**47**] whether the existence of a generic automorphism of the limit of a Fraïssé class $\mathcal{K}$ could be expressed in terms of AP and JEP for $\mathcal{K}_p$. He gave a partial answer to this showing that the existence of a generic automorphism implied JEP for $\mathcal{K}_p$, and, moreover, if $\mathcal{K}_p$ satisfies JEP and the so-called *cofinal* AP (CAP), then there is indeed a generic automorphism. Here we say that $\mathcal{K}_p$ satisfies the CAP if it has a cofinal subclass, under embeddability, that has the AP.

This is clearly equivalent to saying that there is a subclass $\mathcal{L} \subseteq \mathcal{K}_p$, cofinal under embeddability, such that for any $\mathcal{S} \in \mathcal{L}$ and $\mathcal{T}, \mathcal{R} \in \mathcal{K}_p$ and embeddings $f : \mathcal{S} \to \mathcal{T}$ and $g : \mathcal{S} \to \mathcal{R}$, there exist $\mathcal{H} \in \mathcal{K}_p$ and embeddings $h : \mathcal{T} \to \mathcal{H}$ and $i : \mathcal{R} \to \mathcal{H}$ such that $h \circ f = i \circ g$. We mention that for any Fraïssé class $\mathcal{K}$, the class $\mathcal{L}$ of $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \in \mathcal{K}_p$, such that $\mathbf{A}$ is generated by $\mathbf{B}$ and $\mathbf{C}$, is cofinal under embeddability. This follows from the ultrahomogeneity of the Fraïssé limit of $\mathcal{K}$.

DEFINITION 3.3. A class $\mathcal{K}$ of finite structures satisfies the *weak amalgamation property* (WAP) if for every $\mathbf{A} \in \mathcal{K}$ there are a $\mathbf{B} \in \mathcal{K}$ and an embedding $e : \mathbf{A} \to \mathbf{B}$ such that for all $\mathbf{C}, \mathbf{D} \in \mathcal{K}$ and embeddings $i : \mathbf{B} \to \mathbf{C}$ and $j : \mathbf{B} \to \mathbf{D}$ there exist an $\mathbf{E} \in \mathcal{K}$ and embeddings $l : \mathbf{C} \to \mathbf{E}$ and $k : \mathbf{D} \to \mathbf{E}$ amalgamating $\mathbf{C}$ and $\mathbf{D}$ over $\mathbf{A}$, that is, such that $l \circ i \circ e = k \circ j \circ e$.

A similar definition applies to the class $\mathcal{K}_p$.

We can now use turbulence concepts to provide the following answer to Truss' question. We have recently found out that Ivanov [**29**] has also proved a similar result, by other techniques, in a somewhat different context, namely that of $\aleph_0$-categorical structures.

THEOREM 3.4. *Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ its Fraïssé limit. Then the following are equivalent:*

(i) $\mathbf{K}$ *has a generic automorphism;*
(ii) $\mathcal{K}_p$ *satisfies the* WAP *and the* JEP.

Proof. (i) $\Rightarrow$ (ii) We know that if $\mathbf{K}$ has a generic automorphism, then some $f \in \mathrm{Aut}(\mathbf{K})$ is turbulent and has dense conjugacy class. So by Theorem 1.1, $\mathcal{K}_p$ has the JEP.

Given now $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \in \mathcal{K}_p$, we can assume that $\mathbf{A} \subseteq \mathbf{K}$. Moreover, as $[\psi : \mathbf{B} \to \mathbf{C}]$ is open non-empty and f has dense conjugacy class, we can suppose that $\psi \subseteq f$. Let $V_{\mathbf{A}} = [\mathrm{id} : \mathbf{A} \to \mathbf{A}]$, which is a clopen subgroup of $\mathrm{Aut}(\mathbf{K})$. By the turbulence of f, we know that $c(V_{\mathbf{A}}, f) = \{g^{-1}fg : g \in V_{\mathbf{A}}\}$ is dense in some open neighborhood $[\phi : \mathbf{E} \to \mathbf{F}]$ of f.

Now, since $f \supseteq \psi$, we can suppose that $\psi \subseteq \phi$. Let $\mathbf{D}$ be the substructure of $\mathbf{K}$ generated by $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{F}$, and put $\mathcal{T} = \langle \mathbf{D}, \phi : \mathbf{E} \to \mathbf{F} \rangle$. So $\mathcal{S}$ is a subsystem of $\mathcal{T}$ and denote by e the inclusion mapping of $\mathcal{S}$ into $\mathcal{T}$.

Assume now that $i : \mathcal{T} \to \mathcal{F} = \langle \mathbf{H}, \chi : \mathbf{M} \to \mathbf{N} \rangle$ and $j : \mathcal{T} \to \mathcal{G} = \langle \mathbf{P}, \xi : \mathbf{Q} \to \mathbf{R} \rangle$ are embeddings. Then we wish to amalgamate $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{S}$.

By using the extension property of $\mathbf{K}$ we can actually assume that $\mathbf{H}$ and $\mathbf{P}$ are substructures of $\mathbf{K}$ and $\mathbf{D} \subseteq \mathbf{H}, \mathbf{P}$, with χ and ξ extending ϕ, and i and j the inclusion mappings. By the density of $c(V_{\mathbf{A}}, f)$ in $[\phi : \mathbf{E} \to \mathbf{F}]$ there are $g, k \in V_{\mathbf{A}} = [\mathrm{id} : \mathbf{A} \to \mathbf{A}]$ such that $g^{-1}fg \supseteq \chi$ and $h^{-1}fh \supseteq \xi$. Let $\mathbf{S}$ be the substructure of $\mathbf{K}$ generated by $g''\mathbf{H}$ and $h''\mathbf{P}$, let $\mathbf{T}$ be the substructure generated by $g''\mathbf{M}$ and $h''\mathbf{Q}$, and let $\mathbf{U}$ be the substructure generated by $g''\mathbf{N}$ and $h''\mathbf{R}$. Finally, put $\theta = f \restriction_{\mathbf{T}} : \mathbf{T} \to \mathbf{U}$ and $\mathcal{E} = \langle \mathbf{S}, \theta : \mathbf{T} \to \mathbf{U} \rangle$.

As f restricts to an isomorphism of $g''\mathbf{M}$ with $g''\mathbf{N}$ and an isomorphism of $h''\mathbf{Q}$ with $h''\mathbf{R}$, it also restricts to an isomorphism of $\mathbf{T}$ with $\mathbf{U}$. So θ is well defined. Moreover, g and h obviously embed $\mathcal{F}$ and $\mathcal{G}$ into $\mathcal{E}$, as $g^{-1}fg \supseteq \chi$ and $h^{-1}fh \supseteq \xi$. And finally, as $g, h \in [\mathrm{id} : \mathbf{A} \to \mathbf{A}]$, we have $g \circ i \circ e = h \circ j \circ e = \mathrm{id}_{\mathbf{A}}$. So $\mathcal{E}$ is indeed an amalgam over $\mathcal{S}$. Therefore $\mathcal{K}_p$ satisfies WAP.

(ii) $\Rightarrow$ (i) Now suppose $\mathcal{K}_p$ satisfies WAP and JEP. We will construct an $f \in \mathrm{Aut}(\mathbf{K})$ with turbulent and dense conjugacy class. Notice that this will be enough to insure that the conjugacy class of f is comeager. For as the conjugacy action is continuous, if the orbit of f is non-meager, then it will be comeager in its closure, that is, comeager in the whole group.

As before for $\psi : \mathbf{B} \to \mathbf{C}$, an isomorphism between finite substructures of $\mathbf{K}$, we let

$$D(\psi : \mathbf{B} \to \mathbf{C}) = \{f \in \mathrm{Aut}(\mathbf{K}) : \exists g \in \mathrm{Aut}(\mathbf{K})(g^{-1}fg \supseteq \psi)\}. \tag{4}$$

For $\mathbf{E}$ a finite substructure of $\mathbf{K}$, we let $V_{\mathbf{E}} = [\mathrm{id} : \mathbf{E} \to \mathbf{E}]$, which is a clopen subgroup of $\mathrm{Aut}(\mathbf{K})$.

We have seen in the proof of Theorem 2.1 that the JEP for $\mathcal{K}_p$ implies that each $D(\psi : \mathbf{B} \to \mathbf{C})$ is open dense in $\mathrm{Aut}(\mathbf{K})$. And moreover, any element in their intersection has dense conjugacy class.

Now for $\psi : \mathbf{B} \to \mathbf{C}$ let $\mathbf{A}$ be the substructure generated by $\mathbf{B}$ and $\mathbf{C}$ and let $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle$. Then by WAP for $\mathcal{K}_p$ and the extension property of $\mathbf{K}$, there is an extension $\hat{\mathcal{S}} = \langle \hat{\mathbf{A}}, \hat{\psi} : \hat{\mathbf{B}} \to \hat{\mathbf{C}} \rangle$ of $\mathcal{S}$ such that any two extensions of $\hat{\mathcal{S}}$ can be amalgamated over $\mathcal{S}$. By extending $\hat{\mathcal{S}}$, we can actually assume that $\hat{\mathbf{A}}$ is generated by $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$. Enumerate all such $\hat{\psi}$ as $\hat{\psi}_1, \hat{\psi}_2, \ldots$. Moreover, list all the possible finite extensions $\theta : \mathbf{M} \to \mathbf{N}$ of $\hat{\psi}_m : \hat{\mathbf{B}} \to \hat{\mathbf{C}}$ as $\theta_1^{\hat{\psi}_m}, \theta_2^{\hat{\psi}_m}, \ldots$. Define

$$E(\psi : \mathbf{B} \to \mathbf{C}) = \{f \in \mathrm{Aut}(\mathbf{K}) : f \supseteq \psi \Rightarrow \text{(for some } \phi \supseteq \psi \text{ and some } m \text{ we have } f \supseteq \hat{\phi}_m)\} \tag{5}$$

and

$$F_{m,n}(\psi : \mathbf{B} \to \mathbf{C}) = \{f \in \mathrm{Aut}(\mathbf{K}) : f \supseteq \hat{\psi}_m \Rightarrow (c(V_{\mathbf{B}}, f) \cap [\theta_n^{\hat{\psi}_m}] \neq \emptyset)\}. \tag{6}$$

Now obviously both $E(\psi : \mathbf{B} \to \mathbf{C})$ and $F_{m,n}(\psi : \mathbf{B} \to \mathbf{C})$ are open and $E(\psi : \mathbf{B} \to \mathbf{C})$ is dense.

Lemma 3.5. *Suppose that $f \in \mathrm{Aut}(\mathbf{K})$ is in $E(\psi : \mathbf{B} \to \mathbf{C})$ and $F_{m,n}(\psi : \mathbf{B} \to \mathbf{C})$ for all $\psi : \mathbf{B} \to \mathbf{C}$ and $m, n \in \mathbb{N}$. Then f is turbulent.*

Proof. Let a clopen subgroup $V_{\mathbf{E}} \leqslant \mathrm{Aut}(\mathbf{K})$ be given. We shall show that $c(V_{\mathbf{E}}, f)$ is somewhere dense.

As $f \in E(f\restriction_{\mathbf{E}} : \mathbf{E} \to f''\mathbf{E})$, there are some $\phi : \mathbf{B} \to \mathbf{C}, \mathbf{E} \subseteq \mathbf{B}$ and m such that $f \supseteq \hat{\phi}_m$. We claim that $c(V_{\mathbf{B}}, f)$ and *a fortiori* $c(V_{\mathbf{E}}, f)$ is dense in $[\hat{\phi}_m : \hat{\mathbf{B}} \to \hat{\mathbf{C}}]$. This is because any basic open subset of $[\hat{\phi}_m : \hat{\mathbf{B}} \to \hat{\mathbf{C}}]$ is of the form $[\theta_n^{\hat{\phi}_m}]$ for some $n \in \mathbb{N}$ and we know that $c(V_{\mathbf{B}}, f) \cap [\theta_n^{\hat{\phi}_m}] \neq \emptyset$. □

So now we only need to show that each $F_{m,n}(\psi : \mathbf{B} \to \mathbf{C})$ is dense, as any element in the intersection of the sets D, E and $F_{m,n}$ will do.

Given a basic open set $[\phi : \mathbf{A} \to \mathbf{D}]$, where we can suppose that $\phi \supseteq \hat{\psi}_m$, we need to show that for some $f \supseteq \phi$, $c(V_{\mathbf{B}}, f) \cap [\theta_n^{\hat{\psi}_m}] \neq \emptyset$. Now $\theta_n^{\hat{\psi}_m}$ and ϕ are both extensions of $\hat{\psi}_m$, so by WAP for $\mathcal{K}_p$, they can be amalgamated over ψ. It follows by the extension property of $\mathbf{K}$ that there is some $g \in \mathrm{Aut}(\mathbf{K})$ fixing $\mathbf{B}$ such that $g^{-1}\phi g$ and $\theta_n^{\hat{\psi}_m}$ are compatible in $\mathrm{Aut}(\mathbf{K})$, that is, for some $f \supseteq \phi$ we have $g^{-1} f g \in [\theta_n^{\hat{\psi}_m}]$.

This finishes the proof. □

Remark. It is easy to verify that the intersection of the sets D, E and $F_{m,n}$ is conjugacy invariant, so actually this intersection is exactly equal to the set of generic automorphisms.

As, obviously, the CAP implies the WAP, we have a corollary.

Corollary 3.6 (Truss [**47**]). *If $\mathcal{K}_p$ has the cofinal amalgamation property, then there is a generic automorphism of $\mathbf{K}$.*

Remark. In [**25**], Hodges developed the Fraïssé theory in the context of a class $\mathcal{K}$ of *finitely generated* (but not necessarily finite) structures satisfying HP, JEP and AP. It is not hard to see that all the arguments and results of §§ 2 and 3 carry over without difficulty to this more general context.

Truss [**47**] calls an automorphism $f \in \mathrm{Aut}(\mathbf{K})$, where $\mathbf{K}$ is the Fraïssé limit of $\mathcal{K}$ and $\mathcal{K}$ is a Fraïssé class, *locally generic* if its conjugacy class is non-meager.

Given a Fraïssé class $\mathcal{K}$, let us say that $\mathcal{K}_p$ satisfies the *local* WAP if there exists $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \in \mathcal{K}_p$ such that WAP holds for the subclass $\mathcal{L}_p$ of $\mathcal{K}_p$ consisting of all $\mathcal{T} \in \mathcal{K}_p$ into which $\mathcal{S}$ embeds.

Using Proposition 3.2 again, we have the following result.

Theorem 3.7. *Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ be its Fraïssé limit. Then the following are equivalent:*

(i) $\mathbf{K}$ *admits a locally generic automorphism;*
(ii) $\mathcal{K}_p$ *satisfies the local* WAP.

Proof. (i) ⇒ (ii) Suppose that $\mathbf{K}$ admits a locally generic automorphism f. Then by Proposition 3.2, the conjugacy class of f is turbulent and of course dense in some open set $U \subseteq \mathrm{Aut}(\mathbf{K})$. So there is some $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \in \mathcal{K}_p$ such that $U \supseteq [\psi : \mathbf{B} \to \mathbf{C}]$. Then, as in the proof of (i) ⇒ (ii) in Theorem 3.4, we can see that WAP holds for all $\mathcal{T} \in \mathcal{K}_p$ into which $\mathcal{S}$ can be embedded.

(ii) $\Rightarrow$ (i) Suppose that $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \to \mathbf{C} \rangle \in \mathcal{K}_p$ witnesses that $\mathcal{K}_p$ has the local WAP. Then we repeat the construction of (ii) $\Rightarrow$ (i) in the proof of Theorem 3.4, by taking as our first approximation ψ (and without of course using the sets $D(\phi : \mathbf{D} \to \mathbf{E})$ whose density was ensured by JEP). $\square$

Similarly one can see that the existence of a conjugacy class that is somewhere dense is equivalent to a local form of JEP defined in an analogous way.

We can also characterize WAP in terms of local genericity.

THEOREM 3.8. *Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ be its Fraïssé limit. Then the following are equivalent:*

(i) *$\mathbf{K}$ admits a dense set of locally generic automorphisms;*
(ii) *$\mathbf{K}$ admits a comeager set of locally generic automorphisms;*
(iii) *$\mathcal{K}_p$ satisfies the* WAP.

Proof. The implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are proved as in Theorems 3.4 and 3.7, beginning with the construction of a turbulent point from any given $\psi : \mathbf{B} \to \mathbf{C}$.

That (ii) $\Rightarrow$ (i) is of course trivial, and (i) $\Rightarrow$ (ii) follows from the fact that if an orbit is non-meager then it is comeager in its closure. $\square$

THEOREM 3.9. *Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbf{K}$ and suppose that $\mathcal{K}_p$ satisfies the* WAP *and the* CJEP. *Then* $\mathrm{Aut}(\mathbf{K})$ *has a neighborhood basis at the identity consisting of clopen subgroups each having a comeager conjugacy class.*

Proof. By Theorem 3.8, $\mathrm{Aut}(\mathbf{K})$ has a dense set of non-meager conjugacy classes. Moreover, by Theorem 2.14, $\mathrm{Aut}(\mathbf{K})$ has a neighborhood basis at the identity consisting of clopen subgroups with dense conjugacy classes. Now assume that G is a clopen subgroup of $\mathrm{Aut}(\mathbf{K})$ with a dense conjugacy class and that $g \in G$ has a non-meager conjugacy class in $\mathrm{Aut}(\mathbf{K})$. Then by Proposition 3.2(ii), g has a non-meager conjugacy class in G; but as the set of group elements with dense conjugacy classes forms a G_δ set, which, when non-empty, is dense, the conjugacy class of g intersects this set. So the conjugacy class of g is both dense and non-meager in G, that is, comeager in G. $\square$

REMARK. As is easily checked, all the structures considered in this paper that are shown to have a comeager conjugacy class actually also satisfy the above theorem. Let us just mention that for the class of ω-stable, $\aleph_0$-categorical structures one just needs to notice that they stay ω-stable, $\aleph_0$-categorical after having been expanded by a finite number of constants (see Hodges [**25**]).

4. *Normal form for isomorphisms of finite subalgebras of* $\mathrm{clop}(2^{\mathbb{N}})$

In this section we will develop some facts needed in the proof of the existence of generic homeomorphisms of $2^{\mathbb{N}}$, which will be given in the next section.

Recall that the Fraïssé limit of the class of finite boolean algebras is the countable atomless boolean algebra $\mathbf{B}_\infty$, which can be concretely realized as $\mathrm{clop}(2^{\mathbb{N}})$, the boolean algebra of clopen subsets of Cantor space $2^{\mathbb{N}}$.

