UNIFORM BOUNDS IN EXCELLENT RINGS AND APPLICATIONS TO SEMICONTINUITY

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Abstract. This is a draft of a paper in preparation on certain uniform behaviours on the spectrum of an excellent $\mathbf{F}_p$-algebra.

1. Summary of results

In this section, let R be an excellent (Noetherian) $\mathbf{F}_p$-algebra.

Theorem 1.0.1 (a uniform version of the Cohen-Gabber theorem; see Theorem 3.4.3)**.** *Assume that R is (R_0). Then there exist constants $\delta, \mu, \Delta \in \mathbf{Z}_{\geq 0}$ depending only on R, and a quasi-finite, syntomic ring map $R \to S$, such that for all $\mathfrak{p} \in \operatorname{Spec}(R)$, there exist a $\mathfrak{q} \in \operatorname{Spec}(S)$ above $\mathfrak{p}$ and a ring map $P \to S^\wedge_\mathfrak{q}$ that satisfy the followings.*

- *(i) $(P, \mathfrak{m}_P)$ is a formal power series ring over a field.*
- *(ii) $P/\mathfrak{m}_P = \kappa(\mathfrak{q})$.*
- *(iii) $P \to S^\wedge_\mathfrak{q}$ is finite and generically étale of generic degree $\leq \delta$.*
- *(iv) $\mathfrak{q}S^\wedge_\mathfrak{q}/\mathfrak{m}_P S^\wedge_\mathfrak{q}$ is generated by at most μ elements.*
- *(v) There exist $e_1, \ldots, e_n \in S^\wedge_\mathfrak{q}$ that map to a basis of $S^\wedge_\mathfrak{q} \otimes_P \operatorname{Frac}(P)$ (as an $\operatorname{Frac}(P)$-vector space), such that $\operatorname{Disc}_{S^\wedge_\mathfrak{q}/P}(e_1, \ldots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.*

The next few results are established in [Smi16] and [Pol18] for F-finite rings or rings essentially of finite type over an excellent local ring.

Theorem 1.0.2 (see Corollary 5.2.5)**.** *For every finite R-module M, there exists a constant $C(M)$ with the following property. For all $\mathfrak{p} \in \operatorname{Spec}(R)$, all ideals $I \subseteq J$ of $R_\mathfrak{p}$ with $l_{R_\mathfrak{p}}(J/I) < \infty$, and all $e \leq e' \in \mathbf{Z}_{\geq 1}$, the following holds.*

$$\left| \frac{1}{p^{e \dim M_\mathfrak{p}}} l_{R_\mathfrak{p}} \left(\frac{J^{[p^e]} M_\mathfrak{p}}{I^{[p^e]} M_\mathfrak{p}} \right) - \frac{1}{p^{e' \dim M_\mathfrak{p}}} l_{R_\mathfrak{p}} \left(\frac{J^{[p^{e'}]} M_\mathfrak{p}}{I^{[p^{e'}]} M_\mathfrak{p}} \right) \right| \leq C(M) p^{-e} l_{R_\mathfrak{p}}(J/I).$$

Here by convention the left hand side is zero if $M_\mathfrak{p} = 0$.

Theorem 1.0.3 (see Theorem 6.1.3)**.** *Assume that R is locally equidimensional. Then the function $\mathfrak{p} \mapsto e_{\mathrm{HK}}(R_\mathfrak{p})$ is upper semi-continuous on R.*

Theorem 1.0.4 (see Theorem 6.2.7 and Corollary 6.2.8; restriction comes from [EY11])**.** *Assume that R is either a quotient of a regular ring, or Gorenstein. Then the function $\mathfrak{p} \mapsto s(R_\mathfrak{p})$ is lower semi-continuous on R, and the strongly F-regular locus of R is open.*

2. Preliminaries

2.1. **Local Bertini.** We recall the following classical theorem. See also [Tri94] and [OS15].

Theorem 2.1.1 ([Fle77, Satz 2.1])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring containing a field.*[1] *Let I be a proper ideal of A. Let $D(I)$ be the open subset $\mathrm{Spec}(A) \setminus V(I)$ of $\mathrm{Spec}(A)$. Let Σ be a finite subset of $D(I)$.*

Then there exists an element $a \in I$ that is not contained in any prime in Σ, and is not contained in $\mathfrak{p}^{(2)}$ for any $\mathfrak{p} \in D(I)$.

We shall use the following consequence.

Lemma 2.1.2. *Let $(A, \mathfrak{m})$ be a Noetherian J-2 local ring that is (R_0). Assume $d := \dim A \geq 1$. Then there exist elements $a_1, \ldots, a_{d-1} \in \mathfrak{m}$ such that*

(i) a_{j+1} is not in any minimal prime of $(a_1, \ldots, a_j)$; and that
(ii) $A/(a_1, \ldots, a_{d-1})$ is (R_0).

Proof. This follows from the argument in [Fle77, §3]. We reproduce the proof for the reader's convenience.

We can assume $d > 1$. By induction, it suffices to find an element $a_1 = a \in \mathfrak{m}$ not in any minimal prime of A such that A/aA is (R_0).

Since A is J-2, the singular locus $\mathrm{Sing}(A)$ is closed in $\mathrm{Spec}(A)$. Since A is (R_0), $\Sigma_1 = \{\mathfrak{p} \in \mathrm{Sing}(A) \mid \mathrm{ht}\,\mathfrak{p} \leq 1\}$ is finite. Let Σ_2 be the set of minimal primes of A. Then $\mathfrak{m} \notin \Sigma_1 \cup \Sigma_2$ since $d > 1$.

By Theorem 2.1.1, we can find $a \in \mathfrak{m}$ such that a is not in any prime in $\Sigma_1 \cup \Sigma_2$ and that $a \notin \mathfrak{p}^{(2)}$ for all $\mathfrak{p} \in \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$. It is then straightforward to verify that A/aA is (R_0). $\square$

2.2. **Discriminant.** We review some basic facts about the discriminant we will use, which is related to the Dedekind different, [Stacks, Tag 0BW0], cf. [Lan86, Chapter III].

We let A be a normal domain, K its fraction field, B a finite A-algebra, and we assume $B \otimes_A K$ finite étale over K of degree n.

Definition 2.2.1. Let $e_1, \ldots, e_n \in B$ be elements that map to a basis of $B \otimes_A K$. The *discriminant* of $e_1, \ldots, e_n$ is

$$\mathrm{Disc}_{B/A}(e_1, \ldots, e_n) = \det\left(\mathrm{Tr}_{B/A}(e_i e_j)_{i,j}\right).$$

where $\mathrm{Tr}_{B/A}$ denotes the Galois-theoretic trace map $B \otimes_A K \to K$.

Since B is integral over A and A is normal, we have $\mathrm{Tr}_{B/A}(B) \subseteq A$, and thus $\mathrm{Disc}_{B/A}(e_1, \ldots, e_n) \in A$. Moreover, it is clear that the discriminant is unchanged along a flat base change $A \to A'$ of normal domains.

This notion is useful to us later because of the following result.

[1] Flenner stated the theorem for mixed characteristic rings as well, but his proof in that case was wrong. This is explained and fixed in [Tri94].

Lemma 2.2.2. *Let $A, B, e_1, \dots, e_n$ be as above. If B is a torsion-free A-module and A contains $\mathbf{F}_p$, then for any $m \in \mathbf{Z}_{\geq 1}$, as subsets of $(B \otimes_A K)^{1/p^m}$*

$$\mathrm{Disc}_{B/A}(e_1, \dots, e_n).B^{1/p^m} \subseteq A^{1/p^m}[B].$$

Proof. This is [HH90, Lemma 6.5] for $(-)^{1/p^\infty}$, but the same proof works in the case of $(-)^{1/p^m}$. □

We need a compatibility result.

Lemma 2.2.3. *Assume that $(A, \mathfrak{m})$ is local and that $A \to B$ is finite étale. Let $e_1, \dots, e_n \in B$ be a basis of B as an A-module. Then*

$$\overline{\mathrm{Disc}_{B/A}(e_1, \dots, e_n)} = \mathrm{Disc}_{(B/\mathfrak{m}B)/(A/\mathfrak{m})}(\overline{e_1}, \dots, \overline{e_n})$$

where $\overline{(-)}$ means mod $\mathfrak{m}$ *or* mod $\mathfrak{m}B$.

Proof. Let z_{ijkl} be elements of B such that $e_i e_j e_k = \sum_l z_{ijkl} e_l$. Then $\mathrm{Tr}_{B/A}(e_i e_j) = \sum_k z_{ijkk}$, so $\mathrm{Disc}_{B/A}(e_1, \dots, e_n) = \det((\sum_k z_{ijkk})_{i,j})$. The same formulas compute the right hand side, showing the desired identity. □

We need an explicit computation.

Lemma 2.2.4. *Let $A \to B$ be a finite map of DVRs. Let $K = \mathrm{Frac}(A)$, $L = \mathrm{Frac}(B)$, and $s = [L : K]$. Assume that $s \in A^\times$, and that the residue fields of A and B are the same (i.e. L/K is totally tamely ramified). Let v_A and v_B be the discrete valuation of A and B respectively.*

Assume that there exists $y \in B$ such that $v_B(y)$ and s are relatively prime, and that $x := y^s \in A$. Then $v_A(\mathrm{Disc}_{B'/A}(y, \dots, y^{s-1}, y^s)) = (s+1)v_A(x)$ for any sub-A-algebra B' of B that contains y.

Proof. By assumptions L/K is separable and $v_B|_A = sv_A$. Since $v_B(y)$ and s are relatively prime, it is clear that $y, \dots, y^{s-1}, y^s$ is a basis of L/K, thus for any sub-A-algebra B' that contains y, $L = \mathrm{Frac}(B')$, so we may assume $B' = B$.

Since $x = y^s \in A$, we can easily write down the matrix of a power of y as a linear transformation with respect to the basis $y, \dots, y^{s-1}, y^s$, and it follows that $\mathrm{Tr}(y^{bs}) = sx^b$ and $\mathrm{Tr}(y^a) = 0$ if s does not divide a. Thus the matrix $\mathrm{Tr}(y^i y^j)$ has exactly one nonzero entry in each row, which is sx in the first $s-1$ rows and sx^2 in the last one. Since $s \in A^\times$, $v_A(\mathrm{Disc}_{B'/A}(y, \dots, y^{s-1}, y^s)) = (s+1)v_A(x)$ as desired. □

2.3. **A non-completed version of Cohen-Gabber.** We will need the following version of the Cohen-Gabber structure theorem [GO08, Théorème 7.1].

Theorem 2.3.1. *Let $(A^{nc}, \mathfrak{m}^{nc}, k)$ be a Noetherian local $\mathbf{F}_p$-algebra and let $(A, \mathfrak{m}, k)$ be the reduction of the completion of A^{nc}. Assume that A is equidimensional, and assume that for each minimal prime $\mathfrak{p}$ of A^{nc}, there is exactly one minimal prime of A above $\mathfrak{p}$.*

Let $d = \dim A$. Then there exists a set $\Lambda \subseteq A^{nc}$ and a system of parameters $t_1, \ldots, t_d \in \mathfrak{m}^{nc}$ with the following properties.

(i) *Λ maps to a p-basis of k.*

(ii) *For the unique coefficient field κ of A containing Λ (see [Bou83, chapitre IX, §2, Théorème 1]), A is finite and generically étale over the subring $\kappa[[t_1, \ldots, t_d]]$.*

Proof. We run the argument in [GO08, §7] for the ring A, while making sure that the elements of concern belong in the ring A^{nc}. We start with the constructions in [GO08, (7.2)]. Let $\mathfrak{p}_1^{nc}, \ldots, \mathfrak{p}_c^{nc}$ be the minimal primes of A^{nc}, so by our assumption, A has exactly c minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_c$ with $\mathfrak{p}_i \cap A^{nc} = \mathfrak{p}_i^{nc}$. Fix a set $\Lambda \subseteq A^{nc}$ that maps to a p-basis of k and let κ be the unique coefficient field of A containing Λ. For a finite set $e \subseteq \Lambda$, let $\kappa_e = \kappa^p(\Lambda \setminus e)$.

