ELEMENTARY SUBRINGS OF NOETHERIAN RINGS AND SINGULARITY LOCI

SHIJI LYU

Stardard notions of first-order theories and elementary substructures can be found in books such as [Hod93] and [BS66]. See [Sch10] and [Lyu23, Appendix] for a specific discussion on the first-order theory of rings.

Lemma 1. *Let A be a Noetherian ring, $\kappa = \max\{|A|, \aleph_0\}$. Then the following hold.*

(*i*) $|\{$*ideals of* $A\}| \leq \kappa$.
(*ii*) $|\operatorname{Spec}(A)| \leq \kappa$.
(*iii*) $|\{$*open subsets of* $\operatorname{Spec}(A)\}| \leq \kappa$.
(*iv*) $|\{$*isomorphism classes of finitely generated A-algebras*$\}| \leq \kappa$.

Proof. (*i*) is clear from Noetherianness. (*ii*) follows as a point of $\operatorname{Spec}(A)$ is a prime ideal of A. (*iii*) follows as an open subset of $\operatorname{Spec}(A)$ is defined by an ideal of A. (*iv*) follows from (*i*) for all polynomial algebras of finitely many variables over A. □

Lemma 2. *Let A be an elementary subring of a ring B. Then the following hold.*

(*i*) *For every ideal I of A, $I = IB \cap A$.*
(*ii*) *If B is Noetherian, so is A.*
(*iii*) *If A is coherent, then the inclusion $A \to B$ is faithfully flat.*
(*iv*) *For every finite and finitely presented A-algebra A', A' is an elementary subring of $A' \otimes_A B$.*
(*v*) *A is an integral domain if and only if B is.*

Proof. For (*i*), we may assume $I = (a_1, \ldots, a_n)$ is finitely generated. An element $a \in A$ (resp. $b \in B$) is in I (resp. IB) if and only if (A, a) (resp. (B, b)) satisfies the formula

$$\varphi(x) = \exists x_1, \ldots, x_d \left(x = \sum x_i a_i \right),$$

giving $IB \cap A = I$.

(*ii*) follows from (*i*) as the ascending chain condition can be checked in B.

For (*iii*), flatness follows from the equational criterion. More precisely, an A-algebra C is flat if and only if for all finitely generated ideals I of A, $I \otimes_A C \to C$ is injective [Stacks, Tag 00HD]. As A is coherent, there exists an exact sequence

$$A^m \xrightarrow{g} A^n \xrightarrow{f} A \longrightarrow A/I \longrightarrow 0 \tag{1}$$

of A-modules, and the injectivity is equivalent to the exactness at the term C^n of the base changed sequence

$$C^m \xrightarrow{g_C} C^n \xrightarrow{f_C} C \longrightarrow C/IC \longrightarrow 0. \tag{2}$$

Let $(g_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ (resp. $(f_j)_{1\leq j\leq n}$) be the matrix of g (resp. f). Then the exactness of (1) (resp. (2)) is the same as A (resp. C) satisfies the sentence

$$\forall x_1 \dots \forall x_n \left(\left(0 = \sum f_j x_j\right) \Rightarrow \exists y_1 \dots \exists y_m \bigwedge_{1\leq j\leq n} \left(x_j = \sum g_{ij} y_i\right)\right).$$

As (1) is exact and A is an elementary subring of B we see (2) is exact for $C = B$, showing B flat over A; then B is faithfully flat over A by *(i)* (cf. [Stacks, Tag 00HP]).

(iv) follows from the fact that A' can be interpreted in A by a presentation, see [Lyu23, §A.3], and the fact such interpretation is compatible with base change. Details omitted.

(v) is clear as the sentence $\forall x \forall y\, ((xy = 0) \Rightarrow (x = 0) \vee (y = 0))$ characterizes integral domains. □

Remark 3. *(iii)* characterizes coherence of A, see [Sch10, Theorem 3.3.4].