We will eventually prove that the class $\mathcal{K}_p$, where $\mathcal{K}$ is the class of finite boolean algebras, has the CAP. Thus, we will first have to describe the cofinal class $\mathcal{L}$ of $\mathcal{K}_p$ over which we can amalgamate. First of all, it is clear that if $\phi \colon \mathbf{B}_\infty \to \mathbf{B}_\infty$ is such that for some $a \in \mathbf{B}_\infty$ we have $\phi(a) < a$, then the orbit of a under ϕ is infinite. This shows that there are partial automorphisms of finite boolean algebras that cannot be extended to full automorphisms of bigger finite boolean algebras. This fact makes the situation a lot messier than in the case of measured boolean algebras and, in order to prove the amalgamation property, requires us to be able to describe the local structure of a partial automorphism of a finite boolean algebra.

Definition 4.1. Let $\mathbf{B}$ and $\mathbf{C}$ be finite subalgebras of $\mathrm{clop}(2^{\mathbb{N}})$ and let $\psi : \mathbf{B} \to \mathbf{C}$ be an isomorphism. A *refinement* of $\psi : \mathbf{B} \to \mathbf{C}$ consists of finite superalgebras $\mathbf{B} \subseteq \mathbf{B}'$ and $\mathbf{C} \subseteq \mathbf{C}'$ and an isomorphism $\psi' : \mathbf{B}' \to \mathbf{C}'$ such that $\psi' \restriction_{\mathbf{B}} = \psi$.

Fix an isomorphism $\psi : \mathbf{B} \to \mathbf{C}$ of finite subalgebras of $\mathrm{clop}(2^{\mathbb{N}})$, and let $\mathbf{B} \vee \mathbf{C}$ be the algebra generated by $\mathbf{B}$ and $\mathbf{C}$.

Definition 4.2. A *cyclic chain* is a sequence $a_1, \dots, a_n$ of distinct atoms of $\mathbf{B} \vee \mathbf{C}$ belonging to both $\mathbf{B}$ and $\mathbf{C}$ such that $\psi(a_1) = a_2,\ \dots,\ \psi(a_{n-1}) = a_n$, $\psi(a_n) = a_1$:

$$\begin{array}{cccccccc} \mathbf{B} & & a_1 & a_2 & & a_{n-1} & a_n \\ & \psi & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \mathbf{C} & & a_2 & a_3 & & a_n & a_1 \end{array} \tag{7}$$

Definition 4.3. A *stable chain* is a sequence $a_0, \dots, a_n$ of distinct atoms of $\mathbf{B} \vee \mathbf{C}$ plus an element c, which we call its *end*, such that one of the following two situations occurs.

(I) (1) $a_0, \dots, a_n$ belong to $\mathbf{B}$,
 (2) $a_1, \dots, a_n, c$ belong to $\mathbf{C}$,
 (3) $\psi(a_0) = a_1,\ \dots,\ \psi(a_{n-1}) = a_n$,
 (4) $c = \psi(a_n) = a_0 \vee b_1 \vee \ldots \vee b_k = a_0 \vee x$, where $b_1, \dots, b_k \neq a_0$ are atoms of $\mathbf{B} \vee \mathbf{C}$ and $k \geqslant 1$.

Diagramatically,

$$\begin{array}{cccccccc} \mathbf{B} & & a_0 & a_1 & & a_{n-1} & a_n \\ & \psi & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \mathbf{C} & & a_1 & a_2 & & a_n & a_0 \vee x \end{array} \tag{8}$$

where $x = b_1 \vee \ldots \vee b_k \neq 0$.

(II) (1) $a_0, \dots, a_n$ belong to $\mathbf{C}$,
 (2) $a_1, \dots, a_n, c$ belong to $\mathbf{B}$,
 (3) $\psi^{-1}(a_0) = a_1,\ \dots,\ \psi^{-1}(a_{n-1}) = a_n$,
 (4) $c = \psi^{-1}(a_n) = a_0 \vee b_1 \vee \ldots \vee b_k = a_0 \vee x$, where $b_1, \dots, b_k \neq a_0$ are atoms of $\mathbf{B} \vee \mathbf{C}$ and $k \geqslant 1$.

Diagramatically,

$$\begin{array}{cccccccc} \mathbf{B} & & a_1 & a_2 & & a_n & a_0 \vee x \\ & \psi & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \mathbf{C} & & a_0 & a_1 & & a_{n-1} & a_n \end{array} \tag{9}$$

where $x = b_1 \vee \ldots \vee b_k \neq 0$.

For a stable chain as above we say that an atom b in $\mathbf{B} \vee \mathbf{C}$ such that $b \neq a_0$ and $b < \psi(a_n)$ (respectively, $b < \psi^{-1}(a_n)$) is *free*. These are the elements $b_1, \dots, b_k$ above. Moreover, the atom a_0 is said to be the *beginning* of the stable chain.

Definition 4.4. A *linking chain* is a sequence $a_1, \dots, a_n$ of distinct atoms of $\mathbf{B} \vee \mathbf{C}$, such that

(1) $a_1, \dots, a_{n-1}$ belong to $\mathbf{B}$,
(2) $a_2, \dots, a_n$ belong to $\mathbf{C}$,
(3) $\psi(a_1) = a_2,\ \dots,\ \psi(a_{n-1}) = a_n$,

(4) a_1 and a_n are free in some stable chains.

Diagramatically,

$$\begin{array}{cccccccc}
\mathbf{B} & & & a_1 & a_2 & & a_{n-1} & a_n \vee y \\
& \psi & & \downarrow & \downarrow & \dots & \downarrow & \\
\mathbf{C} & & a_1 \vee z & a_2 & a_3 & & a_n &
\end{array} \tag{10}$$

DEFINITION 4.5. An isomorphism $\psi : \mathbf{B} \to \mathbf{C}$ is said to be *normal* if and only if

(i) every atom of $\mathbf{B}$ or of $\mathbf{C}$ that is not an atom in $\mathbf{B} \vee \mathbf{C}$ is the end of a stable chain,

(ii) every atom of $\mathbf{B} \vee \mathbf{C}$ is a term in either a stable, linking or cyclic chain, or is free in some stable chain.

For any finite subalgebras $\mathbf{B}$ and $\mathbf{C}$ of a common algebra, we let $n(\mathbf{B}, \mathbf{C})$ be the number of atoms in $\mathbf{B}$ that are not atoms in $\mathbf{B} \vee \mathbf{C}$ plus the number of atoms in $\mathbf{C}$ that are not atoms in $\mathbf{B} \vee \mathbf{C}$.

LEMMA 4.6. *For any isomorphism $\psi : \mathbf{B} \to \mathbf{C}$ between finite subalgebras $\mathbf{B}, \mathbf{C} \subseteq \mathrm{clop}(2^{\mathbb{N}})$, there is a refinement $\psi' : \mathbf{B}' \to \mathbf{C}'$ satisfying condition* (i) *of normality.*

Proof. The proof is by induction on $n(\mathbf{B}, \mathbf{C})$. For the basis of the induction, if $n(\mathbf{B}, \mathbf{C}) = 0$, then every atom of $\mathbf{B}$ and every atom of $\mathbf{C}$ is an atom of $\mathbf{B} \vee \mathbf{C}$ and there is nothing to prove. In this case, we see that the structure of ψ is particularly simple, since it is then just an automorphism of $\mathbf{B} = \mathbf{C} = \mathbf{B} \vee \mathbf{C}$ and thus one can easily split $\mathbf{B} \vee \mathbf{C}$ into cyclic chains, namely the ψ-orbits of atoms.

Now for the induction step, suppose $x \in \mathbf{B}$ is some atom of $\mathbf{B}$ that is not an atom of $\mathbf{B} \vee \mathbf{C}$ (the case when x is an atom of $\mathbf{C}$ that is not an atom of $\mathbf{B} \vee \mathbf{C}$ is of course symmetric), and trace the longest chain of atoms of $\mathbf{B} \vee \mathbf{C}$, $a_i \in \mathbf{B}$, $b_i \in \mathbf{C}$, such that

$$x \overset{\psi}{\longmapsto} b_0 = a_0 \overset{\psi}{\longmapsto} b_1 = a_1 \overset{\psi}{\longmapsto} b_2 = a_2 \overset{\psi}{\longmapsto} \cdots . \tag{11}$$

If for some $i < j$, $a_i = a_j$, then also $b_i = b_j$, and by the injectivity of ψ, $a_{i-1} = a_{j-1}$, etc. So in that case, by induction, we have $b_0 = b_{j-i}$, whence $a_{j-i-1} = x$, contradicting the fact that x was not an atom in $\mathbf{B} \vee \mathbf{C}$.

So as $\mathbf{B} \vee \mathbf{C}$ is finite, the chain has to stop either

(i) on some a_n or

(ii) on some b_n

(if it stops with x we let $x = a_{-1}$ and treat it as in case (i)).

Case (i). Let $y = \psi(a_n)$, which is an atom in $\mathbf{C}$ but not in $\mathbf{B} \vee \mathbf{C}$. Now, because $\bigvee_{i=1}^n a_i = \bigvee_{i=1}^n b_i$, both x and y are disjoint from $\bigvee_{i=0}^n a_i$; let $x = c_1 \vee \ldots \vee c_p$ and $y = d_1 \vee \ldots \vee d_q$ be the decompositions into atoms of $\mathbf{B} \vee \mathbf{C}$. Suppose that $p \leqslant q$ (the case when $q \leqslant p$ is similar). Split $a_i = b_i$ into non-zero elements $a_i^1, \ldots, a_i^p$ of $\mathrm{clop}(2^{\mathbb{N}})$ for each $i \leqslant n$. Then we let $\mathbf{B}'$ be the smallest algebra containing $\mathbf{B}$ and new atoms a_i^l, for $l \leqslant p$ and $i \leqslant n$, and $c_1, \ldots, c_p$. Also let $\mathbf{C}'$ be the smallest algebra containing $\mathbf{C}$ and new elements a_i^l, for $l \leqslant p$ and $i \leqslant n$, and $d_1, d_2, \ldots, d_{p-1}, d_p \vee \ldots \vee d_q$. Finally, let ψ' be the unique extension of ψ satisfying $\psi'(c_l) = a_0^l$, $\psi'(a_i^l) = a_{i+1}^l$ for $i < n$ and $l \leqslant p$, $\psi'(a_n^l) = d_l$ for $l < p$, and $\psi'(a_n^p) = d_p \vee \ldots \vee d_q$. We remark that $n(\mathbf{B}', \mathbf{C}') < n(\mathbf{B}, \mathbf{C})$.

Case (ii). Notice that $x \wedge \bigvee_{i=1}^{n-1} b_i = x \wedge \bigvee_{i=1}^{n-1} a_i = 0$, and, as b_n is an atom of $\mathbf{B} \vee \mathbf{C}$, either $b_n < x$ or $b_n \wedge x = 0$. If $b_n < x$, then x is the end of a stable chain. If $b_n \wedge x = 0$, then we proceed as follows.

Suppose $x = c_1 \vee \ldots \vee c_p$ is the decomposition into atoms of $\mathbf{B} \vee \mathbf{C}$ and split b_i $(i \leqslant n)$ into $b_i^1, \ldots, b_i^p$ in $\mathrm{clop}(2^{\mathbb{N}})$. Now let $\mathbf{B}'$ be the algebra generated by $\mathbf{B}$ and $b_i^1, \ldots, b_i^p$, for $i < n$, and $c_1, \ldots, c_p$. Let $\mathbf{C}'$ be the algebra generated by $\mathbf{C}$ and $b_i^1, \ldots, b_i^p$ for $i \leqslant n$, and let ψ' be the unique extension of ψ satisfying $\psi'(c_l) = b_0^l$ ($l \leqslant p$), and $\psi(b_i^l) = b_{i+1}^l$ ($l \leqslant p$). Again we remark that $n(\mathbf{B}', \mathbf{C}') < n(\mathbf{B}, \mathbf{C})$. So this concludes the induction step. □

PROPOSITION 4.7. *Any isomorphism* $\psi : \mathbf{B} \to \mathbf{C}$ *between finite subalgebras* $\mathbf{B}, \mathbf{C} \subseteq \mathrm{clop}(2^{\mathbb{N}})$ *has a finite normal refinement.*

Proof. By the lemma we can suppose that $\psi : \mathbf{B} \to \mathbf{C}$ satisfies condition (i) of normality. We will then see that it actually satisfies condition (ii) too.

So consider an atom a_0 of $\mathbf{B} \vee \mathbf{C}$. Find an atom c of $\mathbf{B}$ such that $a_0 \leqslant c$. Now if $a_0 < c$, then we know from condition (i) that c is the end of a stable chain and a_0 is therefore either the beginning of a stable chain or is free. Otherwise, if $a_0 = c \in \mathbf{B}$, then we find the maximal chain of atoms of $\mathbf{B} \vee \mathbf{C}$, $a_i \in \mathbf{B}$, $b_i \in \mathbf{C}$ such that

$$\cdots \overset{\psi}{\longmapsto} b_i = a_i \overset{\psi}{\longmapsto} b_{i+1} = a_{i+1} \overset{\psi}{\longmapsto} b_{i+2} = a_{i+2} \overset{\psi}{\longmapsto} \cdots, \tag{12}$$

where the indices run over an interval of $\mathbb{Z}$ containing 0. We have now various cases.

Case (i): $a_i = a_j$ *for some* $i < j$. Then obviously $b_{i+1} = b_{j+1}$ and $b_{i-1} = b_{j-1}$ by the injectivity of ψ (notice also that these elements are indeed defined, that is, are atoms in $\mathbf{B} \vee \mathbf{C}$ and not only in $\mathbf{C}$). However, then also $a_{i+1} = a_{j+1}$ and $a_{i-1} = a_{j-1}$, etc. So the chain is bi-infinite and periodic and we can write it as

$$\begin{array}{ccccccc} \mathbf{B} & & a_0 & a_1 & & a_n \\ & \psi & \downarrow & \downarrow & \cdots & \downarrow \\ \mathbf{C} & & a_1 & a_2 & & a_0 \end{array} \tag{13}$$

So a_0 is a term in a cyclic chain.

Case (ii): *the chain ends with some* a_k. Then $c = \psi(a_k)$ is an atom of $\mathbf{C}$ that is not an atom of $\mathbf{B} \vee \mathbf{C}$ and must therefore be the end of some stable chain. So using the bijectivity of ψ one sees that all the terms of the above chain are terms in a stable chain and, in particular, a_0 is a term in a stable chain.

Case (iii): *the chain begins with some* b_k. Let $c = \psi^{-1}(b_k)$. Then c must be some atom of $\mathbf{B}$ that is not an atom of $\mathbf{B} \vee \mathbf{C}$, so it must be the end of some stable chain. Now looking at ψ^{-1} instead of ψ and switching the roles of $\mathbf{B}$ and $\mathbf{C}$, we see as before that a_0 is a term in some stable chain.

Case (iv): *the chain begins with some* a_n *and ends with some* b_k. Then $a_n \neq b_k$ and there must be atoms c of $\mathbf{C}$ and d of $\mathbf{B}$ that are ends of stable chains such that $a_n \wedge c \neq 0$ and $b_k \wedge d \neq 0$, but, as a_n and b_k are atoms, this means that $a_n < c$ and $b_k < d$, so the chain is linking. □

5. *Generic homeomorphisms of Cantor space*

A *generic homeomorphism* of the Cantor space $2^{\mathbb{N}}$ is a homeomorphism whose conjugacy class in $H(2^{\mathbb{N}})$ is dense G_δ. As $H(2^{\mathbb{N}})$ (with the uniform topology) is isomorphic as a topological group to $\mathrm{Aut}(\mathbf{B}_\infty)$, where $\mathbf{B}_\infty$ is the countable atomless boolean algebra, the existence of a generic homeomorphism of $2^{\mathbb{N}}$ is equivalent to the existence of a generic automorphism of $\mathbf{B}_\infty$.

THEOREM 5.1. *Let $\mathcal{K}$ be the class of finite boolean algebras. Then $\mathcal{K}_p$ has the* CAP, *and therefore there exist a generic automorphism of the countable atomless boolean algebra and a generic homeomorphism of the Cantor space.*

Before we embark on the proof, let us first mention that one can construct counter-examples to $\mathcal{K}_p$ having the AP, and that therefore the added complications are necessary.

Proof. We will show that the class $\mathcal{L}$ of $\mathcal{S} = \langle \mathbf{B} \vee \mathbf{C}, \psi : \mathbf{B} \to \mathbf{C}\rangle$, where ψ is normal, has the AP. Notice first that $\mathcal{L}$ is cofinal in $\mathcal{K}_p$ by Proposition 4.7.

Suppose now that $\mathcal{S} = \langle \mathbf{A}, \psi \colon \mathbf{B} \to \mathbf{C}\rangle \in \mathcal{L}$. Let

$$\mathcal{S}^l = \langle \mathbf{A}^l, \psi^l \colon \mathbf{B}^l \to \mathbf{C}^l\rangle \quad \text{and} \quad \mathcal{S}^r = \langle \mathbf{A}^r, \psi^r : \mathbf{B}^r \to \mathbf{C}^r\rangle$$

be two refinements of $\mathcal{S}$, which we do not necessarily demand belong to $\mathcal{L}$.

We claim that they can be amalgamated over $\mathcal{S}$. We remark first that it is, trivially, enough to amalgamate any two refinements of $\mathcal{S}^l$ and $\mathcal{S}^r$ over $\mathcal{S}$. List the atoms of $\mathbf{A}$ as $a_1, \dots, a_n$. Then by refining $\mathcal{S}^l$ and $\mathcal{S}^r$ we can suppose they have atoms

$$a^l_1(1), \dots, a^l_1(k), \dots, a^l_n(1), \dots, a^l_n(k) \tag{14}$$

and

$$a^r_1(1), \dots, a^r_1(k), \dots, a^r_n(1), \dots, a^r_n(k) \tag{15}$$

respectively, where $a^v_t(1), \dots, a^v_t(k)$ (for $v = l, r$ and $t = 1, \dots, n$) is a splitting of a_t. So $a_t \mapsto \bigvee_{j=1}^k a^l_t(j)$ and $a_t \mapsto \bigvee_{j=1}^k a^r_t(j)$ are canonical embeddings of $\mathbf{A}$ into $\mathbf{A}^l$ and $\mathbf{A}^r$ and we can furthermore suppose that both $\mathbf{B}^l$, $\mathbf{C}^l$ and $\mathbf{B}^r$, $\mathbf{C}^r$ contain the image of $\mathbf{A}$ by these embeddings. This can be done by further refining $\mathcal{S}^l$ and $\mathcal{S}^r$. This means that any atom of $\mathbf{B}^v$ and $\mathbf{C}^v$ $(v = l, r)$ will be of the form $\bigvee_{i\in\Gamma} a^v_t(i)$, where $\Gamma \subseteq \{1, \dots, k\}$ and $1 \leqslant t \leqslant n$.

