THE TREE PROPERTY AT $\aleph_{\omega+1}$

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Abstract. We show that given ω many supercompact cardinals, there is a generic extension in which there are no Aronszajn trees at $\aleph_{\omega+1}$. This is an improvement of the large cardinal assumptions. The previous hypothesis was a huge cardinal and ω many supercompact cardinals above it, in Magidor-Shelah [7].

1. Introduction

The tree property at κ^+ states that there are no Aronszajn trees at κ^+ i.e. that every κ^+-tree has an unbounded branch. In 1996, Magidor and Shelah in [7] showed the consistency of the tree property at $\aleph_{\omega+1}$. They start with a ground model with a huge cardinal and ω many supercompact cardinals above it. Here we reduce the large cardinal hypothesis. We present a proof for the consistency of the tree property at $\aleph_{\omega+1}$ starting only from ω many supercompact cardinals. Our construction is motivated by Gitik-Sharon [5] and Neeman [8]. In particular, we will show the following theorem:

Theorem 1. *Suppose that in V, $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals and GCH holds. Then there is a generic extension in which:*

(1) $\kappa_0 = \aleph_\omega$,
(2) *the tree property holds at* $\aleph_{\omega+1}$.

Furthermore, there is a bad scale at κ.

Scales play a central role in PCF theory. Let τ be a singular cardinal, $\tau = \sup_{\eta < \mathrm{cf}(\tau)} \tau_\eta$, where each τ_η is regular. A *scale of length* τ^+ is a sequence of functions $\langle f_\alpha \mid \alpha < \tau^+ \rangle$ from $\prod_{\eta < \mathrm{cf}(\tau)} \tau_\eta$ which is increasing and cofinal with respect to the eventual domination ordering. The scale $\langle f_\alpha \mid \alpha < \tau^+ \rangle$ is *good* if for almost all $\gamma < \tau^+$ with $\mathrm{cf}(\gamma) > \mathrm{cf}(\tau)$ there exists an unbounded set $A \subseteq \gamma$, such that $\{f_\beta(\eta) \mid \beta \in A\}$ is strictly increasing for all large η. If "unbounded in γ" is replaced by "club in γ", the scale is *very good. Bad scales* that are those that are not good. The existence of a bad scale implies the failure of the approachability property, and so it implies the failure of weak square. For more details on scales and their connection to singular cardinal combinatorics see Cummings-Foreman-Magidor [2], [3], and [4].

The rest of the paper presents the proof of Theorem 1. In section 2 we describe the forcing notion and some basic properties about the forcing. In section 3 we will show that there is a bad scale at κ in the resulting model. Section 4 deals with a preservation lemma, which will be used to show the tree property. Finally, in section 5 we prove that there are no Aronszajn trees at $\aleph_{\omega+1}$.

2. The forcing

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals. We start by using Laver's forcing to make κ_0 indestructably supercompact. Let V be the resulting model. Denote $\kappa = \kappa_0$, $\nu = \sup_n \kappa_n$ and $\mu = \nu^+$. First we force with the full support iterated collapse $\mathbb{C}$ to make each κ_n be the n-th successor of κ. Let H be $\mathbb{C}$-generic over V. Work in $V[H]$. Let U be a normal measure on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$. For each n, let U_n be the projection of U to $\mathcal{P}_\kappa(\kappa^{+n})$ and let $j_n = j_{U_n}$.

Proposition 2. *For each $n < \omega$ there exists a $(Col((\kappa^{+n+2})^{N_n}, < j_n(\kappa)))^{N_n}$ - generic filter, K_n, over $N_n = Ult(V[H], U_n)$.*

Proof. $Col((\kappa^{+n+2})^{N_n}, < j_n(\kappa))^{N_n}$ has at most κ^{+n+1} many antichains. Using that the poset is $< \kappa^{+n+1}$ closed and N_n is closed under sequences of length κ^{+n}, build a decreasing sequence of conditions $\langle q_i \mid i < \kappa^{+n+1} \rangle$ meeting these antichains. We use this sequence to construct a generic filter K_n in $V[H]$.

□

Fix generic filters $\langle K_n \mid 0 < n < \omega \rangle$ as in the above lemma. By a similar argument, we can choose a $(Col((\kappa^{+\omega+2})^{N_0}, < j_0(\kappa)))^{N_0}$ - generic filter, K_0, over $N_0 = Ult(V[H], U_0)$.

By standard arguments using the fact that κ is supercompact, we have that in $V[H]$, there is a bad scale $\langle g^*_\beta \mid \beta < \mu \rangle$ in $\prod_n \kappa^{+n+1}$ (see [9]). Furthermore we can fix a U_0 - measure one set $B \subset \kappa$, such that for all $\delta \in B$, $\langle g^*_\beta \mid \beta < \mu \rangle$ has stationary many bad points of cofinality $\delta^{+\omega+1}$:

Lemma 3. *Suppose $\langle g_\beta \mid \beta < \mu \rangle$ is a scale in $\prod_n \kappa^{+n+1}$. There exists a U_0-measure one set B, such that for all $\delta \in B$, there are stationary many bad points $\beta < \mu$ with $\mathrm{cf}(\beta) = \delta^{+\omega+1}$.*

Proof. The proof builds on parts of the arguments in [9]. Let $B = \{\delta < \kappa \mid$ there are stationary many bad points with cofinality $\delta^{+\omega+1}\}$. Suppose for contradiction that B is not measure one. Then $A = \kappa \setminus B \in$

U_0. For each $\delta \in A$ fix a club C_δ in μ, such that all points in C_δ with cofinality $\delta^{+\omega+1}$ are good for the scale. Let $C = \bigcap_{\delta \in A} C_\delta$, which is also club in μ since $\mathrm{cf}(\mu) = \mu > \kappa$.

Recall that U_0 is the projection of U to κ, where U is a normal measure on $\mathcal{P}_\kappa(\mu)$. Let $j = j_U : V[H] \longrightarrow M$ and let $\rho = \sup(j"\mu)$. Then we have

$$M \models \rho \in j(C), \mathrm{cf}(\rho) = \mathrm{cf}(\mu) = \kappa^{+\omega+1}$$

and since $\kappa \in j(A)$ we have by elementarity that $M \models \rho$ is good.

Define f to be the function $\alpha \mapsto \sup(j"\kappa^{+n+1})$; we claim that f is an exact upper bound for $\langle j(g)_\eta \mid \eta < \rho \rangle$ with non-uniform cofinality: If $\eta < \rho$, let $\beta < \mu$ be such that $\eta < j(\beta)$. Then $j(g)_\eta <^* j(g)_{j(\beta)} = j(g_\beta) <^* f$ since for each $n < \omega$, $j(g_\beta)(n) = j(g_\beta(n)) < \sup(j"\kappa^{+n+1})$. Also, if $h <^* f$, without loss of generality assume that for all n, $h(n) < \sup(j"\kappa^{+n+1})$. Define $\overline{h} \in \prod_n \kappa^{+n+1}$ by $\overline{h}(n)$ to be the least $\gamma < \kappa^{+n+1}$ such that $h(n) < j(\gamma)$. Let $\beta < \mu$ be such that $\overline{h} <^* g_\beta$. Then $h <^* j(g)_{j(\beta)}$.

It follows that ρ cannot be good. Contradiction. $\square$

Using standard reflection arguments, we choose sets $X_n \in U_n$ for $n < \omega$ with $X_0 \subset B$, such that for all $x \in X_n$:

- $\kappa \cap x = \kappa_x$ is κ_x^{+n} -supercompact.
- $(\forall k \leq n) o.t.(x \cap \kappa^{+k}) = \kappa_x^{+k}$. In particular, $o.t.(x) = \kappa_x^{+n}$.