By [GO08, (7.3)], we can find an e such that for each ring $B = A/\mathfrak{p}_i$, we have

$$\operatorname{rank} \hat{\Omega}^1_{B/\kappa_e} = d + |e|.$$

Now, we observe that for every ideal I of A^{nc}, the sets $\{\mathrm{d}(i) \mid i \in I\}$ and $\{\mathrm{d}(i) \mid i \in IB\}$ generates the same submodule of $\hat{\Omega}^1_{B/\kappa_e}$. This is because $\hat{\Omega}^1_{B/\kappa_e}$ is a finite module over B, so all its submodules are closed, and because $\{\mathrm{d}(i) \mid i \in I\}$ is dense in $\{\mathrm{d}(i) \mid i \in IB\}$, since $\mathrm{d}(\mathfrak{m}^N B) \subseteq \mathfrak{m}^{N-1}\hat{\Omega}^1_{B/\kappa_e}$. Applying this to $I = \mathfrak{p}_1^{nc} \cap \ldots \cap \mathfrak{p}_j^{nc}$, noting that $IA \not\subseteq \mathfrak{p}_{j+1}$ since $\mathfrak{p}_{j+1} \cap A^{nc} = \mathfrak{p}_{j+1}^{nc}$, we see that the elements m_i, m_i' in [GO08, (7.4)] can be chosen to be in A^{nc}. Finally, applying the observation to $I = A^{nc}$, we see that the elements f_i in [GO08, (7.5)] can be chosen in A^{nc}. This concludes the proof. □

3. Tame ramification

3.1. **A condition of one-dimensional local rings.** We consider the following condition of a Noetherian local ring A of dimension 1.

Condition 3.1.1.

(i) $A^\wedge$ is (R_0).

(ii) $(A/\mathfrak{p})^\nu$ is local for all minimal primes $\mathfrak{p}$ of A.[2]

(iii) The map $A \to (A/\mathfrak{p})^\nu$ induces an isomorphism of residue fields for all minimal primes $\mathfrak{p}$ of A.[3]

Note that if A is complete, or more generally Henselian, then *(ii)* is automatic; see [Stacks, Tag 0BQ0].

Lemma 3.1.2. *Let A be a Noetherian local ring of dimension 1. The followings are equivalent:*

(i) *the completion $A^\wedge$ of A is (R_0); and*

(ii) *A is (R_0), and the normalization of A is finite.*

[2]In other words, $A/\mathfrak{p}$ is unibranch [Stacks, Tag 0BPZ].

[3]In particular, $A/\mathfrak{p}$ is geometrically unibranch [Stacks, Tag 0BPZ].

If this holds, then $(A^\wedge)_{\mathrm{red}} = (A_{\mathrm{red}})^\wedge$ *and* $A^{\wedge\nu} = A^\nu \otimes_A A^\wedge$.

Proof. Assume first that $A^\wedge$ is (R_0), so A is also (R_0). Let $\mathfrak{N}$ be the nilradical of A. Then $\mathfrak{N}(A^\wedge)_P = 0$ for all minimal primes P of $A^\wedge$, thus $A^\wedge/\mathfrak{N}A^\wedge = (A_{\mathrm{red}})^\wedge$ is (R_0). Since A is one-dimensional, A_{red} is Cohen-Macaulay, thus $(A_{\mathrm{red}})^\wedge$ is Cohen-Macaulay, thus reduced since it is (R_0). Finiteness of normalization is then classical, see for example [Stacks, Tag 032Y].

Now assume *(ii)*. We need to show *(i)* and $A^{\wedge\nu} = A^\nu \otimes_A A^\wedge$. Since A^ν is finite over A, we see $A^\nu \otimes_A A^\wedge$ is the completion of A^ν as a semi-local ring. Since A^ν is normal of dimension 1, it is regular, hence so is $A^\nu \otimes_A A^\wedge$. Since A is (R_0), for any $f \in \mathfrak{m}$, $A_f = (A^\nu)_f$, thus $(A^\wedge)_f = (A^\nu \otimes_A A^\wedge)_f$, so $A^\wedge$ is (R_0) and $A^\nu \otimes_A A^\wedge = A^{\wedge\nu}$. □

Condition 3.1.1 implies desired tame behavior, Proposition 3.1.4 below. Before that, some notations.

Notation 3.1.3. Let A be a Noetherian local ring of dimension 1 that satisfies Condition 3.1.1.

Let $\mathfrak{p}$ be a minimal prime of A. $(A/\mathfrak{p})^\nu$ is finite over A by Lemma 3.1.2, thus a DVR.[4] Denote by $v_\mathfrak{p} : A \to \mathbf{Z}_{\geq 0} \cup \{\infty\}$ the corresponding valuation composed with the map $A \to (A/\mathfrak{p})^\nu$; by Condition 3.1.1*(iii)*, we see $v_\mathfrak{p}(a) = l_A((A/\mathfrak{p})^\nu / a(A/\mathfrak{p})^\nu)$. Let $\beta(\mathfrak{p}) \in \mathbf{Z}_{\geq 0}$ be the minimal β such that there exists an element $s \in A$ in all minimal primes of A other than $\mathfrak{p}$ and that $v_\mathfrak{p}(s) = \beta$.

Denote by $\mathfrak{c}_\mathfrak{p}$ the conductor of the extension $A/\mathfrak{p} \to (A/\mathfrak{p})^\nu$, *i.e.*, $\mathfrak{c}_\mathfrak{p} = \{a \in (A/\mathfrak{p})^\nu \mid a(A/\mathfrak{p})^\nu \subseteq A/\mathfrak{p}\}$. Note that $A/\mathfrak{p} \to (A/\mathfrak{p})^\nu$ is finite by Lemma 3.1.2, so $\mathfrak{c}_\mathfrak{p}$ is nonzero. Denote by $\gamma_0(\mathfrak{p})$ the number $l_A((A/\mathfrak{p})^\nu/\mathfrak{c}_\mathfrak{p})$.

Assume now that A contains $\mathbf{F}_p$. We denote by $\gamma(\mathfrak{p})$ the minimal integer γ such that $\gamma \geq \gamma_0(\mathfrak{p}) + \beta(\mathfrak{p})$ and that γ is not divisible by p.

Finally, let $\delta(A) = \sum_\mathfrak{p} \gamma(\mathfrak{p})$ and $\Delta(A) = \sum_\mathfrak{p} (\gamma(\mathfrak{p}) + 1)^2$, where the sum is over all minimal primes.

We now present the main result of this subsection. Our idea has some overlap with [Ska16].

Proposition 3.1.4. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* 1 *that satisfies Condition 3.1.1. Then the followings hold.*

- *(i) Let* $\mathfrak{p}$ *be a minimal prime of* A*, and let* $n_\mathfrak{p} \in \mathbf{Z}$*,* $n_\mathfrak{p} \geq \gamma_0(\mathfrak{p})$*. Then there exists an element* $t_\mathfrak{p} \in A$ *lying in all minimal primes other than* $\mathfrak{p}$*, such that* $v_\mathfrak{p}(t_\mathfrak{p}) = n_\mathfrak{p} + \beta(\mathfrak{p})$*.*
- *(ii) Assume that* A *contains* $\mathbf{F}_p$*. Then there exists* $t \in \mathfrak{m}$ *such that for all minimal primes* $\mathfrak{p}$ *of* A*,* $v_\mathfrak{p}(t) = \gamma(\mathfrak{p})$*.*
- *(iii) Assume that* A *is complete and contains* $\mathbf{F}_p$*. For any* t *as in (ii), and any choice of a coefficient field* $k \subseteq A$*, the map* $k[[T]] \to A$ *mapping* T *to* t *is finite and generically étale of generic degree* $n = \delta(A)$*.*

[4] In fact, the normalization of a one-dimensional Noetherian domain is always Noetherian by the theorem of Krull-Akizuki [Stacks, Tag 00PG].

(*iv*) *For any $k[[T]] \to A$ as in (iii), there exist elements $e_1, ..., e_n$ of A mapping to a basis of $A[\frac{1}{T}]$ over $k((T))$ such that the T-adic valuation of the discriminant $\mathrm{Disc}_{A/k[[T]]}(e_1, \dots, e_n)$ is $\Delta(A)$.*

Proof. By the definition of the conductor, we see that there exists $r_{\mathfrak{p}} \in A$ such that $v_{\mathfrak{p}}(r_{\mathfrak{p}}) = n_{\mathfrak{p}}$. Let $s_{\mathfrak{p}}$ be an element of A contained in all other minimal primes of A and satisfies $v_{\mathfrak{p}}(s_{\mathfrak{p}}) = \beta(\mathfrak{p})$. Then $t_{\mathfrak{p}} = s_{\mathfrak{p}} r_{\mathfrak{p}}$ satisfies $v_{\mathfrak{p}}(t) = n_{\mathfrak{p}} + \beta(\mathfrak{p})$, showing (*i*).

For (*ii*), let $n_{\mathfrak{p}} = \gamma(\mathfrak{p}) - \beta(\mathfrak{p})$ for each $\mathfrak{p}$, and let $t_{\mathfrak{p}}$ be as in (*i*). Then $t = \sum_{\mathfrak{p}} t_{\mathfrak{p}}$ works. Note that t must be in $\mathfrak{m}$ since $\gamma(\mathfrak{p}) > 0$.

Now we prove (*iii*). Let $t \in \mathfrak{m}$ be such that for all minimal primes $\mathfrak{p}$ of A, $v_{\mathfrak{p}}(t) = \gamma(\mathfrak{p})$. In particular, t is a parameter of A. Let $k \subseteq A$ be an arbitrary coefficient field, so the map $k[[T]] \to A$ mapping T to t is finite. Since $v_{\mathfrak{p}}(t) = \gamma(\mathfrak{p})$ is not divisible by p and since the residue field of $(A/\mathfrak{p})^{\nu}$ is k (Condition 3.1.1(*iii*)), we see that $k[[T]] \to (A/\mathfrak{p})^{\nu}$ is totally tamely ramified of index $\gamma(\mathfrak{p})$. In particular, $k[[T]] \to A/\mathfrak{p}$ is generically étale, thus so is $k[[T]] \to A$ since A is (R_0). That $k[[T]] \to A$ has generic degree $\delta(A)$ is clear.

It remains to show (*iv*). Let $\mathfrak{p}$ be a minimal prime of A. Let s be an element of A contained in all other minimal primes of A and satisfies $v_{\mathfrak{p}}(s) = \beta(\mathfrak{p})$. In $(A/\mathfrak{p})^{\nu}$ we can write $s^{\gamma(\mathfrak{p})} = t^{\beta(\mathfrak{p})} u$, where $u \in (A/\mathfrak{p})^{\nu\times}$. Then we can write $u = v w_1^{-1}$, with $v \in k^{\times}$ and w_1 has residue class 1 in the residue field of $(A/\mathfrak{p})^{\nu}$, since the residue field of $(A/\mathfrak{p})^{\nu}$ is k (Condition 3.1.1(*iii*)). Since p does not divide $\gamma(\mathfrak{p})$, by Hensel's Lemma $w_1 = w^{\gamma(\mathfrak{p})}$ for some $w \in (A/\mathfrak{p})^{\nu}$ with residue class 1. Then $(ws)^{\gamma(\mathfrak{p})} = t^{\beta(\mathfrak{p})} v$.

Let $y \in A$ be such that the image of y in $A/\mathfrak{p}$ is in $\mathfrak{c}_{\mathfrak{p}}$ and that $v_{\mathfrak{p}}(y) = n_{\mathfrak{p}} := \gamma(\mathfrak{p}) - \beta(\mathfrak{p}) + 1$. This is possible because $n_{\mathfrak{p}} \geq \gamma_0(\mathfrak{p})$. Then similarly we can write $(w'y)^{\gamma(\mathfrak{p})} = t^{n_{\mathfrak{p}}} v' \in (A/\mathfrak{p})^{\nu}$, where $v' \in k^{\times}$ and w' has residue class 1. Now, since the image of y in $A/\mathfrak{p}$ is in $\mathfrak{c}_{\mathfrak{p}}$, there exists $z \in A$ such that the $z = yww' \in A/\mathfrak{p}$. Finally, let $x_{\mathfrak{p}}$ be the element sz. Then $x_{\mathfrak{p}}$ is in all minimal ideals other than $\mathfrak{p}$, and $x_{\mathfrak{p}}^{\gamma(\mathfrak{p})} = t^{\gamma(\mathfrak{p})+1} v v' \in (A/\mathfrak{p})^{\nu}$, where $v, v' \in k^{\times}$.