Take new formal atoms $a^l_m(i) \otimes a^r_m(j)$ for $m \leqslant n$ and $i, j \leqslant k$. Our amalgam

$$\mathcal{S}^a = \langle \mathbf{A}^a, \psi^a : \mathbf{B}^a \to \mathbf{C}^a\rangle$$

will be such that the atoms in $\mathbf{A}^a$ will be a subset E of these new formal atoms and the embeddings $l : \mathbf{A}^l \to \mathbf{A}^a$ and $r : \mathbf{A}^r \to \mathbf{A}^a$ will be defined by

$$l(a^l_t(i)) = \bigvee\{a^l_t(i) \otimes a^r_t(j) \in E\}, \tag{16}$$

$$r(a^r_t(i)) = \bigvee\{a^l_t(j) \otimes a^r_t(i) \in E\}. \tag{17}$$

The atoms of $\mathbf{B}^a$ will be

$$E^{\Gamma,\Delta}_t = \bigvee\{a^l_t(i) \otimes a^r_t(j) \in E : (i, j) \in \Gamma \times \Delta\}, \tag{18}$$

where $\bigvee_{i\in\Gamma} a^l_t(i)$ is an atom in $\mathbf{B}^l$ and $\bigvee_{j\in\Delta} a^r_t(j)$ is an atom in $\mathbf{B}^r$. The atoms of $\mathbf{C}^a$ are similar.

Now obviously it is enough to define ψ^a between the atoms of $\mathbf{B}^a$ and $\mathbf{C}^a$.

(1) Suppose $\Gamma, \Delta, \Theta, \Lambda \subseteq \{1, \dots, k\}$ are such that

$$\bigvee_{i\in\Gamma} a^l_t(i), \quad \bigvee_{i\in\Delta} a^r_t(i), \quad \bigvee_{i\in\Theta} a^l_m(i), \quad \bigvee_{i\in\Lambda} a^r_m(i) \tag{19}$$

are atoms in $\mathbf{B}^l$, $\mathbf{B}^r$, $\mathbf{C}^l$, $\mathbf{C}^r$, respectively, such that

$$\psi^l\Big(\bigvee_\Gamma a^l_t(i)\Big) = \bigvee_\Theta a^l_m(i) \quad \text{and} \quad \psi^r\Big(\bigvee_\Delta a^r_t(i)\Big) = \bigvee_\Lambda a^r_m(i). \tag{20}$$

Then we let $\psi^a(E^{\Gamma,\Delta}_t) = E^{\Theta,\Lambda}_m$.

(2) Now suppose that $\bigvee_\Gamma a^l_t(i)$, $\bigvee_\Delta a^r_t(i)$, $\bigvee_\Theta a^l_m(i)$ and $\bigvee_\Lambda a^r_s(i)$, with $m \neq s$, are atoms in $\mathbf{B}^l$, $\mathbf{B}^r$, $\mathbf{C}^l$ and $\mathbf{C}^r$, respectively, such that

$$\psi^l\Big(\bigvee_\Gamma a^l_t(i)\Big) = \bigvee_\Theta a^l_m(i) \quad \text{and} \quad \psi^r\Big(\bigvee_\Delta a^r_t(i)\Big) = \bigvee_\Lambda a^r_s(i). \tag{21}$$

We notice first that, since $\mathcal{S} = \langle \mathbf{A}, \psi \colon \mathbf{B} \to \mathbf{C}\rangle \in \mathcal{L}$ is normal with $\mathbf{A} = \mathbf{B} \vee \mathbf{C}$, the only time ψ sends an atom of $\mathbf{A}$ to a non-atom of $\mathbf{A}$, or inversely sends a non-atom of $\mathbf{A}$ to an atom of $\mathbf{A}$, is in the last steps of a stable chain corresponding to $a_n \mapsto a_0 \vee x$ in diagram (8) or $a_0 \vee x \mapsto a_0$ in diagram (9). Moreover, whenever x is an atom of $\mathbf{B}$, then either x or $\psi(x)$ is an atom of $\mathbf{A}$.

Using this, we see from equations (21) and $m \neq s$ that, since ψ^l and ψ^r are refinements of ψ, a_t must be an atom of $\mathbf{B}$, while $a_m \vee a_s$ must be below the end of a stable chain in $\mathcal{S}$.

Now, in this situation we cannot have $a_t^l(i) \otimes a_t^r(j) \in E$, for any $(i,j) \in \Gamma \times \Delta$, or, in other words, we must have $E_t^{\Gamma,\Delta} = 0$. This is because we have $\psi^l\big(\bigvee_\Gamma a_t^l(i)\big) \leqslant a_m$, while $\psi^r\big(\bigvee_\Delta a_t^r(i)\big) \leqslant a_s$, which would force

$$\psi^a(E_t^{\Gamma,\Delta}) \leqslant \bigvee\{a_m^l(i) \otimes a_m^r(j) \in E\} \tag{22}$$

and similarly

$$\psi^a(E_t^{\Gamma,\Delta}) \leqslant \bigvee\{a_s^l(i) \otimes a_s^r(j) \in E\}, \tag{23}$$

which leaves only the possibility $\psi^a(E_t^{\Gamma,\Delta}) = 0$; or, said in another way, it would force

$$\psi^a(E_t^{\Gamma,\Delta}) = \psi^a\Big(\bigvee\{a_t^l(i) \otimes a_t^r(j) \in E : (i,j) \in \Gamma \times \Delta\}\Big) \tag{24}$$

to be the join of elements on the form $a_m^l(p) \otimes a_s^r(q)$, but we do not include any such elements in our amalgam.

There is also a dual version of this problem, namely when

$$\psi^l\Big(\bigvee_\Gamma a_m^l(i)\Big) = \bigvee_\Theta a_t^l(i) \quad \text{and} \quad \psi^r\Big(\bigvee_\Delta a_s^r(i)\Big) = \bigvee_\Lambda a_t^r(i) \tag{25}$$

for distinct s and m.

With (1) and (2) in mind, we can now formulate the necessary and sufficient conditions on E for this procedure to work out.

(a) For l to be well defined as an embedding from $\mathbf{A}^l$ to $\mathbf{A}^a$:

$$\forall t \leqslant n \; \forall i \leqslant k \; \exists j \leqslant k \;\; a_t^l(i) \otimes a_t^r(j) \in E.$$

(b) For r to be well defined as an embedding from $\mathbf{A}^r$ to $\mathbf{A}^a$:

$$\forall t \leqslant n \; \forall j \leqslant k \; \exists i \leqslant k \;\; a_t^l(i) \otimes a_t^r(j) \in E.$$

(c) If Γ, Δ, Θ, Λ and t, m are as in (1), then $E_t^{\Gamma,\Delta} \neq 0$ if and only if $E_m^{\Theta,\Lambda} \neq 0$.

(d) If Γ, Δ, Θ, Λ and t, m, s are as in (2), then $E_t^{\Gamma,\Delta} = 0$.

We will define E separately for terms of stable, linking and cyclic chains.

Suppose we are given a cyclic chain

$$\begin{array}{lccccc} \mathbf{B} & & a_{t_1} & a_{t_2} & & a_{t_w} \\ & \psi & \downarrow & \downarrow & \cdots & \downarrow \\ \mathbf{C} & & a_{t_2} & a_{t_3} & & a_{t_1} \end{array} \tag{26}$$

Then we let $a_{t_v}^l(i) \otimes a_{t_v}^r(j) \in E$ for all $i, j \leqslant k$ and $1 \leqslant v \leqslant w$.

Given a linking chain

$$\begin{array}{lcccccc} \mathbf{B} & & & a_{t_1} & a_{t_2} & & a_{t_{w-1}} & a_{t_w} \vee y \\ & \psi & & \downarrow & \downarrow & \cdots & \downarrow & \\ \mathbf{C} & & x \vee a_{t_1} & a_{t_2} & a_{t_3} & & a_{t_w} & \end{array} \tag{27}$$

we let $a_{t_v}^l(i) \otimes a_{t_v}^r(j) \in E$ for all $i, j \leqslant k$ and $1 \leqslant v \leqslant w$.

So, since we have included all the new formal atoms in these two cases, we easily see that (a), (b) and (c) are verified. Moreover, condition (d) is only relevant for the stable chains as it only pertains to (2).

Finally, suppose we have a stable chain

$$\begin{array}{lccccccc} \mathbf{B} & & a_{t_1} & a_{t_2} & & a_{t_{w-1}} & a_{t_w} \\ & \psi & \downarrow & \downarrow & \ldots & \downarrow & \downarrow \\ \mathbf{C} & & a_{t_2} & a_{t_3} & & a_{t_w} & a_{t_1} \vee a_{t_{w+1}} \vee \ldots \vee a_{t_{w+q}} \end{array} \tag{28}$$

The case when $a_{t_1} \in \mathbf{C}$ and $a_{t_1} \vee a_{t_{w+1}} \vee \ldots \vee a_{t_{w+q}} \in \mathbf{B}$ is symmetric to this one and is taken care of in the same manner.

By choosing k big enough and refining the partitions of $a_{t_1}, \ldots, a_{t_w}$ in $\mathbf{B}^l$ and $\mathbf{B}^r$, the partitions of $a_{t_2}, \ldots, a_{t_w}, a_{t_1} \vee a_{t_{w+1}} \vee \ldots \vee a_{t_{w+q}}$ in $\mathbf{C}^l$ and $\mathbf{C}^r$, and subsequently extending ψ^l and ψ^r to these refined partitions, we can suppose that there are partitions of $\{1, \ldots, k\}$ as follows:

$$\begin{aligned} \{1, \ldots, k\} = \Gamma^l(\gamma, 1) \sqcup \ldots \sqcup \Gamma^l(\gamma, p) \sqcup \Delta^l(\gamma, 1, 1) \sqcup \ldots \\ \sqcup \Delta^l(\gamma, 1, p) \sqcup \ldots \sqcup \ \Delta^l(\gamma, q, 1) \sqcup \ldots \sqcup \Delta^l(\gamma, q, p) \end{aligned} \tag{29}$$

for $\gamma = 1, \ldots, w$, and

$$\Lambda^l(1) \sqcup \ldots \sqcup \Lambda^l(p) = \{1, \ldots, k\}, \tag{30}$$

$$\Theta^l(\beta, 1) \sqcup \ldots \sqcup \Theta^l(\beta, p) = \{1, \ldots, k\}, \tag{31}$$

for $\beta = 1, \ldots, q$, such that the following hold. For $e = 1, \ldots, p$, $\gamma = 1, \ldots, w$ and $\beta = 1, \ldots, q$,

$$\bigvee_{\Gamma^l(\gamma,e)} a^l_{t_\gamma}(i) \quad \text{and} \quad \bigvee_{\Delta^l(\gamma,\beta,e)} a^l_{t_\gamma}(i) \quad \text{are atoms in } \mathbf{B}^l;$$

for $e = 1, \ldots, p$, $\gamma = 2, \ldots, w$ and $\beta = 1, \ldots, q$,

$$\bigvee_{\Gamma^l(\gamma,e)} a^l_{t_\gamma}(i) \quad \text{and} \quad \bigvee_{\Delta^l(\gamma,\beta,e)} a^l_{t_\gamma}(i) \quad \text{are atoms in } \mathbf{C}^l;$$

for $e = 1, \ldots, p$,

$$\bigvee_{\Lambda^l(e)} a^l_{t_1}(i) \quad \text{is an atom in } \mathbf{C}^l;$$

for $e = 1, \ldots, p$ and $\beta = 1, \ldots, q$,

$$\bigvee_{\Theta^l(\beta,e)} a^l_{t_{w+\beta}}(i) \quad \text{is an atom in } \mathbf{C}^l;$$

for $e = 1, \ldots, p$ and $\gamma = 1, \ldots, w-1$,

$$\psi^l\Big(\bigvee_{\Gamma^l(\gamma,e)} a^l_{t_\gamma}(i)\Big) = \bigvee_{\Gamma^l(\gamma+1,e)} a^l_{t_{\gamma+1}}(i); \tag{32}$$

for $e = 1, \ldots, p$, $\gamma = 1, \ldots, w-1$ and $\beta = 1, \ldots, q$,

$$\psi^l\Big(\bigvee_{\Delta^l(\gamma,\beta,e)} a^l_{t_\gamma}(i)\Big) = \bigvee_{\Delta^l(\gamma+1,\beta,e)} a^l_{t_{\gamma+1}}(i); \tag{33}$$

for $e = 1, \ldots, p$,

$$\psi^l\Big(\bigvee_{\Gamma^l(w,e)} a^l_{t_w}(i)\Big) = \bigvee_{\Lambda^l(e)} a^l_{t_w}(i); \tag{34}$$

for $e = 1, \ldots, p$ and $\beta = 1, \ldots, q$,

$$\psi^l\Big(\bigvee_{\Delta^l(w,\beta,e)} a^l_{t_w}(i)\Big) = \bigvee_{\Theta^l(\beta,e)} a^l_{t_{w+\beta}}(i). \tag{35}$$

The same holds for r, that is, with the same constants k and p, though different partitions.

Let us see how this can be obtained. Suppose $\psi^l : \mathbf{B}^l \to \mathbf{C}^l$ is given. First refine the partition of a_{t_2} in $\mathbf{B}^l$ to be finer than the partition of a_{t_2} in $\mathbf{C}^l$. Then refine the partition of a_{t_3} in $\mathbf{C}^l$ so that we can extend ψ^l to be an isomorphism of the refined partitions of a_{t_2} in $\mathbf{B}^l$ and a_{t_3} in $\mathbf{C}^l$, respectively. Again we refine the partition of a_{t_3} in $\mathbf{B}^l$ to be finer than the partition of a_{t_3} in $\mathbf{C}^l$; and refine the partition of a_{t_4} in $\mathbf{C}^l$ so that we can extend ψ^l to be an isomorphism of the refined partitions of a_{t_3} in $\mathbf{B}^l$ and a_{t_4} in $\mathbf{C}^l$ respectively. We continue like this until we have refined the partition of a_{t_w} in $\mathbf{B}^l$. Extend now the partition of $a_{t_1} \vee a_{t_{w+1}} \vee \ldots \vee a_{t_{w+q}}$ in $\mathbf{C}^l$, so we can let ψ^l be an isomorphism of the refined partition of a_{t_w} in $\mathbf{B}^l$ with the refined partition of $a_{t_1} \vee a_{t_{w+1}} \vee \ldots \vee a_{t_{w+q}}$ in $\mathbf{C}^l$. We will now refine the elements of the chain in the other direction. We recall first that the partition of $a_{t_1} \vee a_{t_{w+1}} \vee \ldots \vee a_{t_{w+q}}$ in $\mathbf{C}^l$ was supposed to be fine enough to contain the elements $a_{t_1}, a_{t_{w+1}}, \ldots, a_{t_{w+q}}$.

To begin, refine the new partition of a_{t_w} in $\mathbf{C}^l$ to be the same as the new partition of a_{t_w} in $\mathbf{B}^l$. Now refine the partition of $a_{t_{w-1}}$ in $\mathbf{B}^l$ so that ψ^l extends to an isomorphism of the refined partitions of $a_{t_{w-1}}$ in $\mathbf{B}^l$ and a_{t_w} in $\mathbf{C}^l$. Again refine the partition of $a_{t_{w-1}}$ in $\mathbf{C}^l$ to be the same as the partition of $a_{t_{w-1}}$ in $\mathbf{B}^l$; and refine the partition of $a_{t_{w-2}}$ in $\mathbf{B}^l$ so that ψ^l extends to an isomorphism of the refined partitions of $a_{t_{w-2}}$ in $\mathbf{B}^l$ and $a_{t_{w-1}}$ in $\mathbf{C}^l$. Continue in this fashion until a_{t_1} has been refined in $\mathbf{B}^l$.

This shows that the above form can be obtained for $\psi^l : \mathbf{B}^l \to \mathbf{C}^l$ individually. Now we can of course also obtain it for $\psi^r : \mathbf{B}^r \to \mathbf{C}^r$. So the only thing lacking is to obtain the same constants k and p for both $\psi^l : \mathbf{B}^l \to \mathbf{C}^l$ and $\psi^r : \mathbf{B}^r \to \mathbf{C}^r$; but this is easily done by some further splitting (this last bit is only to reduce complexity of notation).

We now define a derivation D on $\mathcal{P}(\{1, \ldots, k\}^2)$ (that is, a mapping satisfying $D(X) \subseteq X$ for all $X \subseteq \{1, \ldots, k\}^2$) as follows. Set

$$D(Y) = \{(i,j) \in Y : (i,j) \in \Gamma^l(1,e) \times \Gamma^r(1,d) \Rightarrow (\Lambda^l(e) \times \Lambda^r(d)) \cap Y \neq \emptyset\}. \tag{36}$$

Let

$$X_0 = \bigcup_{e,d} \Gamma^l(1,e) \times \Gamma^r(1,d) \cup \bigcup_{\beta=1}^{q} \bigcup_{e,d} \Delta^l(1,\beta,e) \times \Delta^r(1,\beta,d). \tag{37}$$

So $D^{k_\infty+1}(X_0) = D^{k_\infty}(X_0)$ for some minimal $k_\infty \in \mathbb{N}$, and we notice first that

$$\bigcup_{\beta=1}^{q} \bigcup_{e,d} \Delta^l(1,\beta,e) \times \Delta^r(1,\beta,d) \subseteq D^{k_\infty}(X_0). \tag{38}$$

Claim. *Suppose that for some e, d, σ and $(i,j) \in \Gamma^l(1,e) \times \Gamma^r(1,d)$, we have $(i,j) \notin D^{\sigma+1}(X_0)$. Then $(\Lambda^l(e) \times \Lambda^r(d)) \cap D^\sigma(X_0) = \emptyset$.*

Proof. By definition of X_0, there is a $\tau \geqslant 0$ such that $(i,j) \in D^\tau(X_0) \setminus D^{\tau+1}(X_0)$ (so $\tau \leqslant \sigma$); and by definition of D, we must have $(\Lambda^l(e) \times \Lambda^r(d)) \cap D^\tau(X_0) = \emptyset$, whence also $(\Lambda^l(e) \times \Lambda^r(d)) \cap D^\sigma(X_0) = \emptyset$. □

Lemma 5.2. *For all $i \in \{1, \ldots, k\}$ we have*

$$\{i\} \times \{1, \ldots, k\} \cap D^{k_\infty}(X_0) \neq \emptyset \quad \textit{and} \quad \{1, \ldots, k\} \times \{i\} \cap D^{k_\infty}(X_0) \neq \emptyset.$$

Proof. If the lemma is not true, we can take τ minimal such that for some i we have, for example, $\{i\} \times \{1, \ldots, k\} \cap D^\tau(X_0) = \emptyset$. Clearly $\tau > 0$, so, as

$$\bigcup_{\beta=1}^{r} \bigcup_{e,d} \Delta^l(1,\beta,e) \times \Delta^r(1,\beta,d) \subseteq D^{k_\infty}(X_0),$$

we must have $i \in \Gamma^l(1,e)$ for some e. Therefore, by the claim, letting $\sigma+1=\tau$, we have $\Lambda^l(e) \times \{1,\dots,k\} \cap D^{k_\infty}(X_0) = \emptyset$. So, in particular, for any $j \in \Lambda^l(e)$ we have

$$\{j\} \times \{1,\dots,k\} \cap D^{\sigma}(X_0) = \emptyset,$$

contradicting the minimality of τ. □

Now put $a^l_{t_1}(i) \otimes a^r_{t_1}(j) \in E$ for all $(i,j) \in D^{k_\infty}(X_0)$. And put $a^l_{t_\gamma}(i) \otimes a^r_{t_\gamma}(j) \in E$ for all $(i,j) \in \Gamma^l(\gamma,e) \times \Gamma^r(\gamma,d)$ such that $(\Gamma^l(1,e) \times \Gamma^r(1,d)) \cap D^{k_\infty}(X_0) \neq \emptyset$ and for all $(i,j) \in \Delta^l(\gamma,\beta,e) \times \Delta^r(\gamma,\beta,d)$ ($\gamma = 2,\dots,w$ and $\beta = 1,\dots,q$).