Recall

Theorem 4 (Löwenheim–Skolem). *Let B be a ring, A an elementary subring of B, and X be a subset of B. For every cardinal κ, if* $\max\{|A|, |X|, \aleph_0\} \leq \kappa \leq |B|$, *then there exists a sub-A-algebra C of B of cardinality κ that contains X, such that both inclusions $A \subseteq C \subseteq B$ are elementary.*

Definition 5. Let $\mathbf{P}$ be a property of Noetherian rings. We say $\mathbf{P}$ is *pointwise* if for every Noetherian ring R, $\mathbf{P}(R)$ is equivalent to $\mathbf{P}(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. For a pointwise property $\mathbf{P}$ and a Noetherian ring R, we write $U_\mathbf{P}(R)$ for the set $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathbf{P}(R_\mathfrak{p})\}$.

Theorem 6. *Let $\mathbf{P}$ be a pointwise property of Noetherian rings. Let R be a Noetherian ring, κ an infinite cardinal. Assume that for every $\mathfrak{p} \in$* $\operatorname{Spec}(R)$ *(resp. every finite R-algebra B that is an integral domain), $U_\mathbf{P}(R/\mathfrak{p})$ (resp. $U_\mathbf{P}(B)$) contains an intersection of no more than κ dense opens. Then there exists a faithfully flat ring map $R_0 \to R$ whose fibers are $\mathbf{P}$ (resp. geometrically $\mathbf{P}$) with $|R_0| = \kappa$.*

Proof. We give the proof of the case for every finite R-algebra B that is an integral domain, $U_\mathbf{P}(B)$ contains an intersection of no more than κ dense opens. The other case is similar, where instead of all $E_{n\alpha}$, we consider all $A_n/\mathfrak{p}$ where $\mathfrak{p} \in \operatorname{Spec}(A_n)$.

Let A_0 be an elementary subring of R of cardinality κ. We know A_0 is Noetherian and $A_0 \to R$ is faithfully flat (Lemma 2*(ii)(iii)*). We will inductively choose increasing subrings A_n $(n < \omega)$ so that all inclusions

$A_n \subseteq A_{n+1}$ and $A_n \subseteq R$ are elementary and $|A_n| = \kappa$, in a way that $R_0 = \bigcup_{n<\omega} A_n$ is what we want.

With every A_n ($n < \omega$) chosen, fix a family $E_{n\alpha}$ ($\alpha < \kappa$) of finite A_n-algebras that are integral domains so that every finite A_n-algebra that is an integral domain is isomorphic to $E_{n\alpha}$ for some α, Lemma 1*(iv)*. The ring $E_{n\alpha} \otimes_{A_n} R$ is an integral domain by Lemma 2*(iv)(v)*, so by assumption, there exists a tuple $(h_{n\alpha\beta})_{\beta<\kappa}$ of nonzero elements in $E_{n\alpha} \otimes_{A_n} R$ so that for all $P \in \bigcap_\beta D(h_{n\alpha\beta})$, $(E_{n\alpha} \otimes_{A_n} R)_P$ satisfies $\mathbf{P}$. We may therefore find a sub-A_n-algebra A'_n of R of cardinality κ, so that $h_{n\alpha\beta}$ is the image of some $h'_{n\alpha\beta} \in E_{n\alpha} \otimes_{A_n} A'_n$ for all $\alpha, \beta < \kappa$.

Let A_{n+1} be a sub-A_n-algebra of R of cardinality κ containing A'_n, so that both inclusions $A_n \subseteq A_{n+1} \subseteq R$ are elementary. This is possible by Löwenheim–Skolem. This completes our inductive construction.

Let $R_0 = \bigcup_{n<\omega} A_n$, so $|R_0| = \kappa$. As all $A_n \to A_{n+1} \to R$ are faithfully flat (Lemma 2*(ii)(iii)*), it is clear that $A_n \to R_0 \to R$ are faithfully flat (cf. [Stacks, Tag `05UU`]). (In fact, R_0 is an elementary subring of R; verification omitted.) In particular, R_0 is Noetherian.