We are ready to define the main forcing. Basically, we take the Gitik-Sharon forcing and add collapses using the filters $\langle K_n \mid n < \omega \rangle$.

Definition 4. *Conditions in $\mathbb{P}$ are of the form $p = \langle d, \langle p_n \mid n < \omega \rangle \rangle$, where setting $l = lh(p)$, we have:*

(1) *For $0 \leq n < l$, $p_n = \langle x_n, c_n \rangle$ such that:*
  - *$x_n \in \mathcal{P}_\kappa(\kappa^{+n})$, $x_n \in X_n$ and for $i < n$, $x_i \prec x_n$,*
  - *$c_0 \in Col(\kappa_{x_0}^{+\omega+2}, < \kappa_{x_1})$ if $1 < l$, and if $l = 1$, $c_0 \in Col(\kappa_{x_0}^{+\omega+2}, < \kappa)$.*
  - *if $1 < l$, for $0 < n < l-1$, $c_n \in Col(\kappa_{x_n}^{+n+2}, < \kappa_{x_{n+1}})$, and $c_{l-1} \in Col(\kappa_{x_{l-1}}^{+l+1}, < \kappa)$.*

(2) *For $n \geq l$, $p_n = \langle A_n, C_n \rangle$ such that:*
  - *$A_n \in U_n$, $A_n \subset X_n$, and $x_{l-1} \prec y$ for all $y \in A_n$.*
  - *C_n is a function with domain A_n, for $y \in A_n$, $C_n(y) \in Col(\kappa_y^{+n+2}, < \kappa)$ if $n > 0$, and $C_n(y) \in Col(\kappa_y^{+\omega+2}, < \kappa)$ if $n = 0$,*
  - *$[C_n]_{U_n} \in K_n$.*

(3) *if $l > 0$, then $d \in Col(\omega, \kappa_{x_0}^{+\omega})$, otherwise $d \in Col(\omega, \kappa)$.*

Here $x \prec y$ denotes that $x \subset y$ and $|x| < \kappa_y$. For a condition p, we will use the notation $p = \langle d^p, \langle p_n \mid n < \omega \rangle \rangle$, $p_n = \langle x^p_n, c^p_n \rangle$ for $0 \leq n < lh(p)$, and $p_n = \langle A^p_n, C^p_n \rangle$ for $n \geq lh(p)$. The stem of p is $h = \langle d^p, \langle p_n \mid n < lh(p) \rangle \rangle$. Sometimes we will also denote the stem of p by $\langle d^p, \langle \vec{x}, \vec{c} \rangle \rangle$, where $\vec{x}$ and $\vec{c}$ are with length $lh(p)$, and for $i < lh(p)$, $p_i = \langle x_i, c_i \rangle$.

$q = \langle d^q, \langle q_n \mid n < \omega \rangle \rangle \leq p = \langle d^p, \langle p_n \mid n < \omega \rangle \rangle$ if $lh(q) \geq lh(p)$ and:

- *$d^q \leq d^p$,*
- *for all $n < lh(p)$, $x^p_n = x^q_n, c^q_n \leq c^p_n$,*
- *for $lh(p) \leq n < lh(q)$, $x^q_n \in A^p_n$ and $c^q_n \leq C^p_n(x^q_n)$,*
- *for $n \geq lh(p)$, $A^q_n \subset A^p_n$ and for all $y \in A^q_n$, $C^q_n(y) \leq C^p_n(y)$*

We say that q is a direct extension of p, denoted by $q \leq^* p$, if $q \leq p$ and $lh(q) = lh(p)$. For two stems h_1 and h_2, we say that h_1 is stronger or an extension of h_2 if there are conditions $p_1 \leq p_2$ with stems h_1 and h_2 respectively.

Let G be $\mathbb{P}$ generic over $V[H]$, and let $\langle x_n \mid n < \omega \rangle$, where each $x_n \in \mathcal{P}_\kappa(\kappa^{+n})$, be the added generic sequence. Set $\lambda_n = x_n \cap \kappa$. By adapting the arguments in [5] to our situation, we get:

Proposition 5.

(1) *If $\langle A_n \mid n < \omega \rangle \in V[H]$ is a sequence of sets such that every $A_n \in U_n$, then for all large n, $x_n \in A_n$.*
(2) *$\bigcup_n x_n = \nu = (\kappa^{+\omega})^{V[H]}$.*
(3) *For each $n \geq 0$, the cofinality of $\kappa_n = (\kappa^{+n})^{V[H]}$ in $V[H][G]$ is ω.*
(4) *Since any two conditions with the same stem are compatible, $\mathbb{P}$ has the $\mu = \kappa^{+\omega+1}$ chain condition. So, cardinals greater than or equal to $\kappa^{+\omega+1}$ are preserved.*
(5) *$\mathbb{P}$ has the Prikry property. I.e. if p is a condition with length at least 1 and ϕ is a formula, then there is a direct extension $p' \leq^* p$ which decides ϕ.*

Remark 6. The main point in the proof of the Prikry property is the diagonal lemma, which states that for $p \in \mathbb{P}$ with length at least 1 and $lh(p) < n < \omega$ if H is a set of stems of length $lh(p)+n$ and $\langle q^h \mid h \in H \rangle$ are conditions stronger than p such that each q^h has a stem h, then there is $q \leq^* p$ such that if $r \leq q$ is a condition of length at least $lh(p) + n$, then $r \leq q^h$ for some $h \in H$.

Remark 7. Using the closure of the collapsing posets, we get the following corollary to the Prikry property: If p is a condition with length

$n+1$ and ϕ is a formula, then there is a direct extension $q \leq^* p$, such that $q \restriction n = p$ and if $r \leq p$ decides ϕ, then $r \restriction n^\frown q \restriction (\omega \setminus n)$ decides ϕ.

The last property implies that all cardinals χ, such that $\lambda_n \leq \chi \leq \lambda_n^{+n+2}$ for some $n > 0$ are preserved. In particular, in $V[H][G]$ every λ_n for $n > 0$ is a cardinal. And so, $\mathbb{P}$ preserves κ. Also, $\mathbb{P}$ preserves $(\lambda_0^{+\omega+1})^{V[H]}$, $(\lambda_0^{+\omega+2})^{V[H]}$, and collapses $(\lambda_0^{+\omega})^{V[H]}$ to ω. So in $V[H][G]$, $\kappa = \aleph_\omega$, $\mu = \kappa^+$, $(\lambda_0^{+\omega+1})^{V[H]} = \aleph_1$.

Before we turn our attention to the tree property, we will show that there is a bad scale in $V[H][G]$.

3. The bad scale

In this section we will show that there is a bad scale in $V[H][G]$. From this it follows that weak square fails. The construction is motivated by a similar construction of a bad scale in Cummings-Foreman [1]. We start with a couple of propositions.

Proposition 8. *Suppose that in $V[H][G]$, $f \in \prod_n \lambda_n^{+n+1}$. Then there is a sequence of functions $\langle H_n \mid n < \omega\rangle$ in $V[H]$, such that* $\mathrm{dom}(H_n) =$ *X_n, $H_n(x) < \kappa_x^{+n+1}$ for all x, and $f(n) < H_n(x_n)$ for all large n.*

Proof. The proof builds on an argument in [1]. Let p force that $\dot{f}$ is as in the statement of the lemma. For simplicity assume that the length of p is 1. Denote $p = \langle d^p, \langle p_n \mid n < \omega\rangle\rangle$ and $p_n = \langle A_n, C_n\rangle$ for $n > 0$.