By Lemma 5.2, (a) and (b) are satisfied for a_{t_1} and therefore also by construction for $a_{t_2},\dots,a_{t_w}$. Moreover, (c) is satisfied between a_{t_1}, a_{t_2}, between $a_{t_2}, a_{t_3}, \dots$, and between $a_{t_{w-1}}, a_{t_w}$. Furthermore, by definition of X_0, condition (d) is satisfied in the only relevant place, namely between a_{t_w} and $a_{t_1} \vee a_{t_{w+1}} \vee \dots \vee a_{t_{w+q}}$. So we only have to check condition (c) between a_{t_w} and $a_{t_1} \vee a_{t_{w+1}} \vee \dots \vee a_{t_{w+q}}$.

Now obviously, since $a^l_{t_w}(i) \otimes a^r_{t_w}(j) \in E$, for $(i,j) \in \Delta^l(w,\beta,e) \times \Delta^r(w,\beta,d)$ ($\beta = 1,\dots,q$), we only have to check the condition for the products $\Gamma^l(w,e) \times \Gamma^r(w,d)$. So suppose that $a^l_{t_w}(i) \otimes a^r_{t_w}(j) \in E$ for some $(i,j) \in \Gamma^l(w,e) \times \Gamma^r(w,d)$. Then we know that

$$(\Gamma^l(1,e) \times \Gamma^r(1,d)) \cap D^{k_\infty}(X_0) \neq \emptyset.$$

Therefore, as $D^{k_\infty}(X_0)$ is D-stable, we must have $(\Lambda^l(e) \times \Lambda^r(d)) \cap D^{k_\infty}(X_0) \neq \emptyset$. This means that if

$$\bigvee\{a^l_{t_w}(i) \otimes a^r_{t_w}(j) \in E : (i,j) \in \Gamma^l(w,e) \times \Gamma^r(w,d)\} \neq 0, \tag{39}$$

then also

$$\bigvee\{a^l_{t_1}(i) \otimes a^r_{t_1}(j) \in E : (i,j) \in \Lambda^l(e) \times \Lambda^r(d)\} \neq 0, \tag{40}$$

confirming (c) in one direction. And, conversely, if

$$\bigvee\{a^l_{t_1} \otimes a^l_{t_1}(j) \in E : (i,j) \in \Lambda^l(e) \times \Lambda^r(d)\} \neq 0, \tag{41}$$

then, in particular, $(\Lambda^l(e) \times \Lambda^r(d)) \cap D^{k_\infty}(X_0) \neq \emptyset$. So, by the claim, we also have $(\Gamma^l(1,e) \times \Gamma^r(1,d)) \cap D^{k_\infty}(X_0) \neq \emptyset$, and therefore

$$\bigvee\{a^l_{t_w}(i) \otimes a^r_{t_w}(j) \in E : (i,j) \in \Gamma^l(w,e) \times \Gamma^r(w,d)\} \neq 0. \tag{42}$$

□

One can also use well-known results (see, for example, Hjorth [**24**]) to see that there is a generic increasing homeomorphism of $[0,1]$.

THEOREM 5.3. *The group $H_+([0,1])$ of increasing homeomorphisms of the unit interval has a comeager conjugacy class.*

Proof. Using the notation of Hjorth one can easily verify that the set of $\pi \in H_+([0,1])$, for which $<^{\mathcal{M}(\pi)}$ is isomorphic to $\mathbb{Q}$, $P_=^{\mathcal{M}(\pi)} = \emptyset$ and $P_+^{\mathcal{M}(\pi)}$ and $P_-^{\mathcal{M}(\pi)}$ are dense and unbounded in both directions in $<^{\mathcal{M}(\pi)}$, is comeager and forms a single conjugacy class. □

Notice that this group is connected, and so is not a topological subgroup of S_∞. (In fact, it has been recently proved by Rosendal and Solecki [**42**] that it is not even an abstract subgroup of S_∞ either.)

6. *Ample generic automorphisms*

6.1. *The general concept*

Suppose a Polish group G acts continuously on a Polish space X. Then there is a natural induced (diagonal) action of G on X^n, for any $n = 1, 2, \ldots$, defined by

$$g \cdot (x_1, x_2, \ldots, x_n) = (g \cdot x_1, g \cdot x_2, \ldots, g \cdot x_n).$$

Now, by the Kuratowski–Ulam Theorem, if there is a comeager G-orbit in X^n, and $k \leqslant n$, then there is also a comeager orbit in X^k. However this is far from being true in the other direction. Let us reformulate the property for $n = 2$.

Recall that $\forall^* x R(x)$ means that $\{x : R(x)\}$ is comeager, where x varies over elements of a topological space X.

Proposition 6.1. *Let a Polish group G act continuously on a Polish space X and suppose X has a comeager orbit $\mathcal{O}$. Then the following are equivalent:*

(i) *there is a comeager orbit in X^2;*
(ii) $\forall x \in \mathcal{O}\ \forall^* y \in \mathcal{O}(G_x \cdot y$ *is comeager in* $X)$, *where* $G_x = \{g \in G : g \cdot x = x\}$ *is the* stabilizer *of x;*
(iii) $\forall x \in \mathcal{O}\ \forall^* y \in \mathcal{O}(G_x G_y$ *is comeager in* $G)$*;*
(iv) $\forall x \in \mathcal{O}\ \forall^* h \in G(G_x h G_x$ *is comeager in* $G)$*;*
(v) $\exists x, y \in \mathcal{O}(G_x \cdot y$ *is comeager*$)$.

Proof. (i) $\Rightarrow$ (ii) Let C be a comeager orbit in X^2. Then by the Kuratowski–Ulam Theorem $\forall^* x\ \forall^* y (x, y) \in C$. Thus $\{x : \forall^* y(x, y) \in C\}$ is comeager and clearly G-invariant, so for every $x \in \mathcal{O}$, $\forall^* y(x, y) \in C$. It follows that $\forall^* y \in \mathcal{O}(G_x \cdot y$ is comeager).

(ii) $\Rightarrow$ (iii) Fix $x \in \mathcal{O}$ and $y \in \mathcal{O}$ such that $G_x \cdot y$ is comeager. Since $\mathcal{O}$ is G_δ in X, by Effros' Theorem (see, for example, Becker and Kechris [**5**]) the map $\pi : G \to \mathcal{O}$ given by $\pi(g) = g \cdot y$ is continuous and open and therefore

$$\pi^{-1}(G_x \cdot y) = \{g : g \cdot y \in G_x \cdot y\} = \{g : \exists h \in G_x(h^{-1} g \in G_y)\} = G_x G_y$$

is comeager in G.

(iii) $\Rightarrow$ (iv) We have for any $x \in \mathcal{O}, \forall^* y \in \mathcal{O}(G_x G_y$ is comeager in $G)$, so, by Effros' Theorem again, applied this time to $\sigma(h) = h \cdot x$, we have $\forall^* h(G_x G_{h \cdot x}$ is comeager in $G)$; but $G_{h \cdot x} = h G_x h^{-1}$, so $\forall^* h(G_x h G_x$ is comeager).

(iv) $\Rightarrow$ (v) Fix $x \in \mathcal{O}$ and $h \in G$ such that $G_x h G_x$ is comeager, so that $G_x h G_x h^{-1}$ is comeager and $G_x G_y$ is comeager, where $y = h \cdot x \in \mathcal{O}$. Since the map $\pi(g) = g \cdot y$ is continuous and open from G to $\mathcal{O}$, it follows that $\pi(G_x G_y) = G_x \cdot y$ is comeager.

(v) $\Rightarrow$ (i) Fix $x, y \in \mathcal{O}$ with $G_x \cdot y$ comeager. If $z = g \cdot x$, then $G_z g \cdot y = G_{g \cdot x} g \cdot y = g G_x \cdot g^{-1} g \cdot y = g G_x \cdot y$ is comeager. However, $\{z\} \times G_z g \cdot y = \{g \cdot x\} \times G_z g \cdot y \subseteq G \cdot (x, y)$, so $\forall z \in \mathcal{O} \forall^* u \in X(z, u) \in G \cdot (x, y)$; thus by the Kuratowski–Ulam Theorem, $G \cdot (x, y)$ is a comeager orbit in X^2. $\square$

In particular, if G is abelian, there cannot be a comeager orbit in X^2, unless X is a singleton.

Notice also that if there is a comeager orbit in X^2 and $\mathcal{O}$ is the comeager orbit in X, then, by (iv), for all $x \in \mathcal{O}, \forall^* h \in G(G_x h G_x$ is comeager). However, since the product of two comeager sets in G is equal to G, we have

$$\forall x \in \mathcal{O}\ \forall^* h \in G\ (G_x G_{h \cdot x} G_x = G_x h G_x \cdot G_x h^{-1} G_x = G). \tag{43}$$

Remark. Given a Polish space X and $n = 1, 2, \ldots$, there is a natural continuous action of S_n (the group of permutations of n) on X^n given by

$$\sigma \cdot (x_1, x_2, \ldots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}). \tag{44}$$

If there is a comeager orbit A in X^n under the action of G, this orbit will be invariant by permutation of the coordinates, that is, invariant under the action by S_n, because for each $\sigma \in S_n$, $\sigma \cdot A$ will be comeager, so $\sigma \cdot A \cap A \neq \emptyset$. Take some $(y_1, y_2, \ldots, y_n) \in \sigma \cdot A \cap A$ and $h \in G$ such that

$$(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) = (h \cdot y_1, \ldots, h \cdot y_n).$$

Then for any

$$(x_1, x_2, \ldots, x_n) = (k \cdot y_1, \ldots, k \cdot y_n) \in A, \tag{45}$$

we have

$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = (k \cdot y_{\sigma(1)}, \ldots, k \cdot y_{\sigma(n)}) = (kh \cdot y_1, \ldots, kh \cdot y_n) \in A. \tag{46}$$

Remark. Naturally one would like to know whether there is any Polish group G acting continuously on a Polish space X, with $\mathrm{card}(X) > 1$, for which there is a comeager orbit in $X^{\mathbb{N}}$. This however turns out not to be the case. For suppose $\mathrm{card}(X) > 1$ and let $\emptyset \neq V \subseteq X$ be open, but not dense. Let $C \subseteq X^{\mathbb{N}}$ be a comeager orbit, towards a contradiction. Then $A = \{(x_n) \in X^{\mathbb{N}} : \exists^\infty n (x_n \in V)\}$ is dense and G_δ, so fix $(x_n) \in A \cap C$. Then $\Omega = \{n \in \mathbb{N} : x_n \in V\}$ is infinite. Put $B = \{(y_n) \in X^{\mathbb{N}} : \{y_n\}_{n \in \Omega} \text{ is dense in } X\}$, which is dense G_δ. So fix $(y_n) \in B \cap C$. Then there is $h \in G$ such that $h \cdot (x_n) = (y_n)$, so that $\{h \cdot x_n\}_{n \in \Omega} \subseteq h \cdot V$ is dense, a contradiction.

If G acts continuously on X, we call $(x_1, \ldots, x_n) \in X^n$ *generic* if its orbit is comeager. We say that the action has *ample generics* if for each n, there is a generic element in X^n. In particular, we say that a Polish group G has *ample generic* elements if there is a comeager orbit in G^n (with the conjugacy action of G on G^n), for each finite n. If $\bar{g} \in G^n$ has a comeager orbit, we will refer to it as a *generic element* of G^n. This is an abuse of terminology as G^n is itself a Polish group and so it makes sense to talk about a generic element of G^n viewed as a group. Note that a generic element in the sense we use here is also generic in the group G^n, so our current notion is stronger, and it will be the only one we will use in the rest of the paper. Note that G^n as a group has a generic element if and only if G has a generic element.

Suppose that $\mathcal{K}$ is a Fraïssé class, and $\mathbf{K}$ is its Fraïssé limit. We would like to characterize as before when $\mathbf{K}$ has ample generic automorphisms.

We know that $\mathrm{Aut}(\mathbf{K})^n$ has a comeager diagonal conjugacy class if and only if some $\bar{f} = (f_1, \ldots, f_n) \in \mathrm{Aut}(\mathbf{K})^n$ is turbulent and has dense diagonal conjugacy class. Therefore we can prove the following result exactly as before.

Theorem 6.2. *Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ be its Fraïssé limit. Then the following are equivalent:*

(i) *there is a comeager diagonal conjugacy class in* $\mathrm{Aut}(\mathbf{K})^n$*;*

(ii) $\mathcal{K}_p^n$ *has the* JEP *and the* WAP.

Thus $\mathbf{K}$ has ample generic automorphisms if and only if for every n, $\mathcal{K}_p^n$ has the JEP and the WAP.

We recall the results of Hodges *et al.* [**26**] stating that many ω-stable, $\aleph_0$-categorical structures and the random graph have ample generic automorphisms, but this also holds for many other structures.

6.2. *The Hrushovski property*

DEFINITION 6.3. We say that a Fraissé class $\mathcal{K}$ satisfies the *Hrushovski property* if any system $\mathcal{S} = \langle \mathbf{A}, \psi_1 : \mathbf{B}_1 \to \mathbf{C}_1, \ldots, \psi_n : \mathbf{B}_n \to \mathbf{C}_n \rangle$ in $\mathcal{K}_p^n$ can be extended to some

$$\mathcal{T} = \langle \mathbf{D}, \varphi_1 : \mathbf{D} \to \mathbf{D}, \ldots, \varphi_n : \mathbf{D} \to \mathbf{D} \rangle \quad \text{in } \mathcal{K}_p^n,$$

that is, to a sequence of automorphisms of the same finite structure.

Let us first reformulate the Hrushovski property in topological terms as a condition on the automorphism group.

PROPOSITION 6.4. *Let $\mathcal{K}$ be a Fraïssé class (of finite structures), $\mathbf{K}$ be its Fraïssé limit, and $\mathrm{Aut}(\mathbf{K})$ be the automorphism group of $\mathbf{K}$. Then $\mathcal{K}$ has the Hrushovski property if and only if there is a countable chain $C_0 \leqslant C_1 \leqslant C_2 \leqslant \ldots \leqslant \mathrm{Aut}(\mathbf{K})$ of compact subgroups whose union is dense in $\mathrm{Aut}(\mathbf{K})$.*

Proof. Suppose that $\mathcal{K}$ has the Hrushovski property and $G = \mathrm{Aut}(\mathbf{K})$. Then each of the sets

$$\begin{aligned} F_n &= \{(f_1, \ldots, f_n) \in G^n : \forall x \in \mathbf{K} \text{ the orbit of } x \text{ under } \langle f_1, \ldots, f_n \rangle \text{ is finite}\} \\ &= \{(f_1, \ldots, f_n) \in G^n : \langle f_1, \ldots, f_n \rangle \text{ is relatively compact in } G\} \end{aligned}$$

is comeager, and hence the generic infinite sequence (f_i) in G will generate a dense subgroup all of whose finitely generated subgroups are relatively compact. In particular, G is the closure of the union of a countable chain of compact subgroups.

Conversely, suppose that $C_0 \leqslant C_1 \leqslant C_2 \leqslant \ldots \leqslant G$ is a chain of compact subgroups whose union is dense in G. Let $\mathbf{A} \subseteq \mathbf{K}$ be a finite substructure of $\mathbf{K}$ and let $p_1, \ldots, p_n$ be a sequence of partial automorphisms of $\mathbf{A}$. By density, we can find some m such that there are $f_1, \ldots, f_n \in C_m$ with $f_i \supseteq p_i$ for each $i \leqslant n$; but then $B = C_m \cdot \mathbf{A}$ is a finite set and if we let $\mathbf{B}$ be the finite substructure of $\mathbf{K}$ generated by B then, as C_m is a group, one can check that $\mathbf{B}$ is invariant under elements of C_m. In particular, $\mathbf{B}$ is closed under the automorphisms f_i, whence $(\mathbf{B}, f_1|\mathbf{B}, \ldots, f_n|\mathbf{B})$ is the extension of the system $(\mathbf{A}, p_1, \ldots, p_n)$ needed. □

Hrushovski [**28**] originally proved the Hrushovski property for the class $\mathcal{K}$ of finite graphs and this was used by Hodges *et al.* in [**26**] to show that the automorphism group of the random graph has ample generic automorphisms. In fact, the Hrushovski property simply means that the class of all $\mathcal{T}$ as above is cofinal and this is often enough to show that $\mathcal{K}_p^n$ has the CAP for all n. This, combined with the JEP for $\mathcal{K}_p^n$, which is usually not hard to verify, implies the existence of ample generics. For example, this easily works for the class of finite graphs. Another case is the following. Solecki [**45**] and, independently, Vershik have shown that the class $\mathcal{K}$ of finite metric spaces with rational distances satisfies the Hrushovski property. From this it easily follows that $\mathcal{K}_p^n$ has the CAP for each n. For notational simplicity, take $n = 1$.