We have that $v_{\mathfrak{p}}(x_{\mathfrak{p}}) = \gamma(\mathfrak{p}) + 1$ and $\gamma(\mathfrak{p})$ are relatively prime, so we see that $x_{\mathfrak{p}}, \dots, x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}$ is a basis of $\mathrm{Frac}(A/\mathfrak{p})$ over $k((T))$. Since $x_{\mathfrak{p}}$ is in all minimal primes other than $\mathfrak{p}$, we see that $\cup_{\mathfrak{p}} \{x_{\mathfrak{p}}, \dots, x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}\}$ is a basis of $A[\frac{1}{T}]$ over $k((T))$. It suffices to show the discriminant of this basis has T-adic valuation $\Delta(A)$; thus it suffices to show the discriminant of the basis $x_{\mathfrak{p}}, \dots, x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}$ of $\mathrm{Frac}(A/\mathfrak{p})$ over $k((T))$ has T-adic valuation $(\gamma(\mathfrak{p})+1)^2$. Since $x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}$ is the image of $T^{\gamma(\mathfrak{p})+1} v v' \in k[[T]]$, this follows from Lemma 2.2.4. $\square$

We will need to move between a local ring and its completion.

Lemma 3.1.5. *Let A be a Noetherian local ring of dimension 1. Assume that A satisfies Condition 3.1.1(i)(ii). Then the followings hold.*

(*i*) *The map* $\mathfrak{p} \mapsto \mathfrak{p}A^\wedge$ *is a bijection between the minimal primes of* A *and those of* $A^\wedge$.

(*ii*) A *satisfies Condition 3.1.1(iii) if and only if* $A^\wedge$ *does.*

(*iii*) *If (ii) is the case, then the map in (i) identifies* β, γ_0 *and* γ. *In particular,* $\delta(A) = \delta(A^\wedge)$ *and* $\Delta(A) = \Delta(A^\wedge)$.

Proof. Let $\mathfrak{p}$ be a minimal prime of A. By Lemma 3.1.2, $A/\mathfrak{p} \to (A/\mathfrak{p})^\nu$ is finite, so by Lemma 3.1.2 again we see that $(A/\mathfrak{p})^{\nu\wedge} = (A/\mathfrak{p})^{\wedge\nu}$ is normal. Condition 3.1.1(*ii*) says that $(A/\mathfrak{p})^\nu$ is local, thus $(A/\mathfrak{p})^{\nu\wedge}$ is local, hence a DVR, and its subring $(A/\mathfrak{p})^\wedge$ is then an integral domain. So $\mathfrak{p}A^\wedge$ is a minimal prime of $A^\wedge$, showing (*i*), and $(A/\mathfrak{p})^{\nu\wedge} = (A^\wedge/\mathfrak{p}A^\wedge)^\nu$, showing (*ii*).

For (*iii*), by previous discussions $v_{\mathfrak{p}A^\wedge}|_A = v_\mathfrak{p}$. Since taking conductor and finite intersection commute with flat base change, it is clear that $\gamma_0(\mathfrak{p}A^\wedge) = \gamma_0(\mathfrak{p})$ and $\beta(\mathfrak{p}A^\wedge) = \beta(\mathfrak{p})$. Therefore $\gamma(\mathfrak{p}A^\wedge) = \gamma(\mathfrak{p})$. □

3.2. **Tame curves.**

Definition 3.2.1. Let A be a Noetherian local ring, $d = \dim A$. We say a proper ideal $\mathfrak{a}$ of A *defines a tame curve* if

(*i*) all minimal primes of $\mathfrak{a}$ have height $d-1$; and

(*ii*) $A/\mathfrak{a}$ satisfies Condition 3.1.1.

Lemma 3.2.2. *Let* A *be a Noetherian local ring,* $\mathfrak{a}$ *a proper ideal of* A. *If* $\mathfrak{a}$ *defines a tame curve, so does* $\mathfrak{a}A^\wedge \subseteq A^\wedge$.

Proof. Since $A \to A^\wedge$ is flat, every minimal prime of $\mathfrak{a}A^\wedge$ has the same height as some minimal prime of $\mathfrak{a}$. This takes care of (*i*) in Definition 3.2.1. For (*ii*), Condition 3.1.1(*i*) for $A/\mathfrak{a}$ and $A^\wedge/\mathfrak{a}A^\wedge$ are the same, (*ii*) is automatic for the complete local ring $A^\wedge/\mathfrak{a}A^\wedge$, and $A^\wedge/\mathfrak{a}A^\wedge$ satisfies (*iii*) by Lemma 3.1.5. □

Theorem 3.2.3. *Let* $(A, \mathfrak{m})$ *be a Noetherian local* $\mathbf{F}_p$*-algebra,* $d = \dim A$. *Assume that there exist elements* $a_1, \ldots, a_{d-1}$ *such that* $\mathfrak{a} = (a_1, \ldots, a_{d-1})$ *defines a tame curve.*

Then there exists $t \in \mathfrak{m}$ *such that for any coefficient field* k *of* $A^\wedge$, *the map* $P := k[[X_1, \ldots, X_{d-1}, T]] \to A^\wedge$ *mapping* X_i *to* a_i *and* T *to* t *is finite of generic degree* $n = \delta(A/\mathfrak{a})$, *and is étale at the prime* $\mathfrak{P} = (X_1, \ldots, X_{d-1})$, *and there exists a basis* $e_1, \ldots, e_n$ *of* $A^\wedge \otimes_P \mathrm{Frac}(P)$ *over* $\mathrm{Frac}(P)$ *such that*

$$\mathrm{Disc}_{A^\wedge/P}(e_1, \ldots, e_n) \notin \mathfrak{P} + T^{\Delta(A/\mathfrak{a})+1}P$$

See Notation 3.1.3 for $\delta(-)$ and $\Delta(-)$ in the statement.

Proof. The completion of A satisfies the same assumptions by Lemmas 3.2.2 and 3.1.5. We will show that if A is complete, and $t \in A$ is such that the image of t in $A/\mathfrak{a}$ is as in Proposition 3.1.4(*ii*), then t works. This proves the theorem, since the set of t indicated in Proposition 3.1.4(*ii*) is open in the adic topology.

Assume A and t are as above. Let k be an arbitrary coefficient field and let $P \to A$ and $\mathfrak{P}$ be as in the statement of our theorem. Note that $P \to A$

is finite and $\mathfrak{a} = \mathfrak{P}A$. Since every minimal prime of $\mathfrak{a}$ has height $d-1$, every maximal ideal of $A_{\mathfrak{P}}$ has height $d-1$. Since $P_{\mathfrak{P}}/\mathfrak{P}P_{\mathfrak{P}} \to A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}$ is finite étale of degree $n = \delta(A/\mathfrak{a})$ (Proposition 3.1.4(*iii*)), and since $P_{\mathfrak{P}}$ is normal of dimension $d-1$, $P_{\mathfrak{P}} \to A_{\mathfrak{P}}$ is finite étale of degree n, see for example [Stacks, Tag `0GSC`].

Find n elements of $A/\mathfrak{a} = A/\mathfrak{P}A$ as in Proposition 3.1.4(*iv*) and lift them to elements $e_1, \ldots, e_n \in A$. Then $e_1, \ldots, e_n$ is a basis of $A_{\mathfrak{P}}$ over $P_{\mathfrak{P}}$, and Lemma 2.2.3 gives $\mathrm{Disc}_{A/P}(e_1, \ldots, e_n) \notin \mathfrak{P} + T^{\Delta(A/\mathfrak{a})+1}P$, as desired. □

3.3. **Finding tame curves.** The goal of this subsection is to show that tame curves in the spectrum of a local ring can be found after a reasonable extension (Proposition 3.3.2).

Lemma 3.3.1. *Let R be a Noetherian local $\mathbf{F}_p$-algebra of dimension 1. Assume that $R^\wedge$ is (R_0), and assume that $R/\mathfrak{p}$ is geometrically unibranch for all minimal primes $\mathfrak{p}$ of R. Then the followings hold.*

- (*i*) *There exist a subset Λ of R and a parameter $t \in R$ that satisfies the conclusions of Theorem 2.3.1.*
- (*ii*) *For Λ, t as in (i), put $\kappa_0 = \mathbf{F}_p(\Lambda)$. Then there exists a finite purely inseparable extension κ'/κ_0 such that $R \otimes_{\kappa_0} \kappa'$ satisfies Condition 3.1.1.*

Proof. For (*i*), we need to verify the conditions of Theorem 2.3.1. Since R is one-dimensional, $A = (R^\wedge)_{\mathrm{red}}$ is equidimensional. Since $R/\mathfrak{p}$ is (geometrically) unibrach for each minimal prime $\mathfrak{p}$ of R and since $R^\wedge$ is (R_0), R satisfies Condition 3.1.1(*i*)(*ii*), so by Lemma 3.1.5, $\mathfrak{p} \mapsto \mathfrak{p}A$ is a bijection between the minimal primes of R and A. Thus the conditions of Theorem 2.3.1 are satisfied.

Now fix Λ and t as in (*i*) and let $\kappa_0 = \mathbf{F}_p(\Lambda)$. Let κ be the unique coefficient field of $R^\wedge$ containing κ_0, so A is finite and generically étale over $\kappa[[t]]$, see Theorem 2.3.1. Since $R^\wedge$ is (R_0), we see $R^\wedge$ is finite and generically étale over $\kappa[[t]]$ as well.

We fix a perfect closure κ_0^{perf} and denote by $\kappa_1, \kappa_2, \ldots$ the finite purely inseparable extensions of κ_0 inside κ_0^{perf}. For a κ_1, denote by R_1 the ring $R \otimes_{\kappa_0} \kappa_1$, so R_1 is a Noetherian local ring with residue field $k \otimes_{\kappa_0} \kappa_1$ where k is the residue field of R. Let $\tilde{R}$ be the local ring $R \otimes_{\kappa_0} \kappa_0^{\mathrm{perf}}$, and let R^* be the ring $R^\wedge \otimes_{\kappa[[t]]} \kappa^{\mathrm{perf}}[[t]]$. Since Λ is a p-basis of κ, we have $\kappa^{\mathrm{perf}} = \kappa \otimes_{\kappa_0} \kappa_0^{\mathrm{perf}}$, so we have canonical maps $R_1 \to \tilde{R} \to R^*$. Note that R^* is finite and generically étale over $\kappa^{\mathrm{perf}}[[t]]$, hence is complete, Noetherian, and (R_0). The map $R_1 \to R^*$ is faithfully flat, and $\tilde{R}$ is the union of all such rings R_1, so $\mathfrak{a} = \mathfrak{a}R^* \cap \tilde{R}$ for every ideal $\mathfrak{a}$ of $\tilde{R}$. Thus $\tilde{R}$ is Noetherian, and it is clear that $\tilde{R}^\wedge = R^*$. By Lemma 3.1.2, the normalization of $\tilde{R}$ is finite, so we can find a κ_1 such that $R_1^\nu \otimes_{\kappa_1} \kappa_0^{\mathrm{perf}} = \tilde{R}^\nu$, hence for all κ_2/κ_1, we have $R_1^\nu \otimes_{\kappa_1} \kappa_2 = R_2^\nu$.

For a κ_2/κ_1, consider the quantity $\lambda(\kappa_2) = l_{R_2^\nu}(R_2^\nu/tR_2^\nu)$. Then $\lambda(\kappa_2) \leq l_{R_2}(R_2^\nu/tR_2^\nu)$, and since $R_1^\nu \otimes_{\kappa_1} \kappa_2 = R_2^\nu$, this latter quantity is equal to $l_{R_1}(R_1^\nu/tR_1^\nu)$. Thus the quantities $\lambda(\kappa_2)$ are bounded, so we may take a κ_2/κ_1 that achives the maximal $\lambda(\kappa_2)$. We claim that κ_2/κ_0 is what we want. Condition 3.1.1(i) is clear since $R_2 \to R^*$ is faithfully flat; we need the rest two items.

Note that $R \to R_2$ is finite and radicial, so for each minimal prime $\mathfrak{p}_2$ of R_2, $\mathfrak{p}_2 \cap R$ is a minimal prime R and we have $(R/\mathfrak{p}_2 \cap R)^{sh} \otimes_{R/\mathfrak{p}_2 \cap R} R_2/\mathfrak{p}_2 = (R_2/\mathfrak{p}_2)^{sh}$. Since $R/\mathfrak{p}_2 \cap R$ is geometrically unibranch, $R_2/\mathfrak{p}_2$ is geometrically unibranch, see [Stacks, Tag 06DM]. In particular, Condition 3.1.1(ii) holds for R_2.