Fix $0 < n < \omega$ and $x \in A_n$. For any stem $h = \langle d^p, \langle \vec{z}, \vec{c}\rangle\rangle$ of length $n+1$ with $z_n = x$ and $c_i = C_i^p(z_i)$ for $i \leq n$, such that there is a condition stronger than p with this stem, let $p^h \leq p$ be a condition with this stem. Then $p^h \Vdash \dot{f}(n) < \kappa_x^{+n+1}$. The closure of $Col(\kappa_x^{+n+2}, < \kappa)$ is more than the number of stems of length n that are stronger than $h \restriction n$. So, by the Prikry property we can build a decreasing sequence $\langle q^\gamma \mid \gamma < \kappa_x^{+n+1}\rangle$, such that for each γ, $q^\gamma \leq^* p^h$ is such that $q^\gamma \restriction n = p^h \restriction n$, and if some $r \leq q^\gamma$ decides "$\dot{f}(n) = \gamma$", then $r \restriction n^\frown q^\gamma \restriction (\omega \setminus n)$ decides $\dot{f}(n) = \gamma$ (see Remark 7 from the previous section).

Let q^h be a direct extension of every q^γ, for $\gamma < \kappa_x^{+n+1}$. Again we use that the closure of $Col(\kappa_x^{+n+2}, < \kappa)$ is big enough. Let $c^h \in Col(\kappa_x^{+n+2}, < \kappa)$ be such that $q_n^h = \langle x, c^h\rangle$. Using induction, we define the q^h's so that the c^h's are decreasing according to some enumeration. Set c_x to be stronger than each c^h, where h is a stem of length $n+1$ ending in $\langle x, C_n^p(x)\rangle$. Define $H_n(x) = \sup\{\gamma < \kappa_x^{+n+1} \mid (\exists q) lh(q) = n+1, q_n = \langle x, c_x\rangle \text{ and } q \Vdash \dot{f}(n) = \gamma\} + 1$. Since the number of such q's is κ_x^{+n}, we have that $H_n(x) < \kappa_x^{+n+1}$.

Apply the diagonal lemma to q^h, for all stems h of length $n+1$ as considered above, to get a condition $q^n \leq^* p$ such that if $r \leq q^n$ has length at least $n+1$, then for some h, $r \leq q^h$ (see Remark 6 from the previous section). Then $q^n \Vdash \dot{f}(n) < H(\dot{x}_n)$. Note that since for each h, $d^{q^h} = d^p$, then $d^{q^n} = d^p$. Let q be stronger than each q^n. Then q forces that $\langle H_n \mid n < \omega \rangle$ is as desired.

$\square$

Proposition 9. *Suppose $V[H][G] \models A \subset ON, o.t.(A) = \aleph_1$. Then there is a set $B \in V[H]$ such that B is an unbounded subset of A.*

Proof. As before, for $p \in \mathbb{P}$, we denote $p = \langle d^p, \langle p_n \mid n < \omega \rangle \rangle$.

For $p \in \mathbb{P}$ let $A_p = \{\alpha \mid p \Vdash \alpha \in \dot{A}\}$; $A = \bigcup_{p \in G} A_p$. Let $\delta < \kappa$ and $q \in G$ with length 1 be such that $q \Vdash \dot{\lambda}_0 = \delta$. Let $\tau = (\delta^{+\omega+1})^{V[H]}$, then $q \Vdash \dot{\aleph}_1 = \tau$. Work below q. Fix $n > 0$, $d \in Col(\omega, \delta^{+\omega})$ such that $\bigcup_{p \in G, lh(p)=n, d^p=d} A_p = A'$ is unbounded in A. In $V[H][G]$ let $h : \tau \to A'$ enumerate A'. Then by definition of A', for each $\gamma < \tau$, we can fix $p_\gamma \in G$ with length n and $d^{p_\gamma} = d$ deciding a value for $h(\gamma)$. Let p be stronger than every p_γ. Then p decides h. $\square$

Next we define the bad scale in $V[H][G]$. Recall that $\langle g^*_\beta \mid \beta < \mu \rangle \in \prod_n \kappa^{+n+1}$ is a bad scale in $V[H]$ fixed in advance, so that it has stationary (in $V[H]$) many bad points of cofinality $\lambda_0^{+\omega+1} = (\aleph_1)^{V[H][G]}$. For all $n < \omega$ and $\eta < \kappa^{+n+1}$, fix $f^\eta_n : X_n \longrightarrow V[H]$, such that $\forall x f^\eta_n(x) < \kappa_x^{+n+1}$, and $[f^\eta_n]_{U_n} = \eta$. Define in $V[H][G]$, $\langle g_\beta \mid \beta < \mu \rangle$ in $\prod_n \lambda_n^{+n+1}$ by:

$$g_\beta(n) = f_n^{g^*_\beta(n)}(x_n)$$

Lemma 10. *$\langle g_\beta \mid \beta < \mu \rangle$ is a bad scale in $V[H][G]$, and so weak square fails.*

Proof. First we check that $\langle g_\beta \mid \beta < \mu \rangle$ is a scale. If $\beta < \gamma < \mu$, then for all large n, $[f_\alpha^{g^*_\beta(n)}]_{U_n} = g^*_\beta(n) < g^*_\gamma(n) = [f_n^{g^*_\gamma(n)}]_{U_n}$. By (1) of Proposition 5, it follows that for all large n, $g_\beta(n) = f_n^{g^*_\beta(n)}(x_n) < f_n^{g^*_\gamma(n)}(x_n) = g_\gamma(n)$, so $\langle g_\beta \mid \beta < \mu \rangle$ is increasing.

To prove that $\langle g_\beta \mid \beta < \mu \rangle$ is cofinal, suppose that in $V[H][G]$, $h \in \prod_n \lambda_n^{+n+1}$. Fix $\langle H_n \mid n < \omega \rangle \in V[H]$ as in the conclusion of Proposition 8. In $V[H]$, define $h^* \in \prod_n \kappa^{+n+1}$ by $h^*(n) = [H_n]_{U_n}$. Let $\beta < \mu$ be such that for all large n, $h^*(n) < g^*_\beta(n)$. Then for all large n, $[H_n]_{U_n} = h^*(n) < g^*_\beta(n) = [f_n^{g^*_\beta(n)}]_{U_n}$, and so for all large n, $h(n) < H_n(x_n) < f_n^{g^*_\beta(n)}(x_n) = g_\beta(n)$.

To show that the scale is not good we need the following claim.

Claim 11. *Suppose that γ is a good point in $V[H][G]$ for $\langle g_\beta \mid \beta < \mu\rangle$ with cofinality $\aleph_1$ in $V[H][G]$. Then γ is a good point in $V[H]$ for $\langle g^*_\beta \mid \beta < \mu\rangle$.*

Proof. Setting $\tau = (\aleph_1)^{V[H][G]} = (\lambda_0^{+\omega+1})^{V[H]}$, note that $\mathrm{cf}(\gamma) = \tau$ in both $V[H]$ and $V[H][G]$. Fix unbounded $A^* \subset \gamma$, and $n < \omega$ witnessing goodness of γ in $V[H][G]$. Then by Proposition 9, there is an unbounded set $A \subset A^*$ in $V[H]$. Then A and n witness goodness, so let $p \in G$ be such that $p \Vdash (\forall k > n)\langle \dot g_\beta(k) \mid \beta \in A\rangle$ is increasing.

Subclaim 12. $\forall k > \max(n, lh(p))$, *for U_k-almost all $y \in A^p_k$, we have that $\langle f_k^{g^*_\beta(\alpha)}(y) \mid \beta \in A\rangle$ is increasing.*

Proof. Otherwise, for some $k > \max(n, lh(p))$, we can find $y_k \in A^p_k$, such that $\langle f_k^{g^*_\beta(k)}(y_k) \mid \beta \in A\rangle$ is not increasing and a condition $q \leq p$, such that $k < lh(q)$ and $q_k = \langle y_k, c\rangle$ for some c. Then:

- $(\forall \beta < \mu) q \Vdash \dot g_\beta(k) = f_k^{g^*_\beta(k)}(y_k)$,
- $q \leq p \Rightarrow q \Vdash \langle \dot g_\beta(k) \mid \beta \in A\rangle$ is increasing.