Let $\mathcal{L} \subseteq \mathcal{K}_p$ be the class of systems $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{A} \to \mathbf{A} \rangle$, with $\mathbf{A} \in \mathcal{K}$. By the Hrushovski property of $\mathcal{K}$ this class is cofinal under embeddability in $\mathcal{K}_p$. We claim that $\mathcal{L}$ has the AP. Let $\psi : \mathbf{A} \to \mathbf{A}$, $\phi : \mathbf{B} \to \mathbf{B}$, and $\chi : \mathbf{C} \to \mathbf{C}$ be in $\mathcal{K}_p$ with $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \subseteq \mathbf{C}$, and $\psi \subseteq \phi$ and $\psi \subseteq \chi$. Let ∂ be the metric on $\mathbf{B}$ and ρ the metric on $\mathbf{C}$ and suppose without loss of generality that $\mathbf{B} \cap \mathbf{C} = \mathbf{A}$.

We define the following metric d on $\mathbf{B} \cup \mathbf{C}$:

- for $x, y \in \mathbf{B}$, let $d(x, y) = \partial(x, y)$,
- for $x, y \in \mathbf{C}$, let $d(x, y) = \rho(x, y)$,
- for $x \in \mathbf{B}$ and $y \in \mathbf{C}$, let $d(x, y) = \min(\partial(x, z) + \rho(z, y) : z \in \mathbf{A})$.

One easily checks that this satisfies the triangle inequality. So now we only need to see that $\theta = \phi \cup \chi$ is actually an automorphism of $\mathbf{B} \cup \mathbf{C}$. Let us first notice that it is well defined as ϕ and χ agree on their common domain $\mathbf{A}$. Moreover, trivially $d(x,y) = d(\theta(x),\theta(y))$ whenever both $x, y \in \mathbf{B}$ or both $x, y \in \mathbf{C}$. So let $x \in \mathbf{B}$ and $y \in \mathbf{C}$ and find $z \in \mathbf{A}$ such that $d(x,y) = d(x,z) + d(z,y)$. Then

$$d(\theta(x),\theta(y)) \leqslant d(\theta(x),\theta(z)) + d(\theta(z),\theta(y)) = d(x,z) + d(z,y) = d(x,y),$$

as $\theta(z) \in \mathbf{A}$; and reasoning with θ^{-1} we get $d(x,y) \leqslant d(\theta(x),\theta(y))$, so θ is indeed an isometry of $\mathbf{B} \cup \mathbf{C}$ with the metric d. This can therefore be taken to be our amalgam.

Since the argument in the proof of Theorem 2.2 also shows that $\mathcal{K}_p^n$ has the JEP, for all n, it follows from Theorem 6.2 that $\mathbf{U}_0$ has ample generic automorphisms.

For a further example of a structure with the Hrushovski property and ample generic automorphisms take $\mathcal{K} = \mathcal{MBA}_{\mathbb{Q}}$, with Fraïssé limit $(\mathbf{F}, \lambda)$. If

$$\mathcal{S} = \langle \mathbf{A}, \psi_1 : \mathbf{B}_1 \to \mathbf{C}_1, \ldots, \psi_n : \mathbf{B}_n \to \mathbf{C}_n \rangle$$

is in $\mathcal{K}_p^n$, then it can be extended to some

$$\mathcal{T} = \langle \mathbf{D}, \phi_1 : \mathbf{D} \to \mathbf{D}, \ldots, \phi_n : \mathbf{D} \to \mathbf{D} \rangle,$$

that is, to a sequence of automorphisms of the same structure. Moreover, $\mathbf{D}$ can be found such that all the atoms of $\mathbf{D}$ have the same measure.

To see this, notice that, as the measure on $\mathbf{A}$ only takes rational values, we can refine $\mathbf{A}$ to some $\mathbf{D}$ such that all its atoms have the same measure. However then, as ψ_i preserves the measure, for any $b \in \mathbf{B}_i$, b is composed of the same number of $\mathbf{D}$ atoms as $\psi_i(b)$. So ψ_i can easily be extended to an automorphism ϕ_i of $\mathbf{D}$.

Let $\mathcal{L}^n$ be the subclass of $\mathcal{K}_p^n$ consisting of systems of the same form as $\mathcal{T}$. We claim that $\mathcal{L}^n$ has the AP. For suppose that $\mathcal{S} = \langle \mathbf{A}, \psi_1, \ldots, \psi_n \rangle$, $\mathcal{T} = \langle \mathbf{B}, \phi_1, \ldots, \phi_n \rangle$, and $\mathcal{R} = \langle \mathbf{C}, \chi_1, \ldots, \chi_n \rangle$ are in $\mathcal{L}^n$ with $\mathcal{S}$ being a subsystem of $\mathcal{T}$ and $\mathcal{R}$. List the atoms of $\mathbf{A}$ as $a_1, \ldots, a_p$, the atoms of $\mathbf{B}$ as

$$b_1(1), \ldots, b_1(k_1), \ldots, b_p(1), \ldots, b_p(k_p),$$

and the atoms of $\mathbf{C}$ as

$$c_1(1), \ldots, c_1(l_1), \ldots, c_p(1), \ldots, c_p(l_p),$$

where

$$a_i = b_i(1) \vee \ldots \vee b_i(k_i) = c_i(1) \vee \ldots \vee c_i(l_i).$$

Then we can amalgamate $\mathcal{T}$ and $\mathcal{R}$ over $\mathcal{S}$ by taking atoms $b_i(e) \otimes c_i(d)$ and sending $b_i(e)$ to $\bigvee_{d \leqslant l_i} b_i(e) \otimes c_i(d)$ and $c_i(d)$ to $\bigvee_{e \leqslant k_i} b_i(e) \otimes c_i(d)$, and letting

$$\nu(b_i(e) \otimes c_i(d)) = \frac{\delta(b_i(e))\gamma(c_i(d))}{\mu(a_i)}$$

where μ, δ and γ are the measures on $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively. Moreover, let

$$\theta_j(b_i(e) \otimes c_i(d)) = \phi_j(b_i(e)) \otimes \chi_j(c_i(d)).$$

One easily checks that this is indeed an amalgam of $\mathcal{T}$ and $\mathcal{R}$ over $\mathcal{S}$. As, in this case, all n-systems have a common subsystem in $\mathcal{L}^n$, AP for $\mathcal{L}^n$ also implies JEP for $\mathcal{K}^n$, and hence we have the following result.

THEOREM 6.5. *Let $(\mathbf{F}, \lambda)$ be the Fraïssé limit of $\mathcal{MBA}_{\mathbb{Q}}$. Then $(\mathbf{F}, \lambda)$ has ample generic automorphisms.*

It is easy to check that the above works for $\mathcal{MBA}_{\mathbb{Q}_2}$ as well, so the group $\mathrm{Aut}(\mathrm{clop}(2^{\mathbb{N}}), \sigma)$ has ample generic elements and so does the group of measure-preserving homeomorphisms of $2^{\mathbb{N}}$.

Another property that has been studied in the context of automorphism groups is the existence of a dense locally finite subgroup. Bhattacharjee and Macpherson [**8**] showed that such a group exists in the automorphism group of the random graph and it is not difficult to see that also $H(2^{\mathbb{N}}, \sigma)$ has one. Let us just mention that if $\mathrm{Aut}(\mathbf{M})$, for $\mathbf{M}$ a (locally finite) Fraïssé structure, has a dense locally finite subgroup H, then $\mathbf{M}$ has the Hrushovski property. This follows directly from Proposition 6.4.

In particular, there is no dense locally finite subgroup of $H(2^{\mathbb{N}})$, since $\mathbf{B}_\infty$ does not even have the Hrushovski property. Vershik [**48**] poses the question of whether $\mathrm{Aut}(\mathbf{U}_0)$ has a locally finite dense subgroup.

6.3. *Two lemmas*

We will now prove two technical lemmas, which generalize and extend some results of Hodges *et al.* [**26**]. The second will be repeatedly used later on.

LEMMA 6.6. *Let G be a Polish group acting continuously on a Polish space X with ample generics. Let $A, B \subseteq X$ be such that A is not meager and B is not meager in any non-empty open set. Then if $\bar{x} \in X^n$ is generic and V is an open neighborhood of the identity of G, there are $y_0 \in A$, $y_1 \in B$ and $h \in V$ such that $(\bar{x}, y_0), (\bar{x}, y_1) \in X^{n+1}$ are generic and $h \cdot (\bar{x}, y_0) = (\bar{x}, y_1)$.*

Proof. Let $C \subseteq X^{n+1}$ be a comeager orbit. Then, by the Kuratowski–Ulam Theorem, $\{\bar{z} \in X^n : \forall^* y(\bar{z}, y) \in C\}$ is comeager and clearly G-invariant, so it contains $\bar{x}$ and thus $\forall^* y(\bar{x}, y) \in C$. If $y \in C_{\bar{x}} = \{z : (\bar{x}, z) \in C\}$, then $C_{\bar{x}} = G_{\bar{x}} \cdot y$, where $G_{\bar{x}}$ is the stabilizer of $\bar{x}$ for the action of G on X^n. Thus for any $y \in C_{\bar{x}}$, $G_{\bar{x}} \cdot y$ is comeager. Fix $y_0 \in A \cap C_{\bar{x}}$. Consider now the action of $G_{\bar{x}}$ on X. Since $G_{\bar{x}} \cdot y_0$ is comeager, it is G_δ, so by Effros' Theorem, the map

$$\pi : G_{\bar{x}} \longrightarrow G_{\bar{x}} \cdot y_0,$$
$$g \longmapsto g \cdot y_0$$

is continuous and open. Thus $\pi(G_{\bar{x}} \cap V) = (G_{\bar{x}} \cap V) \cdot y_0$ is open in $G_{\bar{x}} \cdot y_0$, so $(G_{\bar{x}} \cap V) \cdot y_0 \cap B \neq \emptyset$. Fix then $y_1 \in (G_{\bar{x}} \cap V) \cdot y_0 \cap B$. Then for some $h \in G_{\bar{x}} \cap V$, we have $h \cdot y_0 = y_1$ and clearly $h \cdot (\bar{x}, y_0) = h \cdot (\bar{x}, y_1)$. □

One can also avoid the use of Effros' Theorem in the above proof and instead give an elementary proof along the lines of Proposition 3.2.

LEMMA 6.7. *Let G be a Polish group acting continuously on a Polish space X with ample generics. Let $A_n, B_n \subseteq X$ be such that, for each n, A_n is not meager and B_n is not meager in any non-empty open set. Then there is a continuous map $a \mapsto h_a$ from $2^{\mathbb{N}}$ into G such that if $a|n = b|n$, $a(n) = 0$, and $b(n) = 1$, we have $h_a \cdot A_n \cap h_b \cdot B_n \neq \emptyset$.*

Proof. Fix a complete metric d on G. For $s \in 2^{<\mathbb{N}}$ define $f_{s^\wedge 0}, f_{s^\wedge 1} \in G$ and $x_{s^\wedge 0}, x_{s^\wedge 1} \in X$ such that if $\bar{x}_s = (x_{s|1}, x_{s|2}, \ldots, x_s)$ and $h_s = f_{s|1} \ldots f_s$ $(s \neq \emptyset)$, we have
(1) $\bar{x}_s$ is generic,
(2) $x_{s^\wedge 0} \in A_{|s|}$ and $x_{s^\wedge 1} \in B_{|s|}$,
(3) $f_{s^\wedge 0} = 1_G$,
(4) $d(h_s, h_s f_{s^\wedge 1}) < 2^{-|s|}$,
(5) $f_{s^\wedge 1} \cdot \bar{x}_{s^\wedge 1} = \bar{x}_{s^\wedge 0}$.
We begin by using Lemma 6.6 to find x_0, x_1, f_0 and f_1 (taking $h_\emptyset = 1$).

Suppose that f_s and x_s are given. By Lemma 6.6 again, we can find $x_{s^\wedge 0} \in A_{|s|}$, so that $\bar{x}_{s^\wedge 0}$ and $\bar{x}_{s^\wedge 1}$ are generic and $g_{s^\wedge 1} \in V = \{f : d(h_s, h_s f) < 2^{-|s|}\}^{-1}$ with $g_{s^\wedge 1} \cdot \bar{x}_{s^\wedge 0} = \bar{x}_{s^\wedge 1}$. Let $f_{s^\wedge 1} = g_{s^\wedge 1}^{-1}$.

By (3) and (4), for any $a \in 2^{\mathbb{N}}$ the sequence $(h_{a|n})$ is Cauchy, and so converges to some $h_a \in G$. Now consider some $a \in 2^{\mathbb{N}}$ and $m < n$. If $a(n-1) = 0$, then by (3), $f_{a|n} = 1_G$ and hence

$$f_{a|n} \cdot x_{a|m} = x_{a|m}.$$

On the other hand, if $a(n-1) = 1$, then by (5) we have

$$f_{a|n} \cdot \bar{x}_{(a|n-1)^\wedge 1} = f_{(a|n-1)^\wedge 1} \cdot \bar{x}_{(a|n-1)^\wedge 1} = \bar{x}_{(a|n-1)^\wedge 0},$$

whence, in particular,

$$f_{a|n} \cdot \bar{x}_{a|n-1} = \bar{x}_{a|n-1};$$

but since $m < n$, $x_{a|m}$ is a term in $\bar{x}_{a|n-1}$ and thus also

$$f_{a|n} \cdot x_{a|m} = x_{a|m}.$$

Fix now $a \in 2^{\mathbb{N}}$. Then, as $h_{a|n} \to h_a$ as $n \to \infty$, we have for any m, $h_{a|n} \cdot x_{a|m} \to h_a \cdot x_{a|m}$ as $n \to \infty$. Thus, for any m,

$$\begin{aligned} h_a \cdot x_{a|m} &= \lim_n h_{a|n} \cdot x_{a|m} \\ &= \lim_n f_{a|1} \cdots f_{a|m} \cdots f_{a|n} \cdot x_{a|m} \\ &= f_{a|1} \cdots f_{a|m} \cdot x_{a|m} \\ &= h_{a|m} \cdot x_{a|m}. \end{aligned} \tag{47}$$

So if $a|n = b|n = s$, $a(n) = 0$, and $b(n) = 1$, we have

$$h_a \cdot x_{s^\wedge 0} = h_{s^\wedge 0} \cdot x_{s^\wedge 0} = h_s f_{s^\wedge 0} \cdot x_{s^\wedge 0} = h_s \cdot x_{s^\wedge 0},$$

as $f_{s^\wedge 0} = 1_G$, and

$$h_b \cdot x_{s^\wedge 1} = h_{s^\wedge 1} \cdot x_{s^\wedge 1} = h_s f_{s^\wedge 1} \cdot x_{s^\wedge 1} = h_s \cdot x_{s^\wedge 0}.$$

Therefore $h_a \cdot x_{s^\wedge 0} = h_b \cdot x_{s^\wedge 1}$, and, because $x_{s^\wedge 0} \in A_n$ and $x_{s^\wedge 1} \in B_n$, we have $h_a \cdot A_n \cap h_b \cdot B_n \neq \emptyset$. □

6.4. *The small index property*

We will next discuss the connection of ample generics with the so-called *small index property* of a Polish group G, which asserts that any subgroup of index less than $2^{\aleph_0}$ is open.

LEMMA 6.8 (Hodges *et al.* [**26**]). *Let G be a Polish group. Then any meager subgroup has index $2^{\aleph_0}$ in G.*

Proof (Solecki). Notice that G is perfect, that is, has no isolated points, as otherwise it would be discrete. Let $E = \{(g, h) \in G^2 : gh^{-1} \in H\}$, where $H \leqslant G$ is a meager subgroup. Then as $(g, h) \mapsto gh^{-1}$ is continuous and open, E must be a meager equivalence relation and therefore by Mycielski's Theorem (see Kechris [**31**, (19.1)]) have $2^{\aleph_0}$ classes. □

THEOREM 6.9. *Let G be a Polish group with ample generics. Then G has the small index property.*

Proof. Suppose that $H \leqslant G$ has index less than $2^{\aleph_0}$ but is not open. Then H is not meager by Lemma 6.8. Also $G \setminus H$ is not meager in any non-empty open set, since otherwise H would be comeager in some non-empty open set and thus, by Pettis' Theorem (see [**31**, (9.9)]), H would be open. We now apply Lemma 6.7 to the action of G on itself by conjugation and $A_n = H$ and $B_n = G \setminus H$. Then, if $a \neq b \in 2^{\mathbb{N}}$, with say $a|n = b|n$, $a(n) = 0$ and $b(n) = 1$, then $h_a H h_a^{-1} \cap h_b(G \setminus H)h_b^{-1} \neq \emptyset$. Therefore $(h_b^{-1}h_a)H(h_a^{-1}h_b) \cap (G \setminus H) \neq \emptyset$, so $h_b^{-1}h_a \notin H$, and thus h_a and h_b belong to different cosets of H, a contradiction. □

We will now apply this to the group of measure-preserving homeomorphisms of the Cantor space.

LEMMA 6.10. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are two finite subalgebras of* $\mathrm{clop}(2^{\mathbb{N}})$ *and that* $G = \mathrm{Aut}(\mathrm{clop}(2^{\mathbb{N}}), \sigma)$. *Then* $\langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle = G_{(\mathbf{A} \cap \mathbf{B})}$, *where* $G_{(\mathbf{A})}$ *is the pointwise stabilizer of* $\mathbf{A}$.

Proof. Since $G_{(\mathbf{A})}$ is open, so is $\langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle$, and it is therefore also closed. Moreover, it is trivially contained in $G_{(\mathbf{A} \cap \mathbf{B})}$, so it is enough to show that $\langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle$ is dense in $G_{(\mathbf{A} \cap \mathbf{B})}$. Suppose $\mathbf{D}$ is a finite subalgebra of $\mathrm{clop}(2^{\mathbb{N}})$ with a set of atoms X all of which have the same measure. Suppose moreover, that $\mathbf{A}, \mathbf{B} \subseteq \mathbf{D}$. Let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_m\}$ and $\{c_1, \ldots, c_k\}$ be the partitions of X given by the atoms of the subalgebras $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A} \cap \mathbf{B} = \mathbf{C}$ of $\mathbf{D} = \mathcal{P}(X)$. Since all of the elements of X have the same measure, it is enough to show that any permutation ρ of X, pointwise fixing $\mathbf{C}$, that is, $\rho'' c_i = c_i$, for $i = 1, \ldots, k$, is in $\langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle$. In fact it is enough to show this for any transposition ρ. Let $\sigma_{x,y}$ denote the transposition switching x and y (here we allow $x = y$). Fix $x_0 \in X$ and let $V_{x_0} \subseteq X$ be the set of atoms $y \in X$ such that $\sigma_{x_0,y} \in \langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle$. Then $V_{x_0} \in \mathbf{C}$. For if, for example, $a_i \cap V_{x_0} \neq \emptyset$, then there is some $y \in a_i \cap V_{x_0}$. So for any $z \in a_i$ we have $\sigma_{x_0,y}, \sigma_{y,z} \in \langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle$, and hence $\sigma_{x_0,z} = \sigma_{x_0,y} \circ \sigma_{y,z} \circ \sigma_{x_0,y} \in \langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle$. This shows that $V_{x_0} \in \mathbf{A}$ and a similar argument shows that $V_{x_0} \in \mathbf{B}$. Therefore, if $\sigma_{x,y}$ is a transposition in $\mathrm{Sym}(X)$ pointwise fixing $\mathbf{C}$, then x and y belong to the same atom of $\mathbf{C}$ and we have $y \in V_x$, showing that $\sigma_{x,y} \in \langle G_{(\mathbf{A})} \cup G_{(\mathbf{B})} \rangle$. □

Let G be a closed subgroup of S_∞. We say that G has the *strong small index property* if whenever $H \leqslant G$ and $[G : H] < 2^{\aleph_0}$ there is a finite set $X \subseteq \mathbb{N}$ such that $G_{(X)} \leqslant H \leqslant G_{\{X\}}$, where $G_{\{X\}}$ is the *setwise stabilizer* of X.