Now we show Condition 3.1.1(iii) holds for R_2. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_c$ be the maximal ideals of R_2^ν. For every κ_3/κ_2, since $R_2^\nu \otimes_{\kappa_2} \kappa_3 = R_3^\nu$, $\mathfrak{n}_i = \sqrt{\mathfrak{m}_i R_3^\nu}$ are exactly the maximal ideals of R_3^ν, and $(R_2^\nu)_{\mathfrak{m}_i} \otimes_{\kappa_2} \kappa_3 = (R_3^\nu)_{\mathfrak{n}_i}$. Thus we see

$$\begin{aligned} l_{R_3^\nu}(R_3^\nu/tR_3^\nu) &= \sum_i l_{R_3^\nu}((R_3^\nu)_{\mathfrak{n}_i}/t(R_3^\nu)_{\mathfrak{n}_i}) \\ &= \sum_i \frac{1}{[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]} l_{R_2^\nu}((R_3^\nu)_{\mathfrak{n}_i}/t(R_3^\nu)_{\mathfrak{n}_i}) \\ &= \sum_i \frac{[\kappa_3 : \kappa_2]}{[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]} l_{R_2^\nu}((R_2^\nu)_{\mathfrak{m}_i}/t(R_2^\nu)_{\mathfrak{m}_i}) \end{aligned}$$

and we always have

$$l_{R_2^\nu}(R_2^\nu/tR_2^\nu) = \sum_i l_{R_2^\nu}((R_2^\nu)_{\mathfrak{m}_i}/t(R_2^\nu)_{\mathfrak{m}_i}).$$

For each i we have $[\kappa_3 : \kappa_2] \geq [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]$ since $\kappa(\mathfrak{n}_i)$ is a quotient of $\kappa(\mathfrak{m}_i) \otimes_{\kappa_2} \kappa_3$. Thus the maximality of $\lambda(\kappa_2)$ gives $[\kappa_3 : \kappa_2] = [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]$ and thus $\kappa(\mathfrak{n}_i) = \kappa(\mathfrak{m}_i) \otimes_{\kappa_2} \kappa_3$. Since κ_3/κ_2 was arbitrary, we must have $\kappa(\mathfrak{m}_i)$ separable over κ_2 for all i. Since $R_2/\mathfrak{p}_2$ is geometrically unibranch for every minimal prime $\mathfrak{p}_2$ of R_2, we see $\kappa(\mathfrak{m}_i)$ is purely inseparable over κ_2, so $\kappa(\mathfrak{m}_i) = \kappa_2$, which is Condition 3.1.1(iii), as desired. □

Proposition 3.3.2. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local $\mathbf{F}_p$-algebra of dimension d. Let $\mathfrak{a}$ be a proper ideal of A. Assume that all minimal primes of $\mathfrak{a}$ has height $d-1$ and that $(A/\mathfrak{a})^\wedge$ is (R_0).*

Then there exist a syntomic-local ring map $A \to B$[5] such that $B/\mathfrak{m}B$ is finite over k and that $\mathfrak{a}B$ defines a tame curve.

Proof. Any étale-local ring map $A/\mathfrak{a} \to E$ is syntomic-local by [Stacks, Tag 00UE], and $E^\wedge$ is (R_0) since it is étale over $(A/\mathfrak{a})^\wedge$. Take an E such that $E/\mathfrak{p}$ is geometrically unibranch for all minimal primes $\mathfrak{p}$ of E, cf. [Stacks, Tag 0CB4]. By Lemma 3.3.1, there exists a finite syntomic E-algebra C that

[5] This, by convention, means that $A \to B$ is a local map of local rings such that B is the localization of a syntomic A-algebra at a prime ideal.

is local and satisfies Condition 3.1.1. Note that $A/\mathfrak{a} \to C$ is also syntomic-local, and $C/\mathfrak{m}C$ is finite over k.

By [Stacks, Tag 07M8], we can lift C to a syntomic-local A-algebra B. By our choice, $B/\mathfrak{a}B = C$ satisfies Definition 3.2.1(*ii*), and $B/\mathfrak{m}B = C/\mathfrak{m}C$ is finite over k. By flatness, $\dim B = d$ and all minimal primes of $\mathfrak{a}B$ have height $d-1$, giving Definition 3.2.1(*i*). □

3.4. **Local uniformity.** The goal of this subsection is to prove the following statement.

Theorem 3.4.1. *Let R be a Noetherian ring, $\mathfrak{p} \in \operatorname{Spec}(R)$, $d = \operatorname{ht}\mathfrak{p}$.*

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be an ideal of R such that $\mathfrak{a}R_\mathfrak{p}$ that defines a tame curve (Definition 3.2.1). Then, upon replacing R by R_g for some $g \notin \mathfrak{p}$, the followings hold.

(*i*) *For all $\mathfrak{P} \in V(\mathfrak{p})$, $\operatorname{ht}(\mathfrak{P}) = d + \operatorname{ht}(\mathfrak{P}/\mathfrak{p})$.*
(*ii*) *For all $\mathfrak{P} \in V(\mathfrak{p})$ such that $R_\mathfrak{P}/\mathfrak{p}R_\mathfrak{P}$ is regular, and all sequence of elements $\pi_1, \ldots, \pi_\delta \in R_\mathfrak{P}$ that maps to a regular system of parameters of $R_\mathfrak{P}/\mathfrak{p}R_\mathfrak{P}$, $\mathfrak{A} := \mathfrak{a}R_\mathfrak{P} + (\underline{\pi})$ defines a tame curve.*
(*iii*) *Notations as in (ii) and Notation 3.1.3. If $R/\mathfrak{a}$ contains $\mathbf{F}_p$, then for $\mathfrak{P}, \underline{\pi}$ as in (ii), we have $\delta(R_\mathfrak{P}/\mathfrak{A}) = \delta(R_\mathfrak{p}/\mathfrak{a})$ and $\Delta(R_\mathfrak{P}/\mathfrak{A}) = \Delta(R_\mathfrak{p}/\mathfrak{a})$.*

Item (*i*) follows from [EGA IV_2, Proposition 6.10.6]. Before going into the proof of (*ii*) and (*iii*), we note the following.

Discussion 3.4.2 (cf. [EY11, Lemmas 3.2 and 3.3])**.** Let R be a Noetherian ring, $\mathfrak{p} \in \operatorname{Spec}(R)$. Let M be a finite R-module. Then, upon replacing R by R_g for some $g \notin \mathfrak{p}$, there is a filtration $M = M_n \supsetneq M_{n-1} \supsetneq \ldots \supsetneq M_1 \supsetneq M_0 = 0$ such that $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ where $\mathfrak{p}_j \subseteq \mathfrak{p}$. In particular, if $M_\mathfrak{p}$ is of finite length, then M is a successive extension of $R/\mathfrak{p}$. Thus if $\mathfrak{P} \in V(\mathfrak{p})$, $\pi_1, \ldots, \pi_h$ elements of $R_\mathfrak{P}$ that are a regular sequence in $R_\mathfrak{P}/\mathfrak{p}R_\mathfrak{P}$, then $\pi_1, \ldots, \pi_h$ is a regular sequence in $M_\mathfrak{P}$. Consequently, if

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M \longrightarrow 0$$

is a short exact sequence of R-modules, then

$$0 \longrightarrow (M_1)_\mathfrak{P}/(\underline{\pi}) \longrightarrow (M_2)_\mathfrak{P}/(\underline{\pi}) \longrightarrow M_\mathfrak{P}/(\underline{\pi}) \longrightarrow 0$$

is exact. Moreover, if $h = \dim R_\mathfrak{P}/\mathfrak{p}R_\mathfrak{P}$ (so in particular $R_\mathfrak{P}/\mathfrak{p}R_\mathfrak{P}$ is Cohen-Macaulay), then looking at the prime filtration we see $l(M_\mathfrak{P}/(\underline{\pi})) = l(M_\mathfrak{p})l(R_\mathfrak{P}/(\mathfrak{p}R_\mathfrak{P} + (\underline{\pi})))$.

Now we continue the proof of Theorem 3.4.1.

Step 1. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the minimal primes of $\mathfrak{a}$. Localize R, we may assume $\mathfrak{q}_i \subseteq \mathfrak{p}$ for all i.

Step 2. For each i, the normalization of $R_\mathfrak{p}/\mathfrak{q}_iR_\mathfrak{p}$ is finite (Lemma 3.1.2). Thus there exists a finite extension R_i' of $R/\mathfrak{q}_i$ in its fraction field such that $(R_i')_\mathfrak{p} = (R_\mathfrak{p}/\mathfrak{q}_iR_\mathfrak{p})^\nu$.

Step 3. By Condition 3.1.1(*ii*), $(R'_i)_{\mathfrak{p}}$ is local, so R'_i has exactly one prime $\mathfrak{p}'_i$ above $\mathfrak{p}$. Localizing R we may assume $\mathfrak{p}'_i = \sqrt{\mathfrak{p}R'_i}$. By Condition 3.1.1(*iii*), $(R'_i)_{\mathfrak{p}}/\mathfrak{p}'_i(R'_i)_{\mathfrak{p}} = \kappa(\mathfrak{p})$, so after localizing R we may assume $R/\mathfrak{p} = R'_i/\mathfrak{p}'_i$. In particular, for each $\mathfrak{P} \in V(\mathfrak{p})$, there is a unique prime $\mathfrak{P}'_i$ of R'_i above $\mathfrak{P}$, and $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}} = (R'_i)_{\mathfrak{P}'_i}/\mathfrak{p}'_i(R'_i)_{\mathfrak{P}'_i}$, in particular $\kappa(\mathfrak{P}'_i) = \kappa(\mathfrak{P})$.

Step 4. Since $(R'_i)_{\mathfrak{p}}$ is a DVR, after localizing R we may assume that $\mathfrak{p}'_i$ is a principal ideal. Let τ_i be a generator, so $R/\mathfrak{p} = R'_i/\tau_i R'_i$. For $\mathfrak{P}, \underline{\pi}$ as in (*ii*), $\tau_i, \underline{\pi}$ is then a regular sequence in $(R'_i)_{\mathfrak{P}'_i} = (R'_i)_{\mathfrak{P}}$ that generates the maximal ideal. Thus $(R'_i)_{\mathfrak{P}}$ is regular and $\tau_i, \underline{\pi}$ is a regular system of parameters. In particular, $\underline{\pi}$ is a regular sequence in $(R'_i)_{\mathfrak{P}}$ and $(R'_i)_{\mathfrak{P}}/(\underline{\pi})$ is a DVR.

Step 5. Apply Discussion 3.4.2 to $M = \frac{R'_i}{R/\mathfrak{q}_i}$, we see that after localizing R, we may assume that for all $\mathfrak{P}, \underline{\pi}$, $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) \to (R'_i)_{\mathfrak{P}}/(\underline{\pi})$ is injective with finite length cokernel. Since $(R'_i)_{\mathfrak{P}}/(\underline{\pi})$ is a DVR (Step 4), it is the normalization of the integral domain $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}}+(\underline{\pi}))$. In particular, $(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi}))$ is a prime ideal, and $\dim R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) = 1$.

Step 6. Apply Discussion 3.4.2 to $M = R/(\mathfrak{q}_i + \mathfrak{q}_j)$ $(i \neq j)$, we see that after localizing R, we may assume that for all $\mathfrak{P}, \underline{\pi}$, $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + \mathfrak{q}_j R_{\mathfrak{P}} + (\underline{\pi}))$ has finite length. Thus $\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi}) \neq \mathfrak{q}_j R_{\mathfrak{P}} + (\underline{\pi})$.

Step 7. Apply Discussion 3.4.2 to $M = \frac{\oplus_i R/\mathfrak{q}_i}{R/\sqrt{\mathfrak{a}}}$, we see that after localizing R, we may assume that for all $\mathfrak{P}, \underline{\pi}$, $\cap_i(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) = \sqrt{\mathfrak{a}} R_{\mathfrak{P}} + (\underline{\pi})$. Thus $\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})$ are precisely all the minimal primes of $\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi})$.