But we assumed that $\langle f_k^{g^*_\beta(k)}(y_k) \mid \beta \in A\rangle$ is not increasing. Contradiction. $\square$

So, we have that for all large k, $\langle [f_k^{g^*_\beta(k)}]_{U_k} \mid \beta \in A\rangle$ is increasing, and so for all large k, $\langle g^*_\beta(k) \mid \beta \in A\rangle$ is increasing. Thus, γ is good for $\langle g^*_\beta \mid \beta < \mu\rangle$ in $V[H]$. $\square$

Then since $\mathbb{P}$ has the μ chain condition and there are stationary many bad points with cofinality $\lambda_0^{+\omega+1}$ in $V[H]$ for $\langle g^*_\gamma \mid \gamma < \mu\rangle$, we get that $\langle g_\gamma \mid \gamma < \mu\rangle$ is not good. $\square$

4. The preservation lemma

In this section we prove a preservation lemma, which will be used to show the tree property. The proof of this lemma is motivated by the Preservation Theorem in Magidor-Shelah [7]. The main difference is that instead of trees, here we are working with narrow systems, which are defined below. Throughout this section V will denote some arbitrary ground model. We start with defining the notion of a narrow system.

Definition 13. *$S = \langle I, \mathcal{R}\rangle$ is a narrow system of height ν^+ and levels of size $\kappa < \nu$ if:*

- *I is an unbounded subset of ν^+, and for each $\alpha \in I$, $S_\alpha = \{\alpha\} \times \kappa$ is the α-level of S,*
- *$\mathcal{R}$ is a set of transitive binary relations on S, such that $|\mathcal{R}| < \nu$,*

- *for every $\alpha < \beta$ in I, there are $u \in S_\alpha$, $v \in S_\beta$, and $R \in \mathcal{R}$ such that $\langle u, v\rangle \in R$,*
- *for all $R \in \mathcal{R}$, if u_1, u_2 are distinct nodes such that $\langle u_1, v\rangle \in R$ and $\langle u_2, v\rangle \in R$, then $\langle u_1, u_2\rangle \in R$ or $\langle u_2, u_1\rangle \in R$.*

For $a_1, a_2 \in S$ and $R \in \mathcal{R}$ we write $a_1 \perp_R a_2$ if $\langle a_1, a_2\rangle \notin R$ and $\langle a_2, a_1\rangle \notin R$, and in that case say that a_1, a_2 are R-incomparable.

A **branch** *of S is a set $b \subset \bigcup_{\alpha\in I} S_\alpha$ such that for every α, $|b \cap S_\alpha| \leq 1$, and for some $R \in \mathcal{R}$, we have that for all $u, v \in b$, $\langle u, v\rangle \in R$ or $\langle v, u\rangle \in R$. In this case we say that b is a branch through R (or with respect to R). We say that b is unbounded if for unboundedly many $\alpha \in I$, $b \cap S_\alpha \neq \emptyset$.*

Theorem 14. *Suppose that ν is a singular cardinal of cofinality ω and $S = \langle I, \mathcal{R}\rangle$ is a narrow system in V of height ν^+, levels of size κ, and with $|\mathcal{R}| = \tau$, where $\kappa, \tau < \nu$. Suppose also that $\mathbb{R}$ is a $<\chi$ closed notion of forcing where $\chi > \max(\kappa,\tau)^+$, and let F be $\mathbb{R}$-generic over V. Suppose that in $V[F]$ there are (not necessarily all unbounded) branches $\langle b_{R,\delta} \mid R \in \mathcal{R},\ \delta < \kappa\rangle$, such that:*

1. *every $b_{R,\delta}$ is a branch through R, and for some $\langle R, \delta\rangle \in \mathcal{R} \times \kappa$, $b_{R,\delta}$ is unbounded;*
2. *for all $\alpha \in I$, there is $\langle R, \delta\rangle \in \mathcal{R} \times \kappa$, such that $S_\alpha \cap b_{R,\delta} \neq \emptyset$.*

Then S has an unbounded branch in V.

Proof. Let λ be a regular cardinal such that $\max(\kappa, \tau) < \lambda < \chi$. First we define a splitting property and prove a lemma.

Definition 15. *Let $r \in \mathbb{R}$. The pair $\langle R, \delta\rangle \in \mathcal{R} \times \kappa$ has the* **splitting property** *below r if for all $\{r_i \mid i < \lambda\} \subset \mathbb{R}$ where each $r_i \leq r$, there is, in V, a set $\{a_i \mid i < \lambda\}$ of nodes of S, such that:*

- *for all $i < \lambda$, there is $r'_i \leq r_i$ with $r'_i \Vdash a_i \in \dot{b}_{R,\delta}$,*
- *for all $i \neq j$ in λ, $a_i \perp_R a_j$.*

Lemma 16. *(Splitting Lemma) Suppose that $r \Vdash$ "$\dot{b}_{R,\delta}$ is an unbounded branch", and suppose S has no unbounded branch in V through R. Then the splitting property below r holds for $\langle R, \delta\rangle$.*

Proof. Let $\{r_i \mid i < \lambda\}$ be conditions stronger than r. For $i < \lambda$, set $E_i = \{a \in S \mid (\exists r \leq r_i) r \Vdash a \in \dot{b}_{R,\delta}\}$. Then $E_i \cap S_\alpha \neq \emptyset$ for unboundedly many α's in I since r_i forces that $\dot{b}_{R,\delta}$ is unbounded. Actually for all $a \in E_i$, $\{c \in E_i \mid \langle a, c\rangle \in R\} \cap S_\alpha \neq \emptyset$ for unboundedly many α's in I. Also, for all $a \in E_i$, there are c_1, c_2 in E_i with $\langle a, c_1\rangle \in R$, $\langle a, c_2\rangle \in R$ such that $c_1 \perp_R c_2$. This is because otherwise $E_i^a = \{c \in E_i \mid \langle a, c\rangle \in R\}$ will be an unbounded branch in V through R. So we get the following claim:

Claim 17. *For every i, there is a sequence $\{c_\eta \mid \eta < \lambda\}$ of pairwise R incomparable nodes in E_i.*

Proof. Fix $i < \lambda$. Let $C \subset \nu^+$ be a club such that for all $\alpha \in C$, if $\gamma \in I \cap \alpha$ and $a \in E_i \cap S_\gamma$, then there are R-incomparable nodes c_1, c_2 in $E_i \cap \bigcup_{\beta \in I \cap \alpha} S_\beta$ with $\langle a, c_1\rangle \in R$ and $\langle a, c_1\rangle \in R$.

Inductively build sequences $\langle \gamma_\eta \mid \eta < \lambda\rangle$, $\langle \alpha_\eta \mid \eta < \lambda\rangle$, $\langle a_\eta \mid \eta < \lambda\rangle$, and decreasing $\langle r_\eta \mid \eta < \lambda\rangle$ in $\mathbb{R}$ stronger than r_i, such that:

- for each η, $\gamma_\eta \in C$, $\alpha_\eta < \gamma_\eta$, and $a_\eta \in E_i$ is a node of level α_η;
- $\langle a_\eta \mid \eta < \lambda\rangle$ are pairwise R-comparable and each $r_\eta \Vdash a_\eta \in \dot{b}_{R,\delta}$.