THEOREM 6.11. *Let* $G = \mathrm{Aut}(\mathrm{clop}(2^{\mathbb{N}}), \sigma) \cong H(2^{\mathbb{N}}, \sigma)$. *Then*
(i) *G has ample generics,*
(ii) *G has the strong small index property.*

Proof. We know that G has ample generics, so it has the small index property.

Suppose that $[G : H] < 2^{\aleph_0}$. Then H is open and therefore for some finite $Y \subseteq \mathrm{clop}(2^{\mathbb{N}})$, we have $G_{(Y)} \leqslant H$. If $\mathbf{A}$ is the subalgebra generated by Y, then obviously $G_{(\mathbf{A})} = G_{(Y)} \leqslant H$. We notice that if $h \in H$ then $G_{(h''\mathbf{A})} = hG_{(\mathbf{A})}h^{-1} \leqslant H$ and by the lemma $G_{(\mathbf{A} \cap h''\mathbf{A})} = \langle G_{(\mathbf{A})} \cup G_{(h''\mathbf{A})} \rangle \leqslant H$. So, as $\mathbf{A}$ is finite, we have, for $\mathbf{B} = \bigcap_{h \in H} h''\mathbf{A}$, that $G_{(\mathbf{B})} \leqslant H$, but also, trivially, $H \leqslant G_{\{\mathbf{B}\}}$. So G has the strong small index property. □

6.5. *Uncountable cofinality*

In the same manner as in [**26**], we can also obtain the following result (which is analogous to [**26**, Theorem 6.1]).

THEOREM 6.12. *Let G be a Polish group with ample generic elements. Then G is not the union of a countable chain of non-open subgroups.*

Proof. Let $G = \bigcup_n A_n$, where $A_0 \subseteq A_1 \subseteq \ldots$ are non-open subgroups. We can then assume that each A_n is non-meager. Also, as in the proof of Theorem 6.9, each $G \setminus A_n$ is non-meager in every non-empty open set. Then apply Lemma 6.7 to A_n and $B_n = G \setminus A_n$ to find h_a, with $a \in 2^{\mathbb{N}}$, in G, so that if $a|n = b|n$, $a(n) = 0$, and $b(n) = 1$, then

$$(h_a A_n h_a^{-1}) \cap (h_b(G \setminus A_n)h_b^{-1}) \neq \emptyset,$$

whence $h_a \notin A_n$ or $h_b \notin A_n$. Find uncountable $C \subseteq 2^{\mathbb{N}}$ and m so that $h_a \in A_m$ for all $a \in C$. Then find $a, b \in C$ such that $h_a, h_b \in A_m$ and $a|n = b|n$, $a(n) = 0$, and $b(n) = 1$ for some $n > m$. As $h_a, h_b \in A_n$ we have a contradiction. □

Let us also mention here the following result of Cameron (see Hodges *et al.* [**26**]). If G is an oligomorphic closed subgroup of S_∞, that is, if G is the automorphism group of some $\aleph_0$-categorical structure or equivalently G has finitely many orbits on each $\mathbb{N}^n$, then any open subgroup of G is only contained in finitely many subgroups of G. Recall also that any open subgroup of a Polish group is actually clopen. Therefore, if G is either connected or an oligomorphic closed subgroup of S_∞, then G is not the union of a chain of open proper subgroups, and therefore if G has ample generic elements, it must have *uncountable cofinality*, that is, G is not the union of a countable chain of proper subgroups.

Notice also that if a Polish group G is topologically finitely generated, then it cannot be the union of a countable chain of proper open subgroups, so, if it also has ample generics, then it has uncountable cofinality.

To summarize: if G is a Polish group with ample generics and one of the following holds:

(i) G is an oligomorphic closed subgroup of S_∞,
(ii) G is connected,
(iii) G is topologically finitely generated,

then G has uncountable cofinality.

6.6. *Coverings by cosets*

One can also use the main lemma to prove an analog of the following well-known group-theoretical result due to B. H. Neumann [**40**]: if a group G is covered by finitely many cosets $\{g_i H_i\}_{i \leqslant n}$, then for some i, H_i has finite index.

Before we consider the main result in this setting, let us briefly look at a simple fact.

Proposition 6.13. *Suppose that G is a Polish group with a comeager conjugacy class. Then the smallest number of cosets of proper subgroups needed to cover G is $\aleph_0$.*

Proof. Assume that $G = g_1 H_1 \cup \ldots \cup g_n H_n$. Then by Neumann's lemma, some H_i is of finite index in G, but then H_i must contain some subgroup N which is normal and of finite index in G. As a subgroup of finite index cannot be meagre, N must intersect and therefore contain the comeager conjugacy class, and thus be equal to G. So $G = N \subseteq H_i$. □

In the same manner one can show that if G is partitioned into finitely many pieces, one of these pieces generates G. We should mention that this result does not generalize to partitions of a generating set. For example, Abért [**1**] has shown that $\mathcal{S}_\infty$ is generated by two abelian subgroups, but evidently $\mathcal{S}_\infty$ is not abelian itself.

Theorem 6.14. *Let G be a Polish group with ample generics. Then for any countable covering of G by cosets $\{g_i H_i\}_{i \in \mathbb{N}}$, there is an i such that H_i is open and thus has countable index.*

The theorem clearly follows from the following more general lemma.

LEMMA 6.15. *Let G be a Polish group with ample generics. If $\{k_iA_i\}_{i\in\mathbb{N}}$ is any covering of G, with $k_i \in G$ and $A_i \subseteq G$, then for some i,*

$$A_i^{-1}A_iA_i^{-1}A_i^{-1}A_iA_i^{-1}A_iA_iA_i^{-1}A_i$$

contains an open neighborhood of the identity.

Proof. First enumerate in a list $\{B_n\}_{n\in\mathbb{N}}$ all non-meager groups A_i, so that each one of them appears infinitely often. Then $\bigcup_{l,n\in\mathbb{N}} k_lB_n = B$ is clearly comeager. Notice now that if for some n the set $B_n^{-1}B_nB_nB_n^{-1}B_n$ is comeager in some non-empty open set, then, by Pettis' Theorem, we have finished, so we can assume that $C_n = G \setminus (B_n^{-1}B_nB_nB_n^{-1}B_n)$ is not meager in any non-empty open set, so, by Lemma 6.7, there is a continuous map $a \mapsto h_a$ from $2^{\mathbb{N}}$ into G such that if $a|n = b|n$, $a(n) = 0$, and $b(n) = 1$, then $h_aB_nh_a^{-1} \cap h_bC_nh_b^{-1} \neq \emptyset$. Since $B_n \cap C_n = \emptyset$, it also follows that $a \mapsto h_a$ is injective, so $K = \{h_a : a \in 2^{\mathbb{N}}\}$ is homeomorphic to $2^{\mathbb{N}}$.

The map $(g, h) \in G \times K \mapsto g^{-1}h \in G$ is continuous and open so, since

$$\bigcup_{l,n\in\mathbb{N}} k_lB_n = B$$

is comeager in G, we have, by the Kuratowski–Ulam Theorem,

$$\forall^* g \in G\ \forall^* h \in K\ (g^{-1}h \in B),$$

so fix $g \in G$ with $\forall^* h \in K(h \in gB)$. Then fix $l, n \in \mathbb{N}$ such that, letting $g_0 = gk_l$, we find that the set

$$\{a \in 2^{\mathbb{N}} : h_a \in g_0B_n\}$$

is non-meager, and so dense in some

$$N_t = \{a \in 2^{\mathbb{N}} : t \subseteq a\},$$

with $t \in 2^{<\mathbb{N}}$. Let then $m > |t|$ be such that $B_n = B_m$, and let a and b be such that $a|m = b|m$, $a(m) = 0$, $b(m) = 1$, and $h_a, h_b \in g_0B_n = g_0B_m$, say $h_a = g_0h_1$ and $h_b = g_0h_2$, with $h_1, h_2 \in B_m$. Then

$$\begin{aligned} h_b^{-1}h_aB_mh_a^{-1}h_b &= h_2^{-1}g_0^{-1}g_0h_1B_mh_1^{-1}g_0^{-1}g_0h_2 \\ &= h_2^{-1}h_1B_mh_1^{-1}h_2 \\ &\subseteq B_m^{-1}B_mB_mB_m^{-1}B_m, \end{aligned} \tag{48}$$

which contradicts the fact that $h_b^{-1}h_aB_mh_a^{-1}h_b \cap C_m \neq \emptyset$. □

COROLLARY 6.16. *Suppose that G is a Polish group with ample generics and that G acts on a set X. Then the following are equivalent:*

(i) *all orbits are of size $2^{\aleph_0}$;*
(ii) *all orbits are uncountable;*
(iii) *for every countable set $A \subseteq X$ there is a $g \in X$ such that $g \cdot A \cap A = \emptyset$.*

Proof. (i) ⇒ (ii) This is trivial.

(ii) ⇒ (iii) Suppose that $A = \{a_0, a_1, \ldots\}$ and that (iii) fails for A. Then clearly, $G = \bigcup_{i,j\in\mathbb{N}} G_{a_i,a_j}$, where $G_{a_i,a_j} = \{g \in G : g \cdot a_i = a_j\}$. Since each G_{a_i,a_j} is a coset of the subgroup G_{a_i}, this implies by Theorem 6.14 that for some i, $[G : G_{a_i}] \leqslant \aleph_0$, and hence $G \cdot a_i$ is countable.

(iii) $\Rightarrow$ (i) Suppose some orbit $\mathcal{O}$ has cardinality strictly smaller than the continuum. Then for any $x \in \mathcal{O}$, $[G : G_x] = |\mathcal{O}| < 2^{\aleph_0}$, so by the small index property of G, G_x is open and hence of countable index in G. Thus $|\mathcal{O}| \leqslant \aleph_0$, and hence $A = \mathcal{O}$ contradicts (iii). $\square$

6.7. *Finite generation of groups*

We will now investigate a strengthening of uncountable cofinality that concerns the finite generation of permutation groups, a subject recently originated by Bergman in [**7**] and also studied by Droste and Göbel in [**12**], and Droste and Holland in [**13**].

Definition 6.17. A group G is said to have the *Bergman property* if and only if for each exhaustive sequence of subsets $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$, there are n and k such that $W_n^k = G$.
If G has the stronger property that, for some k and each exhaustive sequence of subsets $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$, there is n such that $W_n^k = G$, we say that G is *k-Bergman.*

Bergman [**7**] proved the k-Bergman property for S_∞ and subsequently Droste and Göbel [**12**] found a sufficient condition for certain permutation groups to have this property.
We will see that ample generics also provide an approach to this problem.

Proposition 6.18. *Let G be a Polish group with ample generic elements and suppose that $A_0 \subseteq A_1 \subseteq \ldots \subseteq G$ is an exhaustive sequence of subsets of G. Then there is an i such that $1 \in \mathrm{Int}(A_i^{10})$.*

Proof. Notice first that also $A_0 \cap A_0^{-1} \subseteq A_1 \cap A_1^{-1} \subseteq \ldots$ exhausts G. For given $g \in G$ find m such that $g, g^{-1} \in A_m$; then $g \in A_m \cap A_m^{-1}$. So we can suppose that each A_n is symmetric. As $\{A_n\}_{n\in\mathbb{N}}$ is a covering of G, there is, by Lemma 6.7, some i such that

$$A_i^{10} = A_i^{-1}A_iA_i^{-1}A_i^{-1}A_iA_i^{-1}A_iA_iA_i^{-1}A_i \tag{49}$$

contains an open neighborhood of the identity. $\square$

In the case of oligomorphic groups we have the following, whose proof generalizes some ideas of Cameron.

Theorem 6.19. *Suppose that G is a closed oligomorphic subgroup of S_∞ with ample generic elements. Then G is 21-Bergman.*

Proof. Suppose that $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$ is an exhaustive chain of subsets of G. Then by Proposition 6.18, there is an n such that W_n^{10} contains an open neighborhood of the identity. Find some finite sequence $\overline{a} \in \mathbb{N}^{<\mathbb{N}}$, with $|\overline{a}| = m$, such that $G_{(\overline{a})} \subseteq W_n^{10}$. Then as G is oligomorphic there are only finitely many distinct orbits of $G_{(\overline{a})}$ on $\mathbb{N}^m$. Choose representatives $\overline{b}_1, \ldots, \overline{b}_k$ for each of the $G_{(\overline{a})}$ orbits on $\mathbb{N}^m$ that intersect $G \cdot \overline{a}$ and find $h_1, \ldots, h_k \in G$ such that $h_i \cdot \overline{a} = \overline{b}_i$. In other words, for each $f \in G$ there are $i \leqslant k$ and $g \in G_{(\overline{a})}$ such that $f \cdot \overline{a} = g \cdot \overline{b}_i = gh_i \cdot \overline{a}$. Now find $l \geqslant n$ sufficiently big such that $h_1, \ldots, h_k \in W_l$. We claim that $W_l^{21} = G$. Let f be any element of G and find $i \leqslant k$ and $g \in G_{(\overline{a})}$ with $f \cdot \overline{a} = g \cdot \overline{b}_i = gh_i \cdot \overline{a}$. Then $h_i^{-1}g^{-1}f \in G_{(\overline{a})}$ and thus $f \in G_{(\overline{a})}W_lG_{(\overline{a})} \subseteq W_l^{21}$, that is, $W_l^{21} = G$. $\square$

Let us now make the following trivial but useful remarks. Suppose $\mathbf{M}$ is some countable structure and $G = \mathrm{Aut}(\mathbf{M})$ has ample generics. If furthermore for any finitely generated substructure $\mathbf{A} \subseteq \mathbf{M}$ there is a $g \in G$ such that $G = \langle G_{(\mathbf{A})}G_{(g''\mathbf{A})}\rangle$, then G has uncountable cofinality. This follows easily from Theorem 6.12, as $G_{(g''\mathbf{A})} = gG_{(\mathbf{A})}g^{-1}$.

If moreover there is a finite n such that $G=(G_{(\mathbf{A})}G_{(g''\mathbf{A})})^n$, then G has the Bergman property. For suppose $W_0\subseteq W_1\subseteq W_2\subseteq\ldots\subseteq G$ is an exhaustive chain of subsets of G. Then by Proposition 6.18, there is an m such that W_m^{10} contains an open neighborhood of the identity. Say that it contains $G_{(\mathbf{A})}$ for some finitely generated substructure $\mathbf{A}\subseteq\mathbf{M}$. Find $g\in G$ and n as above. Then for $k\geqslant m$ big enough, $g,g^{-1}\in W_k$ and

$$G=(G_{(\mathbf{A})}G_{(g''\mathbf{A})})^n=(G_{(\mathbf{A})}gG_{(\mathbf{A})}g^{-1})^n\subseteq(W_m^{10}W_kW_m^{10}W_k)^n\subseteq W_k^{22n}. \tag{50}$$

This situation is less rare than one might think. In fact we have the following.

THEOREM 6.20. *The group $H(2^{\mathbb{N}},\sigma)$ of measure-preserving homeomorphisms of the Cantor space is 32-Bergman.*

We will, as always, identify $H(2^{\mathbb{N}},\sigma)$ and $\mathrm{Aut}(\mathrm{clop}(2^{\mathbb{N}}),\sigma)$.

Proof. We recall that $2^{\mathbb{Z}}$ is measure-preservingly homeomorphic to $2^{\mathbb{N}}$, and so we will temporarily work on the former space. Let g be the Bernoulli shift on $2^{\mathbb{Z}}$ seen as an element of $\mathrm{Aut}(\mathrm{clop}(2^{\mathbb{Z}}),\sigma)$.

Suppose $\mathbf{A}$ is a finite subalgebra of $\mathrm{clop}(2^{\mathbb{Z}})$. By refining $\mathbf{A}$, we can suppose that $\mathbf{A}$ has atoms $N_s=\{x\in 2^{\mathbb{Z}}: x\restriction_{[-k,k]}=s\}$ for $s\in 2^{2k+1}$. Then $(g^{2k+1})''\mathbf{A}$ is independent of $\mathbf{A}$. Let $h=g^{2k+1}$. We wish to show that $G_{(\mathbf{A})}G_{(h''\mathbf{A})}G_{(\mathbf{A})}=G$. Since $G_{(\mathbf{A})}$ is an open neighborhood of the identity, it is enough to show that $G_{(\mathbf{A})}G_{(h''\mathbf{A})}G_{(\mathbf{A})}$ is dense in G, and for this it suffices to show that whenever $\mathbf{D}$ is a finite subalgebra and $f_0\in G_{\{\mathbf{D}\}}$, the setwise stabilizer of $\mathbf{D}$, then there is some $f_1\in G_{(\mathbf{A})}G_{(h''\mathbf{A})}G_{(\mathbf{A})}$ agreeing with f_0 on $\mathbf{D}$, that is, $f_0\restriction_{\mathbf{D}}=f_1\restriction_{\mathbf{D}}$. We notice first that by refining $\mathbf{D}$, we can suppose that $\mathbf{D}$ is the subalgebra of $\mathrm{clop}(2^{\mathbb{Z}})$ generated by $\mathbf{A}$ and some finite subalgebra $\mathbf{B}\supseteq h''\mathbf{A}$ that is independent of $\mathbf{A}$ and all of whose atoms have the same measure. For concreteness, we can suppose that $\mathbf{B}$ has atoms

$$N_{t,r}=\{x\in 2^{\mathbb{Z}}: x\restriction_{[-l,-k[}=t \text{ and } x\restriction_{]k,l]}=r\}\quad\text{for } t,r\in 2^{l-k},$$

where l is some number greater than k. List the atoms of $\mathbf{A}$ as $a_1,\ldots,a_n$ and the atoms of $\mathbf{B}$ as $b_1,\ldots,b_m$. Then the atoms of $\mathbf{D}$, namely $a_i\cap b_j$, all have the same measure and can be identified with formal elements $a_i\otimes b_j$, with $i\leqslant n$ and $j\leqslant m$. So an element of $G_{\{\mathbf{D}\}}$ gives rise to an element of $S=\mathrm{Sym}(\{a_i\otimes b_j: i\leqslant n,\, j\leqslant m\})$, whereas elements of $G_{(\mathbf{A})}\cap G_{\{\mathbf{D}\}}$ and $G_{(\mathbf{B})}\cap G_{\{\mathbf{D}\}}$ give rise to elements of S that preserve respectively the first and the second coordinates.