Step 8. Apply Discussion 3.4.2 to $M = \sqrt{\mathfrak{a}}/\mathfrak{a}$, we see that after localizing R, we may assume that for all $\mathfrak{P}, \underline{\pi}$, $\frac{\sqrt{\mathfrak{a}}R_{\mathfrak{P}}+(\underline{\pi})}{\mathfrak{a}R_{\mathfrak{P}}+(\underline{\pi})}$ has finite length. Thus $R_{\mathfrak{P}}/(\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi}))$ is (R_0).

At this point, with the characterization of minimal primes and normalizations in the previous steps, and with Lemma 3.1.2, we conclude that for all $\mathfrak{P}, \underline{\pi}$, the ring $R_{\mathfrak{P}}/(\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi}))$ has dimension 1 and satisfies Condition 3.1.1. To see $\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi})$ defines a tame curve, we must show $\mathrm{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) = \mathrm{ht}(\mathfrak{P}) - 1$ for all i.

By what we have done in Steps 4 and 5, $\underline{\pi}$ is a regular sequence in both $(R'_i)_{\mathfrak{P}}$ and $\left(\frac{R'_i}{R/\mathfrak{q}_i}\right)_{\mathfrak{P}}$. Thus $\underline{\pi}$ is a regular sequence in $R_{\mathfrak{P}}/\mathfrak{q}_i R_{\mathfrak{P}}$. Therefore $\mathrm{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) \geq \mathrm{ht}(\mathfrak{q}_i R_{\mathfrak{P}}) + h$, where $h = \mathrm{ht}(\mathfrak{P}/\mathfrak{p})$. Since $\mathrm{ht}(\mathfrak{q}_i R_{\mathfrak{P}}) = \mathrm{ht}(\mathfrak{q}_i R_{\mathfrak{p}}) = d - 1$ (Definition 3.2.1(*i*)), we see $\mathrm{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) \geq d + h - 1$. Since $\mathrm{ht}(\mathfrak{P}) = d + h$ (by (*i*)), we see that $\mathrm{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) = d + h - 1$, thus $\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi})$ defines a tame curve.

It remains to show the agreement of δ and Δ, assuming $R/\mathfrak{a}$ contains $\mathbf{F}_p$. By definition (Notation 3.1.3), it suffices to show, for each i, that $\beta(\overline{\mathfrak{q}}_i) = \beta(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a} R_{\mathfrak{p}})$ and same for γ_0. Here $\overline{\mathfrak{q}}_i$ denotes $\frac{\mathfrak{q}_i R_{\mathfrak{P}}+(\underline{\pi})}{\mathfrak{a} R_{\mathfrak{P}}+(\underline{\pi})}$.

Step 9. Fix an index i. Let $\mathfrak{r}_i = \cap_{j \neq i} \mathfrak{q}_j$, so $\beta(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a} R_{\mathfrak{p}}) = l_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, where M is the finite R-module $R'_i/\mathfrak{r}_i R'_i$. ($R_{\mathfrak{p}}/\mathfrak{a} R_{\mathfrak{p}}$ satisfies Condition 3.1.1(*iii*) so we can calculate the length over $R_{\mathfrak{p}}$.) As in Step 7, after localizing R we may assume for all $\mathfrak{P}, \underline{\pi}$, $\mathfrak{c}_i R_{\mathfrak{P}} + (\underline{\pi}) = \cap_{j \neq i} (\mathfrak{q}_j R_{\mathfrak{P}} + (\underline{\pi}))$. Thus $\beta(\overline{\mathfrak{q}}_i) = l(M_{\mathfrak{P}}/(\underline{\pi})M_{\mathfrak{P}})$. Apply Discussion 3.4.2, we see $\beta(\overline{\mathfrak{q}}_i) = \beta(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a} R_{\mathfrak{p}})$ after localizing R again.

Step 10. Again, fix an index i. Let $\mathfrak{c}_i = \{a \in R \mid aR'_i \subseteq R/\mathfrak{q}_i\}$, the conductor of R'_i over $R/\mathfrak{q}_i$ computed in R, and let $M = \frac{R'_i}{R/\mathfrak{q}_i}$. If $x_1, \ldots, x_l$ generate M as an R-module, then we have an injection

$$\begin{aligned} R/\mathfrak{c}_i &\to M^{\oplus l} \\ a &\mapsto (ax_1, \ldots, ax_l) \end{aligned}$$

The cokernel of this map has finite length at $\mathfrak{p}$ since M does. Apply Discussion 3.4.2, we see that after localizing R, $\mathfrak{c}_i R_{\mathfrak{P}} + (\underline{\pi})$ is the conductor of the normalization over $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi}))$ computed in $R_{\mathfrak{P}}$. We have $\gamma(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a} R_{\mathfrak{p}}) = l_{R_{\mathfrak{p}}}((R'_i/\mathfrak{c}_i R'_i)_{\mathfrak{p}})$ and similar for $\gamma(\overline{\mathfrak{q}}_i)$. Thus applying Discussion 3.4.2 again, we see that after localizing R again, $\gamma(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a} R_{\mathfrak{p}}) = \gamma(\overline{\mathfrak{q}}_i)$.

The proof of Theorem 3.4.1 is now finished.

We arrive at the main theorem of the section.

Theorem 3.4.3. *Let R be a Noetherian $\mathbf{F}_p$-algebra. Assume the followings hold.*

(1) R is J-2.
(2) For all primes $\mathfrak{p}' \subset \mathfrak{p}$ of R with $\mathrm{ht}(\mathfrak{p}/\mathfrak{p}') = 1$, $R^\wedge_{\mathfrak{p}}/\mathfrak{p}' R^\wedge_{\mathfrak{p}}$ is (R_0).
(3) R is (R_0).

Then there exist constants $\delta, \mu, \Delta \in \mathbf{Z}_{\geq 0}$ depending only on R, and a quasi-finite syntomic ring map $R \to S$, such that for all $\mathfrak{p} \in \mathrm{Spec}(R)$, there exist a $\mathfrak{q} \in \mathrm{Spec}(S)$ above $\mathfrak{p}$ and a ring map $P \to S^\wedge_{\mathfrak{q}}$ that satisfy the followings.

(i) $(P, \mathfrak{m}_P)$ is a formal power series ring over a field.
(ii) $P/\mathfrak{m}_P = \kappa(\mathfrak{q})$.
(iii) $P \to S^\wedge_{\mathfrak{q}}$ is finite and generically étale of generic degree $\leq \delta$.
(iv) $\mathfrak{q} S^\wedge_{\mathfrak{q}}/\mathfrak{m}_P S^\wedge_{\mathfrak{q}}$ is generated by at most μ elements.
(v) There exist $e_1, \ldots, e_n \in S^\wedge_{\mathfrak{q}}$ that map to a basis of $S^\wedge_{\mathfrak{q}} \otimes_P \mathrm{Frac}(P)$, such that $\mathrm{Disc}_{S^\wedge_{\mathfrak{q}}/P}(e_1, \ldots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(R)$, $d = \mathrm{ht}(\mathfrak{p})$. We shall find constants $\delta_{\mathfrak{p}}, \mu_{\mathfrak{p}}, \Delta_{\mathfrak{p}}$, a syntomic ring map $R \to S(\mathfrak{p})$, and a constructible subset $\mathcal{C}(\mathfrak{p}) \subseteq \mathrm{Spec}(R)$ containing $\mathfrak{p}$ such that for all $\mathfrak{P} \in \mathcal{C}(\mathfrak{p})$, there exists $\mathfrak{Q} \in \mathrm{Spec}(S(\mathfrak{p}))$ above $\mathfrak{P}$ and a ring map $P \to S(\mathfrak{p})^\wedge_{\mathfrak{Q}}$ that satisfy (i)–(v). If this is possible, then we win since the constructible topology is compact [Stacks, Tag 0901] and we can take a finite product of $S(\mathfrak{p})$'s with the corresponding $\mathcal{C}(\mathfrak{p})$'s covering $\mathrm{Spec}(R)$.

By assumptions, $R_{\mathfrak{p}}$ is J-2 and (R_0). If $d = 0$, then there is some $f \notin \mathfrak{p}$ such that R_f is regular, so we can just take $\mathcal{C}(\mathfrak{p}) = D(f), S(\mathfrak{p}) = R_f, \delta_{\mathfrak{p}} = 1$, $\mu_{\mathfrak{p}} = \Delta_{\mathfrak{p}} = 0$.

Assume $d \geq 1$. By Lemma 2.1.2, there exist elements $a_1, \ldots, a_{d-1} \in \mathfrak{p}R_{\mathfrak{p}}$ such that all minimal primes of $(\underline{a})$ are of height $d-1$ and that $R_{\mathfrak{p}}/(\underline{a})$ is (R_0). Note that then $(R_{\mathfrak{p}}/(\underline{a}))^{\wedge}$ is (R_0) by assumption (2).

Let $R_{\mathfrak{p}} \to B$ be as in Proposition 3.3.2 for the ideal $\mathfrak{a} = (\underline{a})$. B is a localization of a syntomic $R_{\mathfrak{p}}$-algebra, and $B/\mathfrak{p}B$ is finite over $\kappa(\mathfrak{p})$, thus $B = S_{\mathfrak{q}}$ for some quasi-finite syntomic R-algebra S and some $\mathfrak{q} \in \operatorname{Spec}(S)$. We also have $\mathfrak{a}B = \mathfrak{b}_0 B$ for some ideal $\mathfrak{b}_0$ of S generated by $d-1$ elements.

Since R is J-2, we can localize S near $\mathfrak{q}$ to assume $S/\mathfrak{q}$ regular. Let $\delta_{\mathfrak{p}} = \delta(B/\mathfrak{b}_0 B), \Delta_{\mathfrak{p}} = \Delta(B/\mathfrak{b}_0 B)$ (Notation 3.1.3), and let $\mu_{\mathfrak{p}}$ be the number of generators of $\mathfrak{q}/\mathfrak{b}_0$. Find $g \notin \mathfrak{q}$ as in Theorem 3.4.1 (for $R = S$, $\mathfrak{a} = \mathfrak{b}_0$, and $\mathfrak{p} = \mathfrak{q}$), and let $S(\mathfrak{p}) = S_g$. Then for all $\mathfrak{Q} \in V(\mathfrak{q}S(\mathfrak{p}))$, there exists an ideal $\mathfrak{B}$ of $S(\mathfrak{p})_{\mathfrak{Q}}$ generated by $\operatorname{ht}(\mathfrak{Q}) - 1$ elements defining a tame curve (Theorem 3.4.1*(i)(ii)*) and satisfying $\delta(S(\mathfrak{p})_{\mathfrak{Q}}/\mathfrak{B}) = \delta_{\mathfrak{p}}$ and $\Delta(S(\mathfrak{p})_{\mathfrak{Q}}/\mathfrak{B}) = \Delta_{\mathfrak{p}}$ (Theorem 3.4.1*(iii)*). The form of $\mathfrak{B}$ as in Theorem 3.4.1*(ii)* tells us that $\mathfrak{Q}/\mathfrak{B}$ is generated by at most $\mu_{\mathfrak{p}}$ elements.

Let $P \to S(\mathfrak{p})^{\wedge}_{\mathfrak{Q}}$ be a map as in Theorem 3.2.3, so *(ii)* is true by construction and *(i)(iii)(v)* follow from the theorem. By construction, $\mathfrak{B}S(\mathfrak{p})^{\wedge}_{\mathfrak{Q}} \subseteq \mathfrak{m}_P S(\mathfrak{p})^{\wedge}_{\mathfrak{Q}}$, so we get *(iv)*. Finally, we let $\mathcal{C}(\mathfrak{p})$ be the image of $V(\mathfrak{q}S(\mathfrak{p}))$ in $\operatorname{Spec}(R)$, which is constructible since $R \to S(\mathfrak{p})$ is of finite type. This finishes the proof. □

4. More preliminaries

4.1. **Local equidimensionality.**

Lemma 4.1.1. *Let R be a Noetherian ring that is locally equidimensional and universally catenary. Let $R \to S$ be a flat ring map of finite type. If all nonempty generic fibers of $R \to S$ are equidimensional and have the same dimension, then S is locally equidimensional.*

Proof. Let d be the generic fiber dimension. Let $\mathfrak{q} \in \operatorname{Spec}(S)$ be above some $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $\mathfrak{q}_0$ be an arbitrary minimal prime of S contained in $\mathfrak{q}$, lying over $\mathfrak{p}_0 \in \operatorname{Spec}(R)$. Then $\mathfrak{p}_0$ is a minimal prime of R by flatness.