Let $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$ and let $L/\kappa(\mathfrak{p}_0)$ be a finite extension. We must show $L \otimes_{R_0} R$ satisfies $\mathbf{P}$. Let $R_0 \to E$ be a finite ring map with kernel $\mathfrak{p}_0$ where E is an integral domain with fraction field L. By [Stacks, Tags `05N9` and `07RG`] there exists an n and a finite ring map $A_n \to E'$ so that $E \cong E' \otimes_{A_n} R_0$. As $A_n \to R_0$ is faithfully flat, we see $E' \to E' \otimes_{A_n} R_0$ is injective, so E' is an integral domain. Therefore $E' \cong E_{n\alpha}$ for some $\alpha < \kappa$. By our choice, the image $h_{n\alpha\beta} \in E_{n\alpha} \otimes_{A_n} R$ of the elements $h'_{n\alpha\beta} \in E_{n\alpha} \otimes_{A_n} A'_n$ are nonzero and are such that for all $P \in \bigcap_\beta D(h_{n\alpha\beta})$, $(E_{n\alpha} \otimes_{A_n} R)_P$ satisfies $\mathbf{P}$. As $R_0 \to R$ is faithfully flat we can view $h_{n\alpha\beta}$ as nonzero elements of $E_{n\alpha} \otimes_{A_n} R_0 \cong E$. Therefore every prime of $E \otimes_{R_0} R$ above $0 \in \operatorname{Spec}(E)$ is in $\bigcap_\beta D(h_{n\alpha\beta})$, showing $L \otimes_{R_0} R$ satisfies $\mathbf{P}$. □

We remark that the same proof, starting with $|A_0| = \aleph_0 < \kappa$, yields

Theorem 7. *Let $\mathbf{P}$ be a pointwise property of Noetherian rings. Let R be a Noetherian ring, κ an infinite cardinal whose cofinality is $> \aleph_0$. Assume that for every $\mathfrak{p} \in \operatorname{Spec}(R)$ (resp. every finite R-algebra B that is an integral domain), $U_{\mathbf{P}}(R/\mathfrak{p})$ (resp. $U_{\mathbf{P}}(B)$) contains an intersection of strictly less than κ dense opens. Then there exists a faithfully flat ring map $R_0 \to R$ whose fibers are $\mathbf{P}$ (resp. geometrically $\mathbf{P}$) with $|R_0| < \kappa$.*

Corollary 8. *Let R be a Noetherian ring. Assume that for every finite R-algebra B that is an integral domain, the regular locus of $\operatorname{Spec}(B)$ contains an intersection of countably many dense opens.*

Then there exists a countable subring $R_0 \to R$ so that the inclusion is faithfully flat with geometrically regular fibers.

Corollary 9. *Let R be a Noetherian $\mathbf{F}_p$-algebra. Assume that for every finite R-algebra B that is an integral domain, the regular locus of $\operatorname{Spec}(B)$ contains an intersection of countably many dense opens.*

Then the splinter and F-pure birational splinter loci of R are an intersection of countably many open subsets.

Proof. Let $R_0 \to R$ be as in Corollary 8. Then the splinter (resp. F-pure birational splinter) locus of R is the preimage of that of R_0 by [Lyu24, Theorem 1.1 and Lemma 2.2] (resp. [Lyu, Theorem 1.6 and Proposition 3.10] and [Has10, Proposition 2.4]).

As the splinter (resp. F-pure birational splinter) locus of R_0 is stable under generalization by [DT22, Lemma 2.1.3] (resp. [Lyu, Lemma 3.9] and [Has10, Proposition 2.4]), it is an intersection of open subsets (this is true in any topological space). We conclude by Lemma 1(*iii*). □

CMFI stands for Cohen-Macaulay and F-injective, see [DM24].

Corollary 10. *Let R be a Noetherian $\mathbf{F}_p$-algebra. Assume that for every finite R-algebra B that is an integral domain, the CMFI locus of* $\operatorname{Spec}(B)$ *contains an intersection of countably many dense opens.*

Then the F-injective locus of R is an intersection of countably many open subsets.

Proof. The property CMFI is pointwise, [DM24, Proposition 3.3], so we can find a subring $R_0 \to R$ so that R_0 is countable and the inclusion is faithfully flat with geometrically CMFI fibers.

The rest of the proof is the same as Corollary 9, using [DM24, Theorem A and Theorem 3.8]. □

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Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045, USA

Email address: slyu@uic.edu

URL: https://homepages.math.uic.edu/~slyu/