We now need the following well-known lemma; see, for example, Abért [**1**] for a proof.

LEMMA 6.21. *Let $\Omega=\{a_i\otimes b_j: i\leqslant n,\, j\leqslant m\}$ and let $F\leqslant\mathrm{Sym}(\Omega)$ be the subgroup that preserves the first coordinates and $H\leqslant\mathrm{Sym}(\Omega)$ be the subgroup that preserves the second coordinates. Then* $\mathrm{Sym}(\Omega)=\mathrm{FHF}$.

This completes the proof of Theorem 6.20. □

Some of the results of this section are related to independent research by A. Ivanov [**30**]. His setup is slightly different from ours as he formulates his results in terms of amalgamation bases.

6.8. *Actions on trees*

Macpherson and Thomas [**38**] have recently found a relationship between the existence of a comeager conjugacy class in a Polish group and actions of the group on trees. This is further connected with Serre's property (FA) that we will verify for the group of (measure-preserving) homeomorphisms of $2^{\mathbb{N}}$.

A *tree* is a graph $T = (V, E)$ that is uniquely path connected, that is, E is a symmetric irreflexive relation on the set of vertices V such that any two vertices are connected by a unique path.

A group G is said to *act without inversions* on a tree T if there is an action of G by automorphisms on T such that for any $g \in G$ there are no two adjacent vertices a and b on T such that $g \cdot a = b$ and $g \cdot b = a$. Now we can state Serre's property (FA).

A group G is said to have *property* (FA) if whenever G acts without inversions on a tree $T = (V, E)$, there is a vertex $a \in V$ such that $g \cdot a = a$ for all $g \in G$.

We say that a free product with amalgamation $G = G_1 *_A G_2$ is *trivial* when one of the G_i is equal to G.

For groups G that are *not* countable we have the following characterization of property (FA), from Serre [**44**]: G has property (FA) if and only if

(i) G is not a non-trivial product with amalgamation,
(ii) $\mathbb{Z}$ is not a homomorphic image of G,
(iii) G is not the union of a countable chain of proper subgroups.

THEOREM 6.22 (Macpherson and Thomas [**38**]). *Let G be a Polish group with a comeager conjugacy class. Then G cannot be written as a free product with amalgamation.*

Moreover, it is trivial to see that if a Polish group has a comeager conjugacy class then (ii) also holds. In fact every element of G is a commutator. For suppose C is the comeager conjugacy class and g is an arbitrary element of G. Then both C and gC are comeager, so $C \cap gC \neq \emptyset$. Take some $h, f \in C$ such that $h = gf$. Then for some $k \in G$ we have

$$gf = h = kfk^{-1} \quad \text{and} \quad g = kfk^{-1}f^{-1}.$$

So G has only one abelian quotient, namely $\{e\}$. For if for some $H \trianglelefteq G$, G/H is abelian, then $G = [G, G] \leqslant H$.

So to verify whether a Polish group with a comeager conjugacy class has property (FA), we only need to show it has uncountable cofinality.

Another way of approaching property (FA) is through the Bergman property, which one can see is actually very strong. For example, one can show that it implies that any action of the group by isometries on a metric space has bounded orbits, but in the case of isometric actions on real Hilbert spaces or automorphisms of trees, having a bounded orbit is enough to ensure that there is a fixed point. So groups with the Bergman property automatically have property (FA) and property (FH) (the latter says that any action by isometries on a real Hilbert space has a fixed point). Thus in particular, $H(2^{\mathbb{N}}, \sigma)$ has both properties (FA) and (FH).

6.9. *Generic freeness of subgroups*

Let us mention next another application of the existence of dense diagonal conjugacy classes in each G^n. Suppose that G is Polish and has a dense diagonal conjugacy class in G^n, for each $n \in \mathbb{N}$. Then there is a dense G_δ subset $C \subseteq G^{\mathbb{N}}$ such that any two sequences $(f_n), (g_n) \in C$ generate isomorphic groups, that is, the mapping $f_n \mapsto g_n$ extends to an isomorphism of $\langle f_n \rangle$ and $\langle g_n \rangle$.

To see this, suppose $w(X_1, \ldots, X_n)$ is a reduced word. Then either $w(g_1, g_2, \ldots, g_n) = e$, for all $g_1, \ldots, g_n$, or $w(g_1, g_2, \ldots, g_n) \neq e$ for an open dense set of $(g_1, \ldots, g_n) \in G^n$. (This is trivial as

$$w(g_1, g_2, \ldots, g_n) = e \iff w(kg_1k^{-1}, kg_2k^{-1}, \ldots, kg_nk^{-1}) = e,$$

for all $k \in G$ and $(g_1, \dots, g_n) \in G^n$.) In any case, taking the intersection over all reduced words, we find that there is a dense G_δ set $C \subseteq G^{\mathbb{N}}$ such that any two sequences (f_n) and (g_n) in C satisfy the same equations $w(X_1, \dots, X_n) = e$, that is, $f_n \mapsto g_n$ extends to an isomorphism.

In particular, if for any non-trivial reduced word $w(X_1, \dots, X_n)$ there are $g_1, \dots, g_n \in G$ such that $w(g_1, \dots, g_n) \neq e$, then any sequence in C freely generates a free group and, using the Kuratowski–Mycielski Theorem (see Kechris [**31**, (19.1)]) we deduce that the generic compact subset of G also freely generates a free group.

Macpherson [**37**] shows that any oligomorphic closed subgroup of S_∞ contains a free subgroup of infinite rank. It is not hard to see that it also holds for the groups $\mathrm{Aut}([0,1], \lambda)$, $H(2^{\mathbb{N}}, \sigma)$, $\mathrm{Aut}(\mathbf{U}_0)$ and $\mathrm{Iso}(\mathbf{U})$. Moreover, $H(2^{\mathbb{N}})$ and these latter groups also have dense conjugacy classes in each dimension (for $\mathrm{Aut}(\mathbf{U}_0)$ and $H(2^{\mathbb{N}})$ it is enough to use the multidimensional version of Theorem 2.1, analogous to Theorem 6.2).

This property has been studied by a number of authors (see, for example, Gartside and Knight [**17**]) and it seems to be a fairly common phenomenon in bigger Polish groups.

6.10. *Automatic continuity of homomorphisms*

We now come to the study of automatic continuity of homomorphisms from Polish groups with ample generics. Notice that the small index property can be seen as a phenomenon of automatic continuity. In fact, if a topological group G has the small index property, then any homomorphism of G into S_∞ will be continuous. This follows because the inverse image of a basic open neighborhood of 1_{S_∞} will be a subgroup of G with countable index and therefore open. However, of course this puts a strong condition on the target group, namely that it should have a neighborhood basis at the identity consisting of open subgroups. We would like to have some less restrictive condition on the target group that still insures automatic continuity of any homomorphism from a Polish group with ample generics. Of course, some restriction is necessary, for the identity function from a Polish group into itself, equipped with the discrete topology, is never continuous unless the group is countable.

LEMMA 6.23. *Let H be a topological group and κ a cardinal number. Then the following are equivalent:*

(i) *for each open neighborhood V of 1_H, H can be covered by fewer than κ many right translates of V;*

(ii) *for each open neighborhood V of 1_H, there are not κ many disjoint right translates of V.*

Proof. (i) $\Rightarrow$ (ii) Suppose (ii) fails and $\{Vf_\xi\}_{\xi<\kappa}$ is a family of κ many disjoint right translates of some open neighborhood V of 1_H. By choosing V smaller, we can suppose that V is actually symmetric. Then if $\{Vg_\xi\}_{\xi<\lambda}$ covers H for some $\lambda < \kappa$, there are f_ξ, f_ζ and g_ν, with $\xi \neq \zeta$, such that $f_\xi, f_\zeta \in Vg_\nu$, and hence $g_\nu \in Vf_\xi \cap Vf_\zeta$, contradicting the fact that $Vf_\xi \cap Vf_\zeta = \emptyset$. So (i) fails.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds and V is some open neighborhood of 1_H. Find some open neighborhood $U \subseteq V$ of 1_H such that $U^{-1}U \subseteq V$ and choose a maximal family of disjoint right translates $\{Uf_\xi\}_{\xi<\lambda}$ of U. By (ii), $\lambda < \kappa$. Suppose that $g \in H$. Then there is a $\xi < \lambda$ such that $Ug \cap Uf_\xi \neq \emptyset$, whereby $g \in U^{-1}Uf_\xi \subseteq Vf_\xi$. So $\{Vf_\xi\}_{\xi<\lambda}$ covers H and (i) holds. □

Recall that the *Souslin number* of a topological space is the least cardinal κ such that there is no family of κ many disjoint open subsets of the space. By analogy, for a topological group H, let the *uniform Souslin number* be the least cardinal κ satisfying the equivalent conditions of the above lemma.

Notice that the uniform Souslin number is at most the Souslin number, which is again at most $\mathrm{density}(H)^+$, where the density of a topological space is the smallest cardinality of a dense subset. In particular, if H is a separable topological group, then its (uniform) Souslin number is at most $\aleph_1 \leqslant 2^{\aleph_0}$. We should mention that groups with uniform Souslin number at most $\aleph_1$ have been studied extensively under the name $\aleph_0$-bounded groups (see the survey article by Tkachenko [**46**]). A well-known result (see, for example, Guran [**21**]) states that a Hausdorff topological group is $\aleph_0$-bounded if and only if it (topologically) embeds as a subgroup into a direct product of second countable groups. Note that $\aleph_0$-bounded groups are easily seen to contain the σ-compact groups. Moreover, the uniform Souslin number is productive in contradistinction to separability, that is, any direct product of groups with uniform Souslin number at most κ has uniform Souslin number at most κ (for any infinite κ).

THEOREM 6.24. *Suppose that G is a Polish group with ample generic elements and that $\pi : G \to H$ is a homomorphism into a topological group with uniform Souslin number at most $2^{\aleph_0}$ (in particular, if H is separable). Then π is continuous.*

Proof. It is enough to show that π is continuous at 1_G. So let W be an open neighborhood of 1_H. We need to show that $\pi^{-1}(W)$ contains an open neighborhood of 1_G. Pick a symmetric open neighborhood V of 1_H such that $V^{20} \subseteq W$ and put $A = \pi^{-1}(V^{-1}V) = \pi^{-1}(V^2)$.

CLAIM 1. *The set A is non-meager.*

Proof. Otherwise, as $(g,h) \mapsto gh^{-1}$ is open and continuous from G^2 to G, there is, by the Mycielski theorem, a Cantor set $C \subseteq G$ such that for any $g \neq h$ in C, $gh^{-1} \notin A$. However,

$$V\pi(g) \cap V\pi(h) = \emptyset \iff \pi(gh^{-1}) \notin V^{-1}V \iff gh^{-1} \notin A.$$

So this means that there are continuum many disjoint translates of V, contradicting the fact that the uniform Souslin number of H is at most $2^{\aleph_0}$. $\square$

CLAIM 2. *The set A covers G by fewer than $2^{\aleph_0}$ many right translates.*

Proof. By the condition on the uniform Souslin number we can find a covering $\{Vf_\xi\}_{\xi<\lambda}$ of H by $\lambda < 2^{\aleph_0}$ many right translates of V. So for each Vf_ξ intersecting $\pi(G)$ take some $g_\xi \in G$ with $\pi(g_\xi) \in Vf_\xi$. Then $Vf_\xi \subseteq VV^{-1}\pi(g_\xi) = V^2\pi(g_\xi)$, so the latter cover $\pi(G)$. Now, if $g \in G$, find $\xi < \lambda$ such that $\pi(g) \in V^2\pi(g_\xi)$, whence $\pi(gg_\xi^{-1}) \in V^2$, that is, $gg_\xi^{-1} \in A$ and $g \in Ag_\xi$. So the Ag_ξ cover G. $\square$

CLAIM 3. *The set A^5 is comeager in some non-empty open set.*

Proof. Otherwise, by Lemma 6.7, we can find h_a, with $a \in 2^{\mathbb{N}}$, in G so that if $a|n = b|n$, $a(n) = 0$, and $b(n) = 1$, then

$$h_a A h_a^{-1} \cap h_b(G \setminus A^5)h_b^{-1} \neq \emptyset$$

or equivalently

$$h_b^{-1}h_a A h_a^{-1} h_b \cap (G \setminus A^5) \neq \emptyset.$$

Since A covers G by fewer than $2^{\aleph_0}$ right translates, and thus left translates (as A is symmetric), there exist uncountable $B \subseteq 2^{\mathbb{N}}$ and $g \in G$ such that for $a \in B$ we have $h_a \in gA$. If $a, b \in B$, with $a|n = b|n$, $a(n) = 0$, and $b(n) = 1$, then let $g_a, g_b \in A$ be such that $h_a = gg_a$ and $h_b = gg_b$. Then

$$h_b^{-1}h_a A h_a^{-1} h_b = g_b^{-1}g^{-1}gg_a A g_a^{-1}g^{-1}gg_b = g_b^{-1}g_a A g_a^{-1} g_b \subseteq A^5,$$

a contradiction. $\square$

So, by Pettis' theorem, $A^{10} \subseteq \pi^{-1}(W)$ contains an open neighborhood of 1_G and π is continuous. $\square$

COROLLARY 6.25. *Suppose G is a Polish group with ample generic elements. Then G has a unique Polish group topology.*

Proof. Suppose that τ and σ are two Polish group topologies on G such that G has ample generics with respect to τ. Then the identity mapping from G, τ to G, σ is continuous, that is, $\sigma \subseteq \tau$; but then τ must be included in the Borel algebra generated by σ and in particular the identity mapping is Baire measurable from G, σ to G, τ, and so continuous. $\square$

We see from the above proof, that the only thing we need is that the two topologies be inter-definable, which is exactly what Theorem 6.24 gives us.

As one can easily show that S_∞ has ample generics, our result applies in particular to this group. So this implies that, for example, any unitary representation of S_∞ on separable Hilbert space is actually a continuous unitary representation. Moreover, whenever S_∞ acts by homeomorphisms on some locally compact Polish space or by isometries on some Polish metric space, then it does so continuously. For these results it is enough to notice that the actions in question correspond to homomorphisms into the unitary group, respectively the homeomorphism and the isometry group, which are Polish when the spaces are separable, respectively locally compact.

In a beautiful paper Gaughan [**18**] proves that any Hausdorff group topology on S_∞ must extend its usual Polish topology. So coupled with the above result this gives us the following rigidity result for S_∞ (we would like to thank V. Pestov for suggesting how to get rid of a Hausdorff condition in a previous version of the result).

THEOREM 6.26. *The group S_∞ has exactly two separable group topologies, namely the trivial one and the usual Polish topology.*

Proof. Suppose that τ is a separable group topology on S_∞ and define

$$N = \bigcap \{U : U \text{ open and } 1 \in U\}. \tag{51}$$

We easily see that N is conjugacy invariant, $N = N^{-1}$ and that N is closed under products. For if $x, y \in N$ and W is any open neighborhood of xy, then by the continuity of the group operations, we can find open sets $x \in U$ and $y \in V$ such that $UV \subseteq W$. However, then as 1 cannot be separated from x by an open set, x cannot be separated from 1 either. Thus $1 \in U$ and similarly $1 \in V$, whence $1 \in W$. So xy cannot be separated from 1 by an open set and hence 1 cannot be separated from xy either. Thus $xy \in N$.

Therefore N is a normal subgroup of S_∞ and hence equal to one of $\{1\}$, Alt, Fin or S_∞ itself. In the first case, we see that τ is Hausdorff and thus that it extends the Polish topology, and in the last case that $\tau = \{\emptyset, S_\infty\}$. Moreover, by the separability of τ, we know that τ is weaker than the Polish topology on S_∞ and thus in the first case we know that it must be exactly equal to the Polish topology. We are therefore left with the two middle cases that we claim cannot occur. The subgroups Alt and Fin are dense in the Polish topology and thus also dense in τ. If $x \notin N$, then we can find some open neighborhood V of x not containing 1. However, then $V \cap N = \emptyset$, because any element of V can be separated from 1 and thus does not belong to N. This shows that N is closed and contains a dense subgroup, so $N = S_\infty$, a contradiction. $\square$

We should also mention the following result that allows us to see the small index property as a special case of automatic continuity.

PROPOSITION 6.27. *Suppose that G is a Polish group such that any homomorphism from G into a group with uniform Souslin number at most $2^{\aleph_0}$ is continuous. Then G has the small index property.*

Proof. Suppose H is a subgroup of G of small index. Then $\mathrm{Sym}(G/H)$, where G/H is the set of left cosets of H, has uniform Souslin number at most $2^{\aleph_0}$ and clearly the action of left translation of G on G/H gives rise to a homomorphism of G into $\mathrm{Sym}(G/H)$. By automatic continuity, this shows that the pointwise stabilizer of the coset H is open in G, that is, H is open in G. □

6.11. *Automorphisms of trees*

We will finally investigate the structure of the group of Lipschitz homeomorphisms of the Baire space $\mathcal{N}$. This group is of course canonically isomorphic to $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$, where $\mathbb{N}^{<\mathbb{N}}$ is seen as a copy of the uniformly countably splitting rooted tree. Though $\mathrm{Age}(\mathbb{N}^{<\mathbb{N}})$ is not a Fraïssé class in its usual relational language, we can still see $\mathbb{N}^{<\mathbb{N}}$ as being the generic limit of the class of finite rooted trees and we will see that the theory goes through in this context. Alternatively, one can change the language by replacing the tree relation by a unary function symbol that assigns to each node its predecessor in the tree ordering. In this way, $\mathrm{Age}(\mathbb{N}^{<\mathbb{N}})$ becomes a Fraïssé class outright.