By [Stacks, Tag 02IJ], $\operatorname{ht}(\mathfrak{q}/\mathfrak{q}_0) = \operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0) + \operatorname{trdeg}_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{q}_0) - \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q})$. By our assumptions, $\operatorname{trdeg}_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{q}_0) = d$ is independent of $\mathfrak{q}_0$ chosen. Also $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0)$ does not depend on the choice of $\mathfrak{q}_0$ since R is locally equidimensional. Thus $\operatorname{ht}(\mathfrak{q}/\mathfrak{q}_0)$ does not depend on the choice of $\mathfrak{q}_0$, so S is locally equidimensional. □

4.2. **Formally (S_1) rings.**

The purpose of this subsection is to relax the excellence hypothesis in our main theorems. The "excellent" reader can skip this subsection.

Definition 4.2.1. Let R be a Noetherian ring. We say R is *formally* (S_1) if $R^{\wedge}_{\mathfrak{p}}$ is (S_1) for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Lemma 4.2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then R is formally (S_1) if and only if $R^{\wedge}$ is (S_1).*

Proof. If $R^\wedge$ is (S_1), then $R^\wedge$ is formally (S_1) since a complete local ring is a G-ring [Stacks, Tag 07PS] and the property (S_1) ascends [Stacks, Tag 0339]. Since $R \to R^\wedge$ is faithfully flat it is clear that R is formally (S_1). ☐

Lemma 4.2.3. *Let R be a Noetherian ring, $R \to S$ a ring map of finite type. Assume that $R \to S$ is flat with (S_1) fibers. Then if R is formally (S_1), so is S.*

Proof. Let $\mathfrak{q} \in \operatorname{Spec}(S)$, we want to show $S^\wedge_\mathfrak{q}$ is (S_1). We may assume $(R, \mathfrak{m})$ local and $\mathfrak{q} \cap R = \mathfrak{m}$. Let $\mathfrak{Q} \in \operatorname{Spec}(S \otimes_R R^\wedge)$ be above $\mathfrak{q}$, so we have

$$\begin{array}{ccccccc} R & \longrightarrow & S & \longrightarrow & S_\mathfrak{q} & \longrightarrow & S^\wedge_\mathfrak{q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R^\wedge & \longrightarrow & S \otimes_R R^\wedge & \longrightarrow & (S \otimes_R R^\wedge)_\mathfrak{Q} & \longrightarrow & (S \otimes_R R^\wedge)^\wedge_\mathfrak{Q} \end{array}$$

where the vertical maps are faithfully flat. It suffices to show $(S \otimes_R R^\wedge)^\wedge_\mathfrak{Q}$ is (S_1). Since $S \otimes_R R^\wedge$ is of finite type over $R^\wedge$, it is a G-ring [Stacks, Tag 07PX], so by [Stacks, Tag 0339] it suffices to show $S \otimes_R R^\wedge$ is (S_1). By [Stacks, Tag 0339] again it suffices to show the fibers of $R^\wedge \to S \otimes_R R^\wedge$ are (S_1).

Since the fibers of $R \to S$ are (S_1), it suffices to show if k is a field, K/k is a field extension, A is a finite type k-algebra that is (S_1), then $A \otimes_k K$ is (S_1). By [Stacks, Tag 0339], applied to the map $A \to A \otimes_k K$, it suffices to show $k' \otimes_k K$ is (S_1) for all finitely generated extensions k'/k. This ring is actually Cohen-Macaulay, see [Stacks, Tag 045M]. ☐

Lemma 4.2.4. *Let R be a Noetherian local ring. If $R^\wedge$ is (S_1), then $R^\wedge/\mathfrak{p}R^\wedge$ is (S_1) for all minimal primes $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. Since $\mathfrak{p} \in \operatorname{Ass}_R(R)$, $\operatorname{Ass}_{R^\wedge}(R^\wedge/\mathfrak{p}R^\wedge) \subseteq \operatorname{Ass}_{R^\wedge}(R^\wedge)$, cf. [Stacks, Tag 0312]. Thus $R^\wedge/\mathfrak{p}R^\wedge$ has no embedded primes, as desired. ☐

4.3. **Number of generators.**

Lemma 4.3.1. *Let P be a normal domain, A a finite torsion-free P-algebra of generic degree $\delta \geq 1$. Let $\mu \in \mathbf{Z}_{\geq 0}$ be such that A is generated by μ elements as a P-algebra.*

Assume that $A \otimes_P \operatorname{Frac}(P)$ is a product of fields. Then A is generated by at most δ^μ elements as a P-module.

Proof. Let $a \in A$. In each factor of $A \otimes_P \operatorname{Frac}(P)$, a has a monic minimal polynomial whose coefficients are in P since P is normal. Since $A \to A \otimes_P \operatorname{Frac}(P)$ is injective, the product of these minimal polynomials is a monic polynomial of degree $\leq \delta$ with coefficients in P that has a as a root. The rest is clear. ☐

4.4. **An easy estimate.**

Lemma 4.4.1. *Let $(P, \mathfrak{m}, k)$ be a regular local ring containing $\mathbf{F}_p$. Let $d = \dim P$. Let $n \in \mathbf{Z}_{\geq 0}$, $F \in P$, $F \notin \mathfrak{m}^{n+1}$. Then for all $e \in \mathbf{Z}_{\geq 1}$, $l(P/((F) + \mathfrak{m}^{[p^e]})) \leq np^{e(d-1)}$.*

Proof. We may assume $F \in \mathfrak{m}^n$ and k infinite. Arguing as in [Nag62, (40.2)], we can find a regular system of parameters $x_1, \dots, x_d$ of P such that $l(P/(F, x_2, \dots, x_d)) = n$ and that $F, x_2, \dots, x_d$ is a regular sequence in P. Then $l(P/((F) + \mathfrak{m}^{[p^e]})) \leq l(P/((F) + (x_2, \dots, x_d)^{[p^e]})) = np^{e(d-1)}$. □

5. Uniform bound

5.1. Bound from a single Cohen-Gabber type normalization.

Lemma 5.1.1 (cf. [Pol18, proof of Corollary 3.4])**.** *Let A be an $\mathbf{F}_p$-algebra, $\mathfrak{m}$ a maximal ideal of A, I an ideal of A, u an element of A such that $(I : u) = \mathfrak{m}$, e a positive integer. Let M be an A-module.*

Write $J = I + (u)$. Then the followings hold.

(i) $M/(I^{[p^e]}M :_M u^{p^e}) \cong J^{[p^e]}M/I^{[p^e]}M$.
(ii) If $A/\mathfrak{m}$ is perfect and M is finitely generated, then for all $t \in \mathbf{Z}_{\geq 0}$,

$$l_A\left(\frac{F^t_*M}{(I^{[p^e]}F^t_*M :_{F^t_*M} u^{p^e})}\right) = l_A\left(\frac{J^{[p^{e+t}]}M}{I^{[p^{e+t}]}M}\right) < \infty.$$

Proof. There is a canonical surjection $M \to J^{[p^e]}M/I^{[p^e]}M$ sending m to $u^{p^e}m$, showing (i). For (ii), finiteness follows from the fact that $\frac{J^{[p^{e+t}]}M}{I^{[p^{e+t}]}M}$ is a finitely generated $(A/\mathfrak{m}^{[p^{e+t}]})$-module. To see the identity, notice that $\frac{F^t_*M}{(I^{[p^e]}F^t_*M :_{F^t_*M} u^{p^e})} = F^t_*\left(\frac{M}{(I^{[p^{e+t}]}M :_M u^{p^{e+t}})}\right)$, and that calculating the length of an $(F^t_*\mathfrak{m})$-primary (F^t_*A)-module over F^t_*A and A are the same since $A/\mathfrak{m}$ is perfect. □

Proposition 5.1.2. *Let $(P, \mathfrak{m}_P, k)$ be a regular local ring of dimension d containing $\mathbf{F}_p$, $K = \mathrm{Frac}(P)$. Let A be a finite, generically étale, and torsion-free P-algebra generated by $m \in \mathbf{Z}_{>0}$ elements as a P-module.*

Let $\Delta \in \mathbf{Z}_{\geq 0}$. Assume that there exist $e_1, \dots, e_n \in A$ that map to a basis of $A \otimes_P K$ such that $D := \mathrm{Disc}_{A/P}(e_1, \dots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.

Then for all $e \leq e' \in \mathbf{Z}_{\geq 1}$, and all ideals $I \subseteq J$ of A with $l_A(J/I) < \infty$, we have

$$\left|\frac{1}{p^{ed}} l_A\left(\frac{J^{[p^e]}}{I^{[p^e]}}\right) - \frac{1}{p^{e'd}} l_A\left(\frac{J^{[p^{e'}]}}{I^{[p^{e'}]}}\right)\right| \leq m\Delta p^{-e} l_A(J/I).$$

Proof. If $I \subseteq J_1 \subseteq J$ and the statement is true for both inclusions, then it is true for $I \subseteq J$ by additivity. Thus we may assume $l_A(J/I) = 1$. In particular $\mathfrak{m}J \subseteq I$ for a unique maximal ideal $\mathfrak{m}$ of A. For a finite $\mathfrak{m}$-primary

A-module X, we have $l_P(X) = [\kappa(\mathfrak{m}) : k] l_A(X)$, so it suffices to show

$$(1) \qquad \left| \frac{1}{p^{ed}} l_P \left(\frac{J^{[p^e]}}{I^{[p^e]}} \right) - \frac{1}{p^{e'd}} l_P \left(\frac{J^{[p^{e'}]}}{I^{[p^{e'}]}} \right) \right| \leq m \Delta p^{-e} l_P(J/I)$$

when $l_A(J/I) = 1$.

Let $(P, \mathfrak{m}_P, k) \to (P', \mathfrak{m}_{P'}, k')$ be a flat map of regular local rings with $\mathfrak{m}_P P' = \mathfrak{m}_{P'}$, and let $A' = A \otimes_P P'$. Then it is clear that A' is a finite, generically étale, and torsion-free P'-algebra generated by $m \in \mathbf{Z}_{>0}$ elements as a P'-module. The discriminant does not change, and $\mathfrak{m}_{P'}^{\Delta+1} \cap P = \mathfrak{m}_P^{\Delta+1}$ by flatness, so all assumptions hold for $P' \to A'$. For any finite length P-module X, $l_P(X) = l_{P'}(X \otimes_P P')$. Thus to show (1) when $l_A(J/I) = 1$, it suffices to show (1) for $A = A'$ with $l_A(J/I)$ arbitrary, and thus it suffices to show (1) for $A = A'$ with $l_A(J/I) = 1$. Thus we may assume P complete and k algebraically closed. In particular, for any finite P-algebra Q and any finite length Q-module Y, we have $l_P(Y) = l_Q(Y)$.

Write $t = e' - e$. Then P^{1/p^t} is a free P-module of rank p^{td}. Write $H = P^{1/p^t} \otimes_P A$. We have an exact sequence

$$H \longrightarrow A^{1/p^t} \longrightarrow L \longrightarrow 0$$

of H-modules, where L is generated by m elements as a P^{1/p^t}-module (since A^{1/p^t} is) and is annihilated by D (Lemma 2.2.2; here we use A torsion-free).