Suppose $\rho < \lambda$ and we have build these sequences for all $\eta < \rho$. Let $\alpha = \sup_\eta \alpha_\eta$ and r^*_ρ be stronger than all r_η. Then since r forces $\dot{b}_{R,\delta}$ to be unbounded, choose $r_\rho \leq r^*_\rho$, $\alpha_\rho > \alpha$ and $a_\rho \in S_\alpha$ such that $r_\rho \Vdash a_\rho \in \dot{b}_{R,\delta}$. Then let $\gamma_\rho \in C$ be greater than α_ρ.

Then, since each a_η is of level below $\gamma_\eta \in C$, we build R-incomparable nodes $\langle c_\eta \mid \eta < \lambda\rangle$, such that for every η, there is η' with $\langle a'_\eta, c_\eta\rangle \in R$ and c_η is of level below $\gamma_{\eta'}$.

□

For each $i < \lambda$, let $\{a^\eta_i \mid \eta < \lambda\}$ be R incomparable nodes in E_i and let $\alpha_i < \nu^+$ be such that each a^η_i is of level below α_i. Let $\alpha \in I$ be such that $\alpha \geq \sup_{i<\nu} \alpha_i$. We will build inductively pairwise R-incomparable nodes $\{a_i \mid i < \lambda\}$ of levels above α, such that each a_i is in E_i. Suppose that we have defined $\{a_i \mid i < j\}$ where $j < \lambda$. For each $i < j$, since $\{a^\eta_j \mid \eta < \lambda\}$ are R incomparable and appear at levels less than the level of a_i, we have that there is at most one $\eta < \lambda$ such that $\langle a^\eta_j, a_i\rangle \in R$. For $i < j$, let η_i be the unique such η if it exists, and $\eta_i = 0$ otherwise. Chose $\eta \in \lambda \setminus \{\eta_i \mid i < j\}$. Then for all $i < j$, $a^\eta_j \perp_R a_i$. Let $a_j \in E_j$ be such that $\langle a^\eta_j, a_j\rangle \in R$ and the level of a_j is above α. Then for all $i < j$, $a_i \perp_R a_j$.

□

We return to the proof of the theorem. Suppose for contradiction that S has no unbounded branch in V. Working in $V[F]$, let $B = \{\langle R, \delta\rangle \in \mathcal{R} \times \kappa \mid b_{\langle R,\delta\rangle}$ is bounded $\}$. Note that since $\mathbb{R}$ is closed enough, $B \in V$. Let $g : B \to \nu^+$ be such that for all $\langle R, \delta\rangle \in B$, $b_{\langle R,\delta\rangle} \subset \bigcup_{\alpha < g(\langle R,\delta\rangle)} S_\alpha$ i.e. $b_{\langle R,\delta\rangle}$ is bounded by $g(\langle R, \delta\rangle)$. Let $\beta_0 = \sup(\mathrm{ran}(g)) < \nu^+$. So, in $V[F]$, each bounded $b_{\langle R,\delta\rangle}$ is below β_0.

Let $\{r_{\langle R,\delta\rangle} \mid \langle R, \delta\rangle \in \mathcal{R} \times \kappa\}$ be a sequence of conditions in F, such that for each $\langle R, \delta\rangle$, either $r_{\langle R,\delta\rangle} \Vdash$ "$\dot{b}_{R,\delta}$ is unbounded", or $r_{\langle R,\delta\rangle} \Vdash$ "$\dot{b}_{R,\delta} \subset \bigcup_{\alpha \in I \cap \beta_0} S_\alpha$". Using the closure of $\mathbb{R}$, choose a condition $r \in F$ which is stronger than every $r_{\langle R,\delta\rangle}$ and let $U = \{\langle R, \delta\rangle \in \mathcal{R} \times \kappa \mid$

$r \Vdash \dot{b}_{\langle R,\delta\rangle}$ is unbounded $\}$. Then $U \in V$ and by the assumption of the theorem, U is nonempty. Also by (2) from the assumptions of the theorem, by strengthening r if necessary, we have that:

$$(\dagger)\ r \Vdash \text{“for all } \alpha \in I \setminus \beta_0, \text{ there is } \langle R,\delta\rangle \in U \text{ such that } S_\alpha \cap \dot{b}_{R,\delta} \neq \emptyset\text{”}.$$

We will define a sequence $\langle r_i \mid i < \lambda\rangle$ as follows. For every $\langle R,\delta\rangle \in U$ we build conditions $\langle r_i^{R,\delta} \mid i < \lambda\rangle$ stronger than r, such that:

(1) for every i, $\langle r_i^{R,\delta} \mid \langle R,\delta\rangle \in U\rangle$ is decreasing according to some enumeration of U,
(2) for every $\langle R,\delta\rangle \in U$, there are pairwise R-incomparable nodes $\langle u_i \mid i < \lambda\rangle$ in S, such that: $r_i^{R,\delta} \Vdash u_i \in \dot{b}_{R,\delta}$ for each i. Denote $u_i = \langle \beta_i^{R,\delta}, \xi_i^{R,\delta}\rangle$. Here we use the Splitting Lemma and the fact that for every $\langle R,\delta\rangle \in U$, $r \Vdash$ “$\dot{b}_{R,\delta}$ is unbounded” to find these nodes.

Then for every $i < \lambda$, set r_i to be stronger than all of $\langle r_i^{R,\delta} \mid \langle R,\delta\rangle \in U\rangle$.

Let $\beta \in I \setminus \beta_0$ be such that $\beta \geq \sup_{i,R,\delta} \beta_i^{R,\delta}$. For all $i < \lambda$, let $r_i' \leq r_i$ be such that for some ξ_i, R_i, δ_i with $\langle R_i, \delta_i\rangle \in U$,

$$r_i' \Vdash \langle \beta, \xi_i\rangle \in \dot{b}_{R_i,\delta_i}.$$

We can find such r_i' by $(\dagger)$.

Since the size of U is less than λ, for some $\xi < \kappa$ and $\langle R,\delta\rangle \in U$, there are distinct $i < j < \lambda$, such that $\xi_i = \xi_j = \xi$, $R_i = R_j = R$ and $\delta_i = \delta_j = \delta$. Then setting $u = \langle \beta_i^{R,\delta}, \xi_i^{R,\delta}\rangle$ and $v = \langle \beta_j^{R,\delta}, \xi_j^{R,\delta}\rangle$ we have that:

- $r_i \Vdash u \in \dot{b}_{R,\delta}$; $r_i \Vdash \langle \beta, \xi\rangle \in \dot{b}_{R,\delta}$
- $r_j \Vdash v \in \dot{b}_{R,\delta}$; $r_j \Vdash \langle \beta, \xi\rangle \in \dot{b}_{R,\delta}$
- $u \perp_R v$
- u, v are of levels less than β.

That is a contradiction. □

5. The tree property

In this section we will show that there is a $\mathbb{P}$-generic filter over $V[H]$, G, such that in $V[H][G]$ the tree property holds at $\aleph_{\omega+1}$. Given a $\mathbb{P}$ - name $\dot{T}$ in $V[H]$ for a $(\nu^+)^{V[H]}$ tree with levels of size at most κ, we denote the α-th level of T by T_α. We may assume that $T_\alpha = \{\alpha\} \times \kappa$ for $\alpha < \nu^+$. Also throughout this section ν^+ will always denote $(\nu^+)^V = (\nu^+)^{V[H]}$ (which becomes $\aleph_{\omega+1}$ in $V[H][G]$ for any $\mathbb{P}$-generic filter G). The outline of our proof is motivated by Neeman [8]. The main difference is that we have to deal with the poset $\mathbb{C}$ and rely on the Preservation Lemma from last section.

We will make use of the following standard fact. We include the proof for completeness.