In the following, $\mathbb{N}^{<\mathbb{N}}$ is considered a tree with root the empty string, $\emptyset$, such that the children of a vertex $s \in \mathbb{N}^{<\mathbb{N}}$ are $s\hat{\ }n$, for all $n \in \mathbb{N}$. A subtree of $\mathbb{N}^{<\mathbb{N}}$ is a subset $T \subseteq \mathbb{N}^{<\mathbb{N}}$ closed under initial segments, that is, if $\langle n_0, n_1, \ldots, n_k\rangle \in T$, then so are $\emptyset, \langle n_0\rangle, \ldots, \langle n_0, n_1, \ldots, n_{k-1}\rangle$. By $m^{\leqslant m}$ we denote the tree of sequences $\langle n_0, \ldots, n_k\rangle$ such that $n_i < m$ and $k < m$. Moreover, if $s, t \in \mathbb{N}^{<\mathbb{N}}$, we write $s \subseteq t$ to denote that t extends s as a sequence.

LEMMA 6.28. *Suppose $\phi : T \to S$ is an isomorphism between finite subtrees of $\mathbb{N}^{<\mathbb{N}}$, say $T, S \subseteq m^{\leqslant m}$. Then there is an automorphism ψ of $m^{\leqslant m}$ extending ϕ.*

Proof. Notice first that ϕ restricts to a bijection between two subsets of

$$m^1 = \{\langle 0\rangle, \langle 1\rangle, \ldots, \langle m-1\rangle\},$$

so can be extended to a permutation ϕ_1 of m^1. Then ϕ_1 is an isomorphism of $T_1 = T \cup m^1$ and $S_1 = S \cup m^1$. Now suppose $\phi_1(\langle 0\rangle) = \langle j\rangle$. Then ϕ_1 restricts to a bijection between two subsets of $\{\langle 0,0\rangle, \langle 0,1\rangle, \ldots, \langle 0,m-1\rangle\}$ and $\{\langle j,0\rangle, \langle j,1\rangle, \ldots, \langle j,m-1\rangle\}$ and can be extended as before to some isomorphism of

$$T_2 = T_1 \cup \{\langle 0,0\rangle, \langle 0,1\rangle, \ldots, \langle 0,m-1\rangle\} \tag{52}$$

and

$$S_2 = S_1 \cup \{\langle j,0\rangle, \langle j,1\rangle, \ldots, \langle j,m-1\rangle\}. \tag{53}$$

Now continue with $\phi_1(\langle 1\rangle) = \langle \ell\rangle$, etc. Eventually, we will obtain an automorphism ψ of $m^{\leqslant m}$ that extends ϕ. □

LEMMA 6.29. *Suppose $m \leqslant n$ and $(\psi_1, \ldots, \psi_k)$ and $(\phi_1, \ldots, \phi_k)$ are sequences of automorphisms of $n^{\leqslant n}$ extending automorphisms $(\chi_1, \ldots, \chi_k)$ of $m^{\leqslant m}$, that is, $\chi_i \subseteq \psi_i, \phi_i$. Then there is an automorphism ξ of $\ell^{\leqslant \ell}$, for $\ell = 2n$, such that $(\psi_1, \ldots, \psi_k)$ and $(\xi \circ \phi_1 \circ \xi^{-1}, \ldots, \xi \circ \phi_k \circ \xi^{-1})$*

can be extended to some common sequence of automorphisms of $\ell^{\leqslant \ell}$. Moreover, ξ fixes $m^{\leqslant m}$ pointwise.

Proof. Let ξ be an automorphism of $\ell^{\leqslant \ell}$ pointwise fixing $m^{\leqslant m}$ and such that

$$\xi(n^{\leqslant n}) \cap n^{\leqslant n} = m^{\leqslant m}.$$

Then for each $i \leqslant k$, ψ_i and $\xi \circ \phi_i \circ \xi^{-1}$ agree on their common domain, and the same applies for ψ_i^{-1} and $(\xi \circ \phi_i \circ \xi^{-1})^{-1}$. So $\psi_i \cup \xi \circ \phi_i \circ \xi^{-1}$ is an isomorphism of finite subtrees of $\ell^{\leqslant \ell}$ and the result follows from Lemma 6.28. □

Though $\mathbb{N}^{<\mathbb{N}}$ is not ultrahomogeneous in its relational language, it is in the functional language, and the preceding results show that its age in the functional language satisfies the conditions of Theorem 6.2. Therefore $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ has ample generics.

LEMMA 6.30. *Suppose that S and T are finite subtrees of $\mathbb{N}^{<\mathbb{N}}$ and $G = \mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$. Then $G_{(S\cap T)} = G_{(T)}G_{(S)}G_{(T)}$.*

Proof. Since $G_{(T)}G_{(T)}G_{(S)}G_{(T)} = G_{(T)}G_{(S)}G_{(T)}$, and $G_{(T)}$ is an open neighborhood of the identity, it is enough to show that $G_{(T)}G_{(S)}G_{(T)}$ is dense in $G_{(S\cap T)}$. By Lemma 6.28, it is enough to show that if ϕ is an automorphism of $m^{\leqslant m}$ for some m such that $S, T \subseteq m^{\leqslant m}$ and ϕ pointwise fixes $S \cap T$, then there is a $g \in G_{(T)}G_{(S)}G_{(T)}$ with $g \supseteq \phi$.

For $s \in S \cap T$, let $A_s = \{n \in \mathbb{N} : s\hat{\ }n \notin S \cap T \text{ and } s\hat{\ }n \in S\}$ and fix a permutation σ_s of $\mathbb{N}$ pointwise fixing $\{0, 1, 2, \ldots, m-1\} \setminus A_s$, but such that $\sigma_s(A_s) \cap A_s = \emptyset$. Define $f \in G_{(T)}$ as follows. If $u \in S \cap T$, let $f(u) = u$. Otherwise if

$$u = s\hat{\ }n\hat{\ }t, \quad \text{where } s \in S \cap T,\ s\hat{\ }n \notin S \cap T, \text{ and } t \in \mathbb{N}^{<\mathbb{N}}, \tag{54}$$

let $f(u) = s\hat{\ }\sigma_s(n)\hat{\ }t$.

Now let $g \supseteq \phi$ be defined as follows. If $u = s\hat{\ }t$, where s is the maximal initial segment such that $s \in m^{\leqslant m}$, let $g(u) = \phi(s)\hat{\ }t$. Then $f^{-1}gf \in G_{(S)}$. For suppose $u = s\hat{\ }n\hat{\ }t \in S$, $s\hat{\ }n \notin T$, and $s \in S \cap T$. Then

$$f^{-1} \circ g \circ f(u) = f^{-1} \circ g(s\hat{\ }\sigma_s(n)\hat{\ }t) = f^{-1}(\phi(s)\hat{\ }\sigma_s(n)\hat{\ }t) = f^{-1}(s\hat{\ }\sigma_s(n)\hat{\ }t) = u, \tag{55}$$

since ϕ fixes $S \cap T$ pointwise. And if $u \in S \cap T$, then $f^{-1} \circ g \circ f(u) = u$, as both $f, g \in G_{(S\cap T)}$. So

$$g = ff^{-1}gff^{-1} \in G_{(T)}G_{(S)}G_{(T)}. \tag{56}$$

□

THEOREM 6.31. *Let $G = \mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$. Then:*

(i) *G has ample generic elements;*
(ii) (*R. Möller* [**39**]) *G has the strong small index property;*
(iii) *G is 32-Bergman;*
(iv) *G has a locally finite dense subgroup.*

Proof. Part (i) has been verified and so G has the small index property. Suppose that $G_{(S)} \leqslant H \leqslant G$ for some finite subtree $S \subseteq \mathbb{N}^{<\mathbb{N}}$ and open subgroup H. Then, by Lemma 6.30, we deduce, for any $h \in H$, that $G_{(S\cap h''S)} = G_{(S)}G_{(h''S)}G_{(S)} \leqslant H$. So, as S is finite, we find, if $T = \bigcap_{h\in H} h''S$, that $G_{(T)} \leqslant H$ and also $H \leqslant G_{\{T\}}$. So G has the strong small index property.

Now suppose $W_0 \subseteq W_1 \subseteq \ldots \subseteq G$ is an exhaustive sequence of subsets. Then, by Proposition 6.18, there is an $n \in \mathbb{N}$ such that W_n^{10} contains an open subgroup $G_{(S)}$, where $S \subseteq \mathbb{N}^{<\mathbb{N}}$ is some finite tree. Take $g \in G$ such that $g''S \cap S = \{\emptyset\}$. Then for $m \geqslant n$ sufficiently big we have

$g, g^{-1} \in W_m$ and

$$G = G_{\{\emptyset\}} = G_{(S)} G_{(g''S)} G_{(S)} = G_{(S)} g G_{(S)} g^{-1} G_{(S)} \subseteq W_m^{32}.$$

This verifies the Bergman property.

Finally, the following is easily seen to be a dense locally finite subgroup of G:

$$K = \{g \in G : \exists m \exists \phi \text{ an automorphism of } m^{\leqslant m} \text{ such that if } u = s\hat{\ }t \in \mathbb{N}^{<\mathbb{N}}, \text{ where } s \text{ is the maximal initial segment of } u \text{ such that } s \in m^{\leqslant m}, \text{ then } g(u) = \phi(s)\hat{\ }t\}. \qquad \square$$

We denote by $\mathbf{T}$ the $\aleph_0$-regular tree on $\mathbb{N}$, that is, the tree in which each vertex has valency $\aleph_0$. Notice that $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}}) \cong \mathrm{Aut}(\mathbf{T}, a_0) = \mathrm{Aut}(\mathbf{T})_{a_0}$, for any $a_0 \in \mathbf{T}$. Hence, $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ is isomorphic to a clopen subgroup of $\mathrm{Aut}(\mathbf{T})$.

Corollary 6.32. *The group* $\mathrm{Aut}(\mathbf{T})$ *satisfies automatic continuity and has the small index property.*

This follows from the general fact that automatic continuity passes from an open subgroup to the whole group.

For the next couple of results, we need some more detailed information about the structure of group actions on trees. If g is an automorphism of a tree S that acts without inversion, then g either has a fixed point, in which case g is said to be *elliptic*, or g acts by translation on some line in the tree, in which case g is said to be *hyperbolic*. Serre's book [**44**] is a good reference for more information on these concepts.

Lemma 6.33. *Suppose that* $\ell_g = (a_i : i \in \mathbb{Z}) \subseteq \mathbf{T}$ *is a line and that* g *is a hyperbolic element of* $\mathrm{Aut}(\mathbf{T})$ *acting by translation on* ℓ_g *with amplitude* 1, *that is* $g \cdot a_i = a_{i+1}$, *for all* $i \in \mathbb{Z}$. *Then* $\mathrm{Aut}(\mathbf{T}) = \langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle$.

Proof. Suppose that t_0 is any element of $\mathbf{T}$. Then there is an $h \in \mathrm{Aut}(\mathbf{T}, a_0)$ such that $t_0 \in h \cdot \ell_g$ and so $hg^{d(t_0,a_0)} \cdot a_0 = t_0$. Put $k = hg^{d(t_0,a_0)}$; then

$$\mathrm{Aut}(\mathbf{T}, t_0) = k\,\mathrm{Aut}(\mathbf{T}, a_0)k^{-1} \subseteq \langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle. \tag{57}$$

This shows that $\langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle$ contains all elliptic elements of $\mathrm{Aut}(\mathbf{T})$.

Now, suppose k is any other hyperbolic element of $\mathrm{Aut}(\mathbf{T})$ acting by translation on a line ℓ_k with amplitude $m = \|k\|$. Let $\alpha = (b_n, \ldots, b_0)$ be the geodesic from ℓ_k to ℓ_g and $a_i = b_0$ be its endpoint.

There are two cases. Either $\ell_k \cap \ell_g \neq \emptyset$, in which case it is easy to find some $h \in \mathrm{Aut}(\mathbf{T}, a_i)$ such that $k = hg^m h^{-1} \in \langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle$. Otherwise, take some $f \in \mathrm{Aut}(\mathbf{T}, a_i)$ such that $f \cdot a_{i+j} = b_j$ for $j = 0, \ldots, n$. So

$$b_n = f \cdot a_{i+n} \in (f \cdot \ell_g) \cap \ell_k = \ell_{fgf^{-1}} \cap \ell_k. \tag{58}$$

Replacing g by fgf^{-1} we can repeat the argument above to find an $h \in \mathrm{Aut}(\mathbf{T}, b_n)$ such that $k = h(fgf^{-1})^m h^{-1} \in \langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle$. Therefore, $\langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle$ contains all hyperbolic elements of $\mathrm{Aut}(\mathbf{T})$.

We now only have the inversions left. So suppose k inverts an edge (a, b) of $\mathbf{T}$. Find some hyperbolic f with amplitude $\|f\| = 1$ such that its characteristic subtree ℓ_f passes through a and b with $f \cdot a = b$. Take also some elliptic $h \in \mathrm{Aut}(\mathbf{T}, b)$ such that $h \cdot (f \cdot b) = a$; then clearly

$$hfk \in \mathrm{Aut}(\mathbf{T}, a) \subseteq \langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle. \tag{59}$$

Therefore, $\mathrm{Aut}(\mathbf{T}) = \langle \mathrm{Aut}(\mathbf{T}, a_0) \cup \{g\} \rangle$. $\square$

THEOREM 6.34. *The group* $\mathrm{Aut}(\mathbf{T})$ *is of uncountable cofinality.*

Proof. Suppose that $H_0 \leqslant H_1 \leqslant \ldots \leqslant \mathrm{Aut}(\mathbf{T})$ exhausts $\mathrm{Aut}(\mathbf{T})$. Then evidently,

$$H_0 \cap \mathrm{Aut}(\mathbf{T}, t_0) \leqslant H_1 \cap \mathrm{Aut}(\mathbf{T}, t_0) \leqslant \ldots \leqslant \mathrm{Aut}(\mathbf{T}, t_0)$$

also exhausts $\mathrm{Aut}(\mathbf{T}, t_0)$ and thus by the uncountable cofinality of the latter, there is some n such that $\mathrm{Aut}(\mathbf{T}, t_0) \leqslant H_n$. Now, fix a line $\ell_g = (a_i : i \in \mathbb{Z}) \subseteq \mathbf{T}$ and a hyperbolic element $g \in \mathrm{Aut}(\mathbf{T})$ acting by translation on ℓ_g with amplitude 1, that is, $g \cdot a_i = a_{i+1}$, for all $i \in \mathbb{Z}$ such that $t_0 = a_0$. We know by Lemma 6.33 that $\mathrm{Aut}(\mathbf{T}) = \langle \mathrm{Aut}(\mathbf{T}, t_0) \cup \{g\} \rangle$, whence if $m \geqslant n$ is such that $g \in H_m$, also $\mathrm{Aut}(\mathbf{T}) = H_m$. □

REMARK. It is easy to see that $\mathrm{Aut}(\mathbf{T})$ fails property (FA). For we can just add a vertex to every edge of $\mathbf{T}$ and extend the action of $\mathrm{Aut}(\mathbf{T})$ to the tree obtained. We now see that $\mathrm{Aut}(\mathbf{T})$ acts without inversion, but does not fix a vertex. An exercise in Serre's book [**44**, p. 34] implies that $\mathrm{Aut}(\mathbf{T})$ is actually a non-trivial free product with amalgamation.

We shall now see that $\mathrm{Aut}(\mathbf{T})$ also satisfies the analogue of Neumann's Lemma. This is a special case of the following general fact.

PROPOSITION 6.35. *Suppose that G is a Polish group having an open subgroup K with ample generics. Then G satisfies Theorem* 6.14 *and Corollary* 6.16.

Proof. Suppose, towards a contradiction, that $\{g_i H_i\}_{i \in \mathbb{N}}$ covers G such that no H_i is open. Then as G has the small index property, every H_i has index $2^{\aleph_0}$ in G and hence when G acts by left translation on

$$X = G/H_0 \sqcup G/H_1 \sqcup \ldots,$$

any orbit is of size $2^{\aleph_0}$. Since K is open in G, there are $f_i \in G$ such that $\{f_i K\}_{i \in \mathbb{N}}$ cover G. So for any $x \in X$, $G \cdot x = \bigcup_i f_i K \cdot x$, whence every K-orbit is of size $2^{\aleph_0}$. Applying Theorem 6.16 to K, we find some $k \in K$ such that $k \cdot A \cap A = \emptyset$, where $A = \{H_i, g_i H_i\}_{i \in \mathbb{N}}$. Thus for each i, $kH_i \neq g_i H_i$, contradicting the fact that $\{g_i H_i\}_{i \in \mathbb{N}}$ covers G. Therefore, some H_i is open, which proves Theorem 6.14 for G. Theorem 6.16 for G now follows from the small index property and Theorem 6.14 for G. □

6.12. *Concluding remarks*

It is interesting to see that there seem to be two different paths to the study of automorphism groups of homogeneous structures. On the one hand, there are the methods of moieties dating back at least to Anderson [**4**] which have been developed and used by a great number of authors, and, on the other hand, there is the use of genericity.

The main tenet of this last section is that although ample genericity can sometimes be quite non-trivial to verify, it is nevertheless a sufficiently powerful tool for it to be worth the effort of looking for. In particular, it provides a uniform approach to proving the small index property, uncountable cofinality, property (FA), the Bergman property and automatic continuity.

6.13. *Some questions*

(1) Are there any examples of Polish groups that are not isomorphic to closed subgroups of S_∞ but have ample generic elements? Are there any that also have the small index property? (*Addendum.* It has now been verified by S. Solecki and the second author that the homeomorphism group of the reals indeed has the small index property, but of course is not a topological subgroup of S_∞.)

(2) Does $\mathrm{Aut}(\mathbf{B}_\infty)$ have ample generic elements?

(3) Can a Polish locally compact group have a comeager conjugacy class?

(4) Characterize the generic elements of $\mathrm{Aut}(\mathbf{B}_\infty)$ and $\mathrm{Aut}(\mathbf{F},\lambda)$, where $(\mathbf{F},\lambda)$ is as in Proposition 2.4. (*Added in proof.* We have recently received a preprint by Akin, Glasner and Weiss [**2**] in which they give another proof of the existence of a comeager conjugacy class in the homeomorphism group of the Cantor space, a proof that also gives an explicit characterization of the elements of this class.)

(5) Is the conjugacy action of $\mathrm{Iso}(\mathbf{U})$ turbulent? Does $\mathrm{Iso}(\mathbf{U}_0)$ have a dense locally finite subgroup (Vershik)?

(6) Suppose that a Polish group has ample generics and acts by homeomorphisms on a Polish space. Is the action necessarily continuous?

(7) Is property (T) somehow related to the existence of dense or comeager conjugacy classes? Concretely, if G is a Polish group, which is not the union of a countable chain of proper open subgroups and such that there is a diagonally dense conjugacy class in $G^{\mathbb{N}}$, does G have property (T)?

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