Write $J = I + (u)$, so $\mathfrak{m}_P u \subseteq I$, and we get an exact sequence

$$\frac{H}{(I^{[p^e]}H :_H u^{p^e})} \longrightarrow \frac{A^{1/p^t}}{(I^{[p^e]}A^{1/p^t} :_{A^{1/p^t}} u^{p^e})} \longrightarrow L' \longrightarrow 0$$

of H-modules with L' a quotient of $L/\mathfrak{m}_P^{[p^e]}L$, see [Pol18, proof of Corollary 3.4] for more details. Note that H is a free A-module of rank p^{td}. Lemma 5.1.1 gives the first inequality in the following chain, and the other two follows from constructions:

$$\begin{aligned} -p^{td} l_P \left(J^{[p^e]}/I^{[p^e]} \right) + l_P \left(J^{[p^{e'}]}/I^{[p^{e'}]} \right) &\leq l_P(L') \\ &\leq l_P \left(L/\mathfrak{m}_P^{[p^e]} L \right) \\ &\leq m l_P \left(\frac{P^{1/p^t}}{\mathfrak{m}_P^{[p^e]} P^{1/p^t} + D.P^{1/p^t}} \right). \end{aligned}$$

Note that $\mathfrak{m}_P^{[p^e]} P^{1/p^t} = (\mathfrak{m}_P^{1/p^t})^{[p^{e'}]}$, and $D \notin (\mathfrak{m}_P^{1/p^t})^{p^t\Delta+1}$ since $D \notin \mathfrak{m}_P^{\Delta+1}$. By Lemma 4.4.1, the last quantity is at most $mp^t \Delta p^{e'(d-1)}$. Therefore (recall $t = e' - e$)

$$(2) \qquad -\frac{1}{p^{ed}} l_P \left(J^{[p^e]}/I^{[p^e]} \right) + \frac{1}{p^{e'd}} l_P \left(J^{[p^{e'}]}/I^{[p^{e'}]} \right) \leq m \Delta p^{-e}.$$

Note that $H \to A^{1/p^t}$ is injective since A is generically étale and torsion-free over P. By Lemma 2.2.2 again, we have an exact sequence

$$D.A^{1/p^t} \longrightarrow H \longrightarrow L_1 \longrightarrow 0$$

where, again, L_1 is generated by m elements over P^{1/p^t} since H is, and is annihilated by D by construction. Since A is torsion-free over P, D^{p^t} is a nonzerodivisor on A, thus $D.A^{1/p^t} \cong A^{1/p^t}$. By the same argument as above, we get (2) with the signs on the left hand side reversed. This shows (1) and thus the proposition. $\square$

Proposition 5.1.3. *Let $(P, \mathfrak{m}_P, k)$ be a regular local ring of dimension d containing $\mathbf{F}_p$, $K = \mathrm{Frac}(P)$. Let A be a P-algebra, $\mathfrak{N}$ an ideal of A, and M a finite A-module. Let $\Delta \in \mathbf{Z}_{\geq 0}, m, e_0, b \in \mathbf{Z}_{\geq 1}$.*

Write $\overline{A} = A/\mathfrak{N}$. Assume the followings hold.

- *(i) $\overline{A}$ is a finite, generically étale, and torsion-free P-algebra generated by m elements as a P-module.*
- *(ii) There exist $e_1, \dots, e_n \in \overline{A}$ that map to a basis of $\overline{A} \otimes_P K$ such that $D := \mathrm{Disc}_{\overline{A}/P}(e_1, \dots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.*
- *(iii) $\mathfrak{N}^{[p^{e_0}]} = 0$.*
- *(iv) M has a filtration $M = M_b \supsetneq M_{b-1} \supsetneq \dots \supsetneq M_0 = 0$ such that $M_j/M_{j-1} \cong \overline{A}$ as A-modules.*

Then for all $e \in \mathbf{Z}, e > e_0$, and all ideals $I \subseteq J$ of A with $l_A(J/I) < \infty$, we have

$$\left| \frac{b}{p^{(e-e_0)d}} l_A \left(\frac{J^{[p^{e-e_0}]}\overline{A}}{I^{[p^{e-e_0}]}\overline{A}} \right) - \frac{1}{p^{ed}} l_A \left(\frac{J^{[p^e]}M}{I^{[p^e]}M} \right) \right| \leq p^{e_0} b^2 m \Delta p^{-e} l_A(J/I).$$

Proof. As before, we may assume $l_A(J/I) = 1$, $J = I + (u)$; and we may assume P complete and k algebraically closed. Calculation of lengths therefore does not depend on the base ring chosen.

Write $H = F_*^{e_0} P \otimes_P A$ and $\overline{H} = F_*^{e_0} P \otimes_P \overline{A}$. As seen in the proof of Proposition 5.1.2, there exists an exact sequence

$$0 \longrightarrow \overline{H} \longrightarrow F_*^{e_0}\overline{A} \longrightarrow L \longrightarrow 0$$

where L is annihilated by D and is generated by m elements as a $F_*^{e_0}P$-module. By *(iv)*, as an $F_*^{e_0}A$-module, $F_*^{e_0}M$ is a successive extension of b isomorphic copies of $F_*^{e_0}\overline{A}$, thus the same is true for $F_*^{e_0}M$ as an H-module. By *(iii)*, $F_*^{e_0}M$ is an $\overline{H}$-module. Thus the exact sequence above implies the existence of an exact sequence of $\overline{H}$-modules

$$0 \longrightarrow \overline{H}^{\oplus b} \longrightarrow F_*^{e_0}M \longrightarrow L' \longrightarrow 0$$

where L' is a successive extension of b isomorphic copies of L. In particular, L' is annihilated by $D^b \notin \mathfrak{m}_P^{b\Delta+1}$ and is generated by bm elements as an $F_*^{e_0}P$-module.

We now proceed as in the proof of Proposition 5.1.2. Taking colon with respect to $I^{[p^{e-e_0}]}$ and $u^{p^{e-e_0}}$, Lemma 5.1.1 gives

$$-bp^{e_0 d} l_P\left(\frac{J^{[p^{e-e_0}]}\overline{A}}{I^{[p^{e-e_0}]}\overline{A}}\right) + l_P\left(\frac{J^{[p^e]}M}{I^{[p^e]}M}\right) \leq l_P\left(\frac{L'}{\mathfrak{m}_P^{[p^{e-e_0}]}L'}\right).$$

By Lemma 4.4.1, $l_P(L'/\mathfrak{m}_P^{[p^{e-e_0}]}L') \leq bmp^{e_0}b\Delta p^{e(d-1)}$. Thus

$$(3) \qquad -\frac{b}{p^{(e-e_0)d}} l_A\left(\frac{J^{[p^{e-e_0}]}\overline{A}}{I^{[p^{e-e_0}]}\overline{A}}\right) + \frac{1}{p^{ed}} l_A\left(\frac{J^{[p^e]}M}{I^{[p^e]}M}\right) \leq p^{e_0} b^2 m\Delta p^{-e}.$$

The exact sequence above gives

$$D^b.F_*^{e_0}M \longrightarrow \overline{H}^{\oplus b} \longrightarrow L'' \longrightarrow 0$$

where L'' is annihilated by D^b by construction, and is generated by bm elements as a $F_*^{e_0}P$-module since $\overline{A}$ is generated by m elements as a P-module. By *(iv)*, M is a torsion-free P-module. Thus D^b is a nonzerodivisor on $F_*^{e_0}M$ and $D^b.F_*^{e_0}M \cong F_*^{e_0}M$. This gives the inequality (3) with signs on the left hand side reversed, showing the proposition. □

Corollary 5.1.4. *Notations and assumptions as in Proposition 5.1.3. Then for all $e \leq e' \in \mathbf{Z}, e > e_0$, and all ideals $I \subseteq J$ of A with $l_A(J/I) < \infty$, we have*

$$\left|\frac{1}{p^{ed}} l_A\left(\frac{J^{[p^e]}M}{I^{[p^e]}M}\right) - \frac{1}{p^{e'd}} l_A\left(\frac{J^{[p^{e'}]}M}{I^{[p^{e'}]}M}\right)\right| \leq (1+(1+p^{e-e'})p^{e_0}b)bm\Delta p^{-e} l_A(J/I).$$

Note that $p^{e-e'} \leq 1$.

Proof. Immediate from Propositions 5.1.2 and 5.1.3. □

5.2. **Uniform bound in excellent and less-excellent rings.** We shall use the following fact.

Theorem 5.2.1. *Let R be a Noetherian $\mathbf{F}_p$-algebra. Assume that $R/\mathfrak{p}$ is J-0 for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Then for every R-module M, there exists a constant $C = C(M)$ such that for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $e \in \mathbf{Z}_{\geq 1}$, $l(M_\mathfrak{p}/\mathfrak{p}^{[p^e]}M_\mathfrak{p}) \leq Cp^{e \dim M_\mathfrak{p}}$.

Proof. This is [PTY, Theorem 2.11], and also follows from [Smi16, Lemma 15], where R is assumed to be excellent. However, both proofs work under the assumption $R/\mathfrak{p}$ is J-0 for all $\mathfrak{p} \in \operatorname{Spec}(R)$. □

Consider the following condition on a Noetherian ring R.

Condition 5.2.2.

(i) R is J-2.
(ii) For all primes $\mathfrak{p}' \subset \mathfrak{p}$ of R with $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}') = 1$, $(R_\mathfrak{p}/\mathfrak{p}'R_\mathfrak{p})^\wedge$ is (R_0).
(iii) R is universally catenary.

Remark 5.2.3. An excellent R, or more generally a J-2, Nagata, and universally catenary R, satisfies Condition 5.2.2; and R_{red} is formally (S_1) (Definition 4.2.1) for such R. See [Stacks, Tag 0BJ0].

Theorem 5.2.4 (cf. [Pol18, Theorem 4.4])**.** *Let R be a Noetherian $\mathbf{F}_p$-algebra that satisfies Condition 5.2.2.*

Then for every finite R-module M, there exists a constant $C(M)$ with the following property. For all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $(R/\sqrt{\operatorname{Ann}_R(M)})^{\wedge}_{\mathfrak{p}}$ is (S_1), all ideals $I \subseteq J$ of $R_{\mathfrak{p}}$ with $l_{R_{\mathfrak{p}}}(J/I) < \infty$, and all $e \leq e' \in \mathbf{Z}_{\geq 1}$, the following holds.

$$\left| \frac{1}{p^{e \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^e]} M_{\mathfrak{p}}}{I^{[p^e]} M_{\mathfrak{p}}} \right) - \frac{1}{p^{e' \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^{e'}]} M_{\mathfrak{p}}}{I^{[p^{e'}]} M_{\mathfrak{p}}} \right) \right| \leq C(M) p^{-e} l_{R_{\mathfrak{p}}}(J/I).$$

Here by convention the left hand side is zero if $M_{\mathfrak{p}} = 0$.

Proof. We may replace R by $R/\operatorname{Ann}_R(M)$, so $\dim M_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p}$ for all $\mathfrak{p}$. Let $\mathfrak{p}_0$ be a minimal prime of R. Then there exists a submodule $N = N(\mathfrak{p}_0)$ of M that is a successive extension of isomorphic copies of $R/\mathfrak{p}_0$, such that $N_{\mathfrak{p}_0} = M_{\mathfrak{p}_0}$, by the theory of associated primes.

Let $N' = \oplus_{\mathfrak{p}_0} N(\mathfrak{p}_0)$ (not necessarily a submodule of M), so M and N' are isomorphic at all minimal primes of R, in particular $\operatorname{Ann}_R(N')$ is nilpotent. Apply the argument in [Pol18, proof of Corollary 3.4], using Theorem 5.2.1 instead of [Pol18, Proposition 3.3], we see that it suffices to prove the result for N'.

In fact, it suffices to prove the result for each $N(\mathfrak{p}_0)$. Indeed, assume the result is true for each $N(\mathfrak{p}_0)$ and let $C(\mathfrak{p}_0)$ be the corresponding constant. Let $C'(\mathfrak{p}_0)$ be the constant as in Theorem 5.2.1 for $N(\mathfrak{p}_0)$ and $C''(\mathfrak{p}_0) = \max\{2C'(\mathfrak{p}_0), C(\mathfrak{p}_0)\}$. We claim that $\sum_{\mathfrak{p}_0} C''(\mathfrak{p}_0)$ works for N'. To see this, let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $(R_{\mathrm{red}})^{\wedge}_{\mathfrak{p}}$ is (S_1). Let $\mathfrak{p}_0$ be a minimal prime of R contained in $\mathfrak{p}$. Since N' is the direct sum of all $N(\mathfrak{p}_0)$, it suffices to show

$$\left| \frac{1}{p^{e \operatorname{ht} \mathfrak{p}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^e]} N(\mathfrak{p}_0)_{\mathfrak{p}}}{I^{[p^e]} N(\mathfrak{p}_0)_{\mathfrak{p}}} \right) - \frac{1}{p^{e' \operatorname{ht} \mathfrak{p}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^{e'}]} N(\mathfrak{p}_0)_{\mathfrak{p}}}{I^{[p^{e'}]} N(\mathfrak{p}_0)_{\mathfrak{p}}} \right) \right| \leq C''(\mathfrak{p}_0) p^{-e} l_{R_{\mathfrak{p}}}(J/I).$$

If $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0) < \operatorname{ht}(\mathfrak{p})$, then this follows from [Pol18, Lemma 3.2] and the choice of $C'(\mathfrak{p}_0)$. Otherwise, $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0) = \operatorname{ht}(\mathfrak{p})$. Since $(R/\mathfrak{p}_0)^{\wedge}_{\mathfrak{p}}$ is (S_1), see Lemma 4.2.4, the desired inequality follows from the choice of $C(\mathfrak{p}_0)$.