Proposition 18. *Suppose that $\mathbb{Q}$ is a poset, $D \subset \mathbb{Q}$ is an open dense set, $\phi(x)$ is a formula, and $\langle \tau_p \mid p \in D\rangle$ are $\mathbb{Q}$ names such that for all $p \in D$, $p \Vdash \phi(\tau_p)$. Then there is a $\mathbb{Q}$ name τ, such that $1_{\mathbb{Q}} \Vdash \phi(\tau)$.*

Proof. Let $A \subset D$ be a maximal antichain. Define a $\mathbb{Q}$-name, $\tau = \{\langle u, r\rangle \mid (\exists p \in A)(r \leq p, u \in \mathrm{dom}(\tau_p), r \Vdash u \in \tau_p)\}$. Then for any $\mathbb{Q}$ - generic F, if $p \in F \cap A$, then $\tau_F = (\tau_p)_F$. It follows that $1_{\mathbb{Q}} \Vdash \phi(\tau)$. □

Now, suppose for contradiction that for any $\mathbb{P}$-generic filter over $V[H]$, G, in $V[H][G]$ the tree property does not hold at $\aleph_{\omega+1}$. Then the set $D = \{p \in \mathbb{P} \mid (\exists \dot{T})(p \Vdash \dot{T}$ is an Aronszajn tree at $\dot{\aleph}_{\omega+1})\}$ is dense. Using the above proposition, we can fix a $\mathbb{P}$-name $\dot{T}$, such that the empty condition forces that $\dot{T}$ is an Aronszajn tree at $\dot{\aleph}_{\omega+1}$.

Lemma 19. *There is $n < \omega$ and an unbounded $I \subset \nu^+$ in $V[H]$, such that for all $\alpha < \beta$ in I, there are $\xi, \delta < \kappa$ and a condition q with length n, such that $q \Vdash \langle \alpha, \xi\rangle <_{\dot{T}} \langle \beta, \delta\rangle$.*

Proof. Recall that U was the normal measure on $\mathcal{P}_\kappa(\nu^+)$ fixed in advance and each U_n is the projection of U to $\mathcal{P}_\kappa(\kappa_n)$. Let $j = j_U : V[H] \to M$. Let G^* be $j(P)$ - generic over M and $T^* = j(\dot{T})_{G^*}$ be such that the first element of the generic sequence added by G^* is κ. We can arrange that since $\kappa \in j(X_0)$. Then ν^+ is preserved by G^*.

Fix a node $u \in T^*$ of level γ, where $\sup(j''\nu^+) < \gamma < j(\nu^+)$. Then for all $\alpha < \nu^+$ let $\xi_\alpha < j(\kappa)$ be such that $\langle j(\alpha), \xi_\alpha\rangle <_{T^*} u$ and $p_\alpha \in G^*$ be such that $p_\alpha \Vdash \langle j(\alpha), \xi_\alpha\rangle <_{j(\dot{T})} u$. Then in $M[G^*]$ there is an unbounded $I^* \subset \nu^+$ and a fixed n, such that for all $\alpha \in I^*$, p_α has length n.

Denoting $p_\alpha = \langle d_\alpha, \langle p_{\alpha i} \mid i < \omega\rangle\rangle$, by further shrinking I^* we can assume that for some $d \in Col(\omega, \kappa^{+\omega})$, for each $\alpha \in I^*$, $d_\alpha = d$. Also, for each $\alpha \in I^*$ and $i < n$, denote $p_{\alpha i} = \langle y_i, c_i^\alpha\rangle$. Note that by choice of G^*, we have that $y_0 = \kappa$. Let $b = \langle d, \langle \vec{y}, \vec{c}\rangle\rangle$ be a stem in $j(\mathbb{P})$ with length n such that $\vec{y} = \langle y_i \mid i < n\rangle$ and $\vec{c} = \langle c_i \mid i < n\rangle$ where each $c_i = \bigcup_\alpha c_i^\alpha$. We can take this union since for $0 < i < n$, c_i belongs to a poset which is $< (j(\kappa) \cap y_i)^{+i+2}$ closed, and $c_0 \in Col(\kappa^{+\omega+2}, < j(\kappa)_{y_1})$. In particular, the closure is larger than $\nu^+ = \kappa^{+\omega+1}$.

Define $I = \{\alpha < \nu^+ \mid \exists p \in j(P)\ \mathrm{stem}(p) = b$, and $\exists \xi < j(\kappa) p \Vdash \langle j(\alpha), \xi\rangle <_{j(\dot{T})} u\}$. Then $I \in V[H]$ and $I^* \subset I$, so I is unbounded. So, I is as desired. □

Let $\bar{n}$ and I be as in the conclusion of the above lemma. We will say that a stem $h \Vdash^* \phi$ if there is a condition p, such that the stem of p is h and $p \Vdash \phi$.

Lemma 20. *There is, in $V[H]$, an unbounded set $J \subset \nu^+$, a stem h of length $\bar{n}$, and a sequence of nodes $\langle u_\alpha \mid \alpha \in J\rangle$ with every u_α of level α, such that for all $\alpha < \beta$ in J there is a condition p with stem h, such that $p \Vdash u_\alpha <_{\dot{T}} u_\beta$.*

Proof. Let $j : V \to N$ be a ν^+ - supercompact embedding with critical point $\kappa_{\bar{n}+2}$. Using standard arguments, extend j to $j^* : V[H] \to N[H^*]$ where $j^* \in V[H^*]$. Here H^* is $j(\mathbb{C}) = \mathbb{C} * \mathbb{C}'$ generic, where $\mathbb{C}'$ is $< \kappa_{\bar{n}+1}$ closed.

Let $\gamma \in j^*(I)$ be such that $\sup(j''\nu^+) < \gamma < j(\nu^+)$. By elementarity for all $\alpha \in I$ we can fix $\xi_\alpha, \delta_\alpha < \kappa$ and $p_\alpha \in j^*(\mathbb{P})$ with length $\bar{n}$ such that $p_\alpha \Vdash_{j^*(\mathbb{P})} \langle j^*(\alpha), \xi_\alpha\rangle <_{j^*(\dot{T})} \langle \gamma, \delta_\alpha\rangle$. Let h_α be the stem of p_α. Note that the function $\alpha \mapsto \langle \xi_\alpha, \delta_\alpha, h_\alpha\rangle$ is in $V[H^*]$.

The number of possible stems in $j^*(\mathbb{P})$ of length $\bar{n}$ is less than $\kappa_{\bar{n}}$, and so since ν^+ is regular in $V[H]$ and $\mathbb{C}'$ does not add sequences of length less than $\kappa_{\bar{n}}$, we have that in $V[H^*]$ there is a cofinal $J \subset I$, $\delta < \kappa$, and a stem h such that for all $\alpha \in J$, $\delta_\alpha = \delta$ and $h_\alpha = h$.

We consider the narrow system $S = \langle I, \mathcal{R}\rangle$ of height ν^+ and levels of size κ, in $V[H]$, where:

- $\mathcal{R} = \langle R_h \mid h$ is a stem of length $\bar{n}\rangle$; $|\mathcal{R}| < \kappa_{\bar{n}}$.
- For nodes a, b, we say that $\langle a, b\rangle \in R_h$ iff $h \Vdash^* a <_{\dot{T}} b$

Apply the preservation lemma to S for $\mathbb{R} = \mathbb{C}'$, which is $< \kappa_{\bar{n}+1}$ closed in $V[H]$, and the branches:

$$b_{R_h,\delta} =_{def} \{\langle \alpha, \xi\rangle \mid h \Vdash^*_{j^*(\mathbb{P})} \langle j(\alpha), \xi\rangle <_{j^*(\dot{T})} \langle \gamma, \delta\rangle\}.$$

We get that S has an unbounded branch in $V[H]$. I.e. in $V[H]$, there are an unbounded $J \subset I$, $\alpha \mapsto \xi_\alpha$ and a stem h such that for all $\alpha, \beta \in J$ with $\alpha < \beta$, we have that $h \Vdash^* \langle \alpha, \xi_\alpha\rangle <_{\dot{T}} \langle \beta, \xi_\beta\rangle$. Setting $u_\alpha = \langle \alpha, \xi_\alpha\rangle$ for $\alpha \in J$, we get that for all $\alpha < \beta$ in J there is a condition p with stem h which forces that $u_\alpha <_{\dot{T}} u_\beta$. $\square$

Fix $\bar{n}$, $\bar{h}$, J, and $\alpha \mapsto u_\alpha$ as in the conclusion of the above lemma. By shrinking J we may assume that for some $\xi < \kappa$, each $u_\alpha = \langle \alpha, \xi\rangle$.

Lemma 21. *Suppose that h is a stem of length k, $L \subset \nu^+$ is unbounded, and for all $\alpha < \beta$ with $\alpha, \beta \in L$, $h \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. Then there are $\rho < \nu^+$ and sets $\langle A_\alpha, C_\alpha : \alpha \in L \setminus \rho\rangle$ in $V[H]$ such that:*

1. *Each $A_\alpha \in U_k$, $\mathrm{dom}(C_\alpha) = A_\alpha$, $C_\alpha(x) \in Col(\kappa_x^{+k+2}, < \kappa)$ for $x \in A_\alpha$, and $[C_\alpha]_{U_k} \in K_k$.*

(2) *For all $\alpha < \beta$ in $L \setminus \rho$, for all $x \in A_\alpha \cap A_\beta$ such that $C_\alpha(x)$ and $C_\beta(x)$ are compatible,*

$$h^\frown\langle x, C_\alpha(x) \cup C_\beta(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

Proof. Let $j : V \to N$ be a ν^+ - supercompact embedding with critical point κ_{k+4}. Using standard arguments, extend j to $j^* : V[H] \to N[H^*]$ where $j^* \in V[H^*]$. Here H^* is $\mathbb{C} * \mathbb{C}'$ generic, where $\mathbb{C}'$ is $< \kappa_{k+3}$ closed. Let $\gamma \in j^*(L)$ be such that $\sup(j''\nu^+) < \gamma < j(\nu^+)$. By elementarity, for all $\alpha \in L$ we can fix $p^\alpha \in j^*(\mathbb{P})$ with stem h such that $p^\alpha \Vdash \langle j^*(\alpha), \xi\rangle <_{j^*(\dot{T})} \langle \gamma, \xi\rangle$. Denote $p^\alpha_k = \langle A^*_\alpha, C^*_\alpha\rangle$. We have that each $A^*_\alpha \in j^*(U_k) = U_k$ and $j^*(A^*_\alpha) = A^*_\alpha$. Note that the function $\alpha \mapsto \langle A^*_\alpha, C^*_\alpha\rangle$ is in $V[H^*]$. The number of possible pairs $\langle A, C\rangle$ in the range of this function is $|\mathcal{P}(\mathcal{P}_\kappa(\kappa_k))| = \kappa_{k+1}$. It follows that since ν^+ is regular in $V[H]$ and $\mathbb{C}'$ adds no sequences of length less than κ_{k+2}, we have that the function $\alpha \mapsto \langle A^*_\alpha, C^*_\alpha\rangle$ is constant on an unbounded subset of L. So, in $V[H^*]$ there is a cofinal $L' \subset L$, $A^* \subset \mathcal{P}_\kappa(\kappa_k)$ and C^*, such that for all $\alpha \in L'$, $A^*_\alpha = A^*$ and $C^*_\alpha = C^*$.

We say that $h^\frown\langle A, C\rangle \Vdash^* \phi$ if there is a condition $p = \langle d^p, \langle p_n \mid n < \omega\rangle\rangle \Vdash \phi$ with stem h and $p_{lh(h)} = \langle A, C\rangle$. Let $\mathcal{R} = \{R_{A,C} \mid A \in U_k, [C]_{U_k} \in K_k, \mathrm{dom}(C) = A, (\forall x \in A) C(x) \in Col(\kappa_x^{+k+2}, < \kappa)\}$. Then $|\mathcal{R}| = \kappa_{k+1}$. We consider the narrow system $S = \langle L, \mathcal{R}\rangle$, where for $\alpha \in L$, $S_\alpha = \{\alpha\} \times \kappa$ and for nodes a, b, we say that $\langle a, b\rangle \in R_{A,C}$ iff $h^\frown\langle A, C\rangle \Vdash^* a <_{\dot{T}} b$. For $R_{A,C} \in \mathcal{R}$, set

$$b_{R_{A,C}} =_{def} \{\langle \alpha, \xi\rangle \mid h^\frown\langle A, C\rangle \Vdash^*_{j^*(\mathbb{P})} \langle j(\alpha), \xi\rangle <_{j^*(\dot{T})} \langle \gamma, \xi\rangle\}.$$

Then $\{b_{R_{A,C}} \mid R_{A,C} \in \mathcal{R}\}$ are branches in $V[H^*]$ such that:

- each $b_{R_{A,C}}$ is a branch through $R_{A,C}$.
- for all $\alpha \in L$, there are A and C, such that $b_{R_{A,C}} \cap S_\alpha \neq \emptyset$,
- for some A and C, $b_{R_{A,C}}$ is unbounded.

We apply the preservation lemma for $\mathbb{R} = \mathbb{C}'$, which is $< \kappa_{k+3}$ closed in $V[H]$, and these branches to get an unbounded branch through S in $V[H]$. So, in $V[H]$ we have $L' \subset L$, A, and C such that for all $\alpha < \beta$ in L', $h^\frown\langle A, C\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.

Finally, let $\rho = \min(L')$. For every $\alpha \in (L \setminus \rho) \setminus L'$, let $\alpha' = \min(L' \setminus \alpha)$ and A'_α, C'_α be such that $h^\frown\langle A'_\alpha, C'_\alpha\rangle \Vdash^* u_\alpha <_{\dot{T}} u_{\alpha'}$. We can find such a pair since by assumption, $h \Vdash^* u_\alpha <_{\dot{T}} u_{\alpha'}$. Let $\alpha \in L \setminus \rho$. If $\alpha \in L'$, define $A_\alpha = A, C_\alpha = C$. If $\alpha \notin L'$, define $A_\alpha = A'_\alpha \cap A, C_\alpha(x) = C'_\alpha(x) \cup C(x)$ for $x \in A_\alpha$ (by shrinking A_α if necessary we may assume that for all $x \in A_\alpha$, $C'_\alpha(x)$ and $C(x)$ are compatible). Then $\alpha \mapsto \langle A_\alpha, C_\alpha\rangle$ is as desired. $\square$

Lemma 22. *There is some $\rho < \nu^+$ and a sequence of conditions $\langle p_\alpha \mid \alpha \in J \setminus \rho\rangle$ with stem $\bar{h}$ such that for $\alpha < \beta$ in $J \setminus \rho$, $p_\alpha \wedge p_\beta \Vdash u_\alpha <_{\dot{T}} u_\beta$. Here $p_\alpha \wedge p_\beta$ denotes the weakest common extension of p_α and p_β.*

Proof. The proof follows closely the argument given in [8].

First we make some remarks on taking diagonal intersections. Let H be a set of stems of length n, and let $\langle A^h \mid h \in H\rangle$ be a sequence of U_n- measure one sets. For a stem $h = \langle d, \langle \vec{y}, \vec{c}\rangle\rangle$ in H and $z \in \mathcal{P}_\kappa(\kappa_n)$, we write $h \prec z$ to denote that $y_{n-1} \prec z$, i.e. that $|y_{n-1}| < \kappa_z$ and $y_{n-1} \subset z$. Note that $h \prec z$ iff for some c, $h^\frown\langle z, c\rangle$ is a stem. Then $A = \triangle_{h\in H} A^h = \{z \in \mathcal{P}_\kappa(\kappa_n) \mid z \in \bigcap_{h\prec z} A^h\}$ is the diagonal intersection of $\langle A^h \mid h \in H\rangle$ and $A \in U_n$.

We will define sequences $\langle \rho_n \mid \bar{n} \le n < \omega\rangle$, $\langle A^n_\alpha \mid \alpha \in J \setminus \rho_n, \bar{n} \le n < \omega\rangle$, and $\langle C^n_\alpha \mid \alpha \in J \setminus \rho_n, \bar{n} \le n < \omega\rangle$ by induction on n, such that for all n:

(1) For all $\alpha \in J \setminus \rho_n$, we have that $A^n_\alpha \in U_n$, $[C^n_\alpha]_{U_n} \in K_n$, $\mathrm{dom}(C^n_\alpha) = A_\alpha$, and $C^n_\alpha(x) \in Col(\kappa_x^{+n+2}, < \kappa)$ for $x \in A^n_\alpha$.
(2) For all $\alpha < \beta$ in $J \setminus \rho_n$, for all stems $h = \langle d, \langle \vec{x}, \vec{c}\rangle\rangle$ of length $n + 1$ extending $\bar{h}$, such that for $\bar{n} \le i \le n$, $x_i \in A^i_\alpha \cap A^i_\beta$ and $c_i \le C^i_\alpha(x_i) \cup C^i_\beta(x_i)$,
$$h \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

Let $\rho_{\bar{n}}$ and $\langle A^{\bar{n}}_\alpha, C^{\bar{n}}_\alpha \mid \alpha \in J \setminus \rho_{\bar{n}}\rangle$ be given by the above lemma applied to $\bar{h}$. Then by the conclusion of the lemma, both of the conditions above for $\bar{n}$ are satisfied. Now suppose we have defined ρ_n and $\langle A^n_\alpha, C^n_\alpha \mid \alpha \in J \setminus \rho_n\rangle$ such that (1) and (2) above hold for n. We have to define ρ_{n+1} and A^{n+1}_α, C^{n+1}_α for $\alpha \in J \setminus \rho_{n+1}$.

For a stem $h = \langle d^h, \langle \vec{x}, \vec{c}\rangle\rangle$ of length $n + 1$ extending $\bar{h}$, we say that $h\, fits\, \alpha$ iff each $x_i \in A^i_\alpha$ and $c_i \le C^i_\alpha(x_i)$ for $i \le n$. Set
$$J^h = \{\alpha \in J \setminus \rho_n \mid h\, fits\, \alpha\}.$$

Define a function $h \mapsto \rho^h$ on stems of length $n + 1$ extending $\bar{h}$ as follows:

- if J^h is bounded in ν^+, let $\rho^h < \nu^+$ be a bound,
- otherwise, let ρ^h and $\langle A^h_\alpha, C^h_\alpha \mid \alpha \in J^h \setminus \rho^h\rangle$ be given by the previous lemma applied to h and J^h (here we use the inductive assumption for n).

Set $\rho_{n+1} = \sup\{\rho^h \mid h$ is a stem of length $n + 1$ extending $\bar{h}\}$. For $\alpha \in J \setminus \rho_{n+1}$, set $H_\alpha(n + 1) = \{h \mid h$ has length $n + 1$, extends $\bar{h}$, and fits $\alpha\}$. For each $\alpha \in J \setminus \rho_{n+1}$, let
$$A^{n+1}_\alpha = \triangle_{h\in H_\alpha(n+1)} A^h_\alpha.$$

Also set $[C_\alpha]_{U_{n+1}} = \bigcup\{[C^h_\alpha]_{U_{n+1}} \mid h \in H_\alpha(n+1)\} \in K_{n+1}$. By shrinking A^{n+1}_α (via diagonal intersections), we can arrange that for all $x \in A^{n+1}_\alpha$, $C_\alpha(x) = \bigcup\{C^h_\alpha(x) \mid h \prec x\}$.

It remains to check that (1) and (2) hold for $n+1$. The first condition holds by construction. For the second condition, we have to show that for all $\alpha < \beta$ in $J \setminus \rho_{n+1}$, for all stems t of length $n+2$, which extend $\bar{h}$ and fit both α and β, we have that

$$t \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

So, suppose that $\alpha < \beta$ are in $J \setminus \rho_{n+1}$ and $h^\frown\langle x, c\rangle$ is a stem as above. Since h fits α and β and $\rho^h < \alpha, \beta$, we have that J^h is unbounded and ρ^h was given by applying the previous lemma. Then $h \in H_\alpha(n+1) \cap H_\beta(n+1)$ and since we took diagonal intersections, it follows that $x \in A^h_\alpha \cap A^h_\beta$ and $c \leq C^h_\alpha(x) \cup C^h_\beta(x)$. So, by construction of $A^h_\alpha, A^h_\beta, C^h_\alpha, C^h_\beta$, we have that $h^\frown\langle x, c\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. This completes the construction.

Now let $\rho = \sup_n \rho_n$ and set p_α for $\alpha \in J \setminus \rho$ to be such that:

- the stem of p_α is $\bar{h}$,
- for all $n \geq \bar{n}$, let $p_{\alpha n} = \langle A^n_\alpha, C^n_\alpha\rangle$.

Then $\langle p_\alpha \mid \alpha \in J \setminus \rho\rangle$ is as desired. For if $\alpha < \beta$ are in $J \setminus \rho$, and $q \leq p_\alpha \wedge p_\beta$, then by the construction, we have that $\mathrm{stem}(q) \Vdash^* u_\alpha <_{\dot{T}} u_\beta$, and so $q \nVdash u_\alpha \not<_{\dot{T}} u_\beta$. It follows that $p_\alpha \wedge p_\beta \Vdash u_\alpha <_{\dot{T}} u_\beta$. $\square$

Let $\langle p_\alpha \mid \alpha \in J \setminus \rho\rangle$ be as in the above lemma. Let $\dot{G}$ be the canonical $\mathbb{P}$-name for a generic ultrafilter. Since $1_{\mathbb{P}} \Vdash$ "$\dot{T}$ is an Aronszajn tree", we have that $1_{\mathbb{P}} \Vdash$ "$\{\alpha < \nu^+ \mid p_\alpha \in \dot{G}\}$ is bounded". Denote $\dot{B} = \{\alpha < \nu^+ \mid p_\alpha \in \dot{G}\}$. $\mathbb{P}$ has the ν^+ chain condition, so for some $\alpha < \nu^+$, $1_{\mathbb{P}} \Vdash \dot{B} \subset \alpha$. Let $\beta \in J \setminus \alpha$. Then $p_\beta \Vdash p_\beta \notin \dot{B}$. Contradiction. So there is a $\mathbb{P}$-generic filter over $V[H]$, G, such that in $V[H][G]$ the tree property holds at $\aleph_{\omega+1}$.

This completes the proof of Theorem 1. To summarize, starting from ω many supercompact cardinals, we have constructed a generic extension in which there are no Aronszajn trees at $\aleph_{\omega+1}$. We conclude this paper with an open problem. Neeman [8] showed that the tree property is consistent with the failure of the singular cardinal hypothesis. It remains open whether it is consistent to have the tree property at $\aleph_{\omega+1}$ and not SCH at $\aleph_\omega$.

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