Thus we may assume M is a successive extension of isomorphic copies of $R/\mathfrak{p}_0$ where $\mathfrak{p}_0$ is a fixed minimal prime of R. Replace R by $R/\operatorname{Ann}_R(M)$ once again, we may assume $\mathfrak{p}_0$ is the nilradical of R. Write $\overline{R} = R/\mathfrak{p}_0$. Let $b = l_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0})$ and let $e_0 \in \mathbf{Z}_{\geq 1}$ be such that $(\mathfrak{p}_0)^{[p^{e_0}]} = 0$. By Theorem 5.2.1 and [Pol18, Lemma 3.2], it suffices to find a constant $C = C(M)$ such that the desired inequality holds for all $e' \geq e > e_0$.

Note that $\overline{R}$ is (R_0) since it is an integral domain. Let $\overline{R} \to \overline{S}$ and δ, μ and Δ be as in Theorem 3.4.3. We shall show that $C = (1 + 2p^{e_0} b) b \delta^{\mu} \Delta$ works. Let $R \to S$ be a syntomic ring map that lifts $\overline{R} \to \overline{S}$, see [Stacks, Tag

07M8]. Then $\mathfrak{p}_0 S$ is a nilpotent ideal of S, so we can identify $\operatorname{Spec}(S)$ and $\operatorname{Spec}(\overline{S})$. Fix $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\overline{R}_{\mathfrak{p}}^{\wedge}$ (S_1), and let $\mathfrak{q} \in \operatorname{Spec}(\overline{S})$, $P \to \overline{S}_{\mathfrak{q}}^{\wedge}$ be as in the statement of Theorem 3.4.3. Lift the map $P \to \overline{S}_{\mathfrak{q}}^{\wedge}$ to a ring map $P \to S_{\mathfrak{q}}^{\wedge}$, possible as P is formally smooth over $\mathbf{F}_p$ [Stacks, Tag 07NL]. Note that $R \to S$ is flat quasi-finite, so $R_{\mathfrak{p}} \to S_{\mathfrak{q}}^{\wedge}$ is flat local with zero-dimensional closed fiber. Thus for all finite length $R_{\mathfrak{p}}$-modules X, $l_{R_{\mathfrak{p}}}(X) l_{S_{\mathfrak{q}}^{\wedge}}(S_{\mathfrak{q}}^{\wedge}/\mathfrak{p}S_{\mathfrak{q}}^{\wedge}) = l_{S_{\mathfrak{q}}^{\wedge}}(X \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}^{\wedge})$. Thus it suffices to prove an estimate as in the statement of Corollary 5.1.4 for the $S_{\mathfrak{q}}^{\wedge}$-module $M \otimes_R S_{\mathfrak{q}}^{\wedge}$ with the correct constants $b, m = \delta^{\mu}$, and Δ.

It thus suffices to verify the assumptions of Corollary 5.1.4 for $P \to S_{\mathfrak{q}}^{\wedge}$. Recall that $\overline{R}$ is an integral domain, and is universally catenary by assumption. Thus $\overline{S}_{\mathfrak{q}}$ is equidimensional (Lemma 4.1.1 since $\overline{R} \to \overline{S}$ is flat quasi-finite) and universally catenary, hence $\overline{S}_{\mathfrak{q}}^{\wedge}$ is equidimensional (Ratliff's result, [Stacks, Tag 0AW3]). Thus all minimal primes of $\overline{S}_{\mathfrak{q}}^{\wedge}$ are above $(0) \subseteq P$. By Lemmas 4.2.2 and 4.2.3, $\overline{S}_{\mathfrak{q}}^{\wedge}$ is (S_1), thus we see $\overline{S}_{\mathfrak{q}}^{\wedge}$ is a torsion-free P-module. Note that $\overline{S}_{\mathfrak{q}}^{\wedge}$ is a finite and generically étale P-algebra (Theorem 3.4.3(*iii*)).

By Theorem 3.4.3(*iv*), we can find $y_1, \ldots, y_{\mu} \in \mathfrak{q}$ such that $\mathfrak{q}\overline{S}_{\mathfrak{q}}^{\wedge} = \mathfrak{m}_P \overline{S}_{\mathfrak{q}}^{\wedge} + (\underline{y})$. Let $S' = P[y_1, \ldots, y_{\mu}] \subseteq \overline{S}_{\mathfrak{q}}^{\wedge}$, and $\mathfrak{m}' = \mathfrak{m}_P S' + (\underline{y})$. Then we see that $(S', \mathfrak{m}')$ is a local ring and that $\overline{S}_{\mathfrak{q}}^{\wedge} = S' + \mathfrak{m}'\overline{S}_{\mathfrak{q}}^{\wedge}$ by Theorem 3.4.3(*ii*). Therefore $S' = \overline{S}_{\mathfrak{q}}^{\wedge}$. By Lemma 4.3.1 (and Theorem 3.4.3(*iii*)), we see that $\overline{S}_{\mathfrak{q}}^{\wedge}$ is generated by at most δ^{μ} elements as a P-module.

Let $e_1, \ldots, e_n \in \overline{S}_{\mathfrak{q}}^{\wedge}$ be as in Theorem 3.4.3(*v*), and let $D = \operatorname{Disc}_{\overline{S}_{\mathfrak{q}}^{\wedge}/P}(e_1, \ldots, e_n)$. Then $D \notin \mathfrak{m}_P^{\Delta+1}$.

We have $(\mathfrak{p}_0 S_{\mathfrak{q}}^{\wedge})^{[p^{e_0}]} = 0$ since $\mathfrak{p}_0{}^{[p^{e_0}]} = 0$. Since M is a successive extension of b isomorphic copies of $\overline{R}$, $M \otimes_R S_{\mathfrak{q}}^{\wedge}$ is a successive extension of b isomorphic copies of $\overline{S}_{\mathfrak{q}}^{\wedge}$. We have verified all assumptions of and checked all constants in Corollary 5.1.4, showing what we want. □

In view of Remark 5.2.3, the following is a special case of the theorem.

Corollary 5.2.5. *Let R be a Noetherian $\mathbf{F}_p$-algebra. Assume that R is excellent, or more generally J-2, Nagata, and universally catenary.*

Then for every finite R-module M, there exists a constant $C(M)$ with the following property. For all $\mathfrak{p} \in \operatorname{Spec}(R)$, all ideals $I \subseteq J$ of $R_{\mathfrak{p}}$ with $l_{R_{\mathfrak{p}}}(J/I) < \infty$, and all $e \le e' \in \mathbf{Z}_{\ge 1}$, the following holds.

$$\left| \frac{1}{p^{e \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}}\left(\frac{J^{[p^e]}M_{\mathfrak{p}}}{I^{[p^e]}M_{\mathfrak{p}}}\right) - \frac{1}{p^{e' \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}}\left(\frac{J^{[p^{e'}]}M_{\mathfrak{p}}}{I^{[p^{e'}]}M_{\mathfrak{p}}}\right) \right| \le C(M) p^{-e} l_{R_{\mathfrak{p}}}(J/I).$$

6. Applications: semi-continuity

6.1. **Hilbert-Kunz multiplicity.** For a Noetherian local $\mathbf{F}_p$-algebra $(R, \mathfrak{m})$, denote by $\lambda_e(R)$ the number $\frac{l(R/m^{[p^e]})}{p^{e \dim R}}$. We have, by definition, $e_{\mathrm{HK}}(R) = \lim_e \lambda_e(R)$, and the limit exists [Mon83].

The following slightly strengthens [SB79].

Lemma 6.1.1. *Let R be a Noetherian $\mathbf{F}_p$-algebra, $\mathfrak{p} \in \operatorname{Spec}(R)$. Assume that $R/\mathfrak{p}$ is J-0.*

Let e be a positive integer. Then for some $g \notin \mathfrak{p}$ and all $\mathfrak{P} \in D(g) \cap V(\mathfrak{p})$, $\lambda_e(R_\mathfrak{p}) = \lambda_e(R_\mathfrak{P})$.

Proof. We may assume $R/\mathfrak{p}$ regular. By Theorem 3.4.1(i), we may assume for all $\mathfrak{P} \in V(\mathfrak{p})$, $\operatorname{ht}(\mathfrak{P}) = \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{P}/\mathfrak{p})$. It remains to apply Discussion 3.4.2 to the module $M = R/\mathfrak{p}^{[p^e]}$ and the regular sequence $\pi_1 = t_1^{p^e}, \ldots, \pi_h = t_h^{p^e}$, where $t_1, \ldots, t_h \in R_\mathfrak{P}$ map to a regular sequence of parameters of $R_\mathfrak{P}/\mathfrak{p}R_\mathfrak{P}$. □

Corollary 6.1.2. *Let R be a Noetherian $\mathbf{F}_p$-algebra. Assume that $R/\mathfrak{p}$ is J-0 for all $\mathfrak{p} \in \operatorname{Spec}(R)$, and that R is catenary and locally equidimensional.*

Let e be a positive integer. Then the function $\mathfrak{p} \mapsto \lambda_e(R_\mathfrak{p})$ is constructible and upper semi-continuous.

Proof. By Lemma 6.1.1 our function is constructible. We have $\operatorname{ht}(\mathfrak{P}) = \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{P}/\mathfrak{p})$ for all $\mathfrak{p} \subseteq \mathfrak{P} \in \operatorname{Spec}(R)$, since R is catenary and locally equidimensional. By [Kun76, Corollary 3.8], our function is non-decreasing along specialization. Thus our function is upper semi-continuous by general topology [Stacks, Tag 0542]. □

Theorem 6.1.3 (cf. [Smi16, Theorem 23])**.** *Let R be a Noetherian $\mathbf{F}_p$-algebra. Assume that R satisfies Condition 5.2.2, and that R_{red} is formally (S_1) (Definition 4.2.1). (For example, if R is excellent, or if R is J-2, Nagata, and universally catenary, see Remark 5.2.3.)*

If R is locally equidimensional, then the function $\mathfrak{p} \mapsto e_{\mathrm{HK}}(R_\mathfrak{p})$ is upper semi-continuous.

Proof. Apply Theorem 5.2.4 to $M = R, I = \mathfrak{p}R_\mathfrak{p}, J = R_\mathfrak{p}$, we see that our function is the uniform limit of the functions $\mathfrak{p} \mapsto \lambda_e(R_\mathfrak{p})$. These functions are upper semi-continuous by Corollary 6.1.2. Thus our function is upper semi-continuous as well. □

6.2. ***F*-signature.** For a Noetherian local $\mathbf{F}_p$-algebra $(R, \mathfrak{m})$, denote by $s_e(R)$ the eth normalized F-splitting number as in [EY11, Definition 1.1]. The limit $s(R) = \lim s_e(R)$ is called the F-signature of R. The limit was first shown to exist in [Tuc12]. (We also recover the existence in Proposition 6.2.4 below.)

We use the following facts.

Fact 6.2.1. Let $(R, \mathfrak{m}) \to (R', \mathfrak{m}')$ be a flat map of Noetherian local $\mathbf{F}_p$-algebras with $\mathfrak{m}R' = \mathfrak{m}'$. Then $s_e(R) = s_e(R')$ for all e, see [Yao06, Remark 2.3(3)].

Fact 6.2.2. For a Noetherian local $\mathbf{F}_p$-algebra $(R, \mathfrak{m})$, $s_e(R) > 0$ for some e if and only if $s_e(R) > 0$ for all e, if and only if R is F-pure. Indeed, using the notations preceding [EY11, Definition 1.1], $s_e(R) > 0$ if and only if $R^{(e)} \otimes_R k \to R^{(e)} \otimes_R E$ is nonzero, if and only if k is not killed in $R^{(e)} \otimes_R E$, if and only if $R \to R^{(e)}$ is pure, see [Fed83, Proposition 1.3(5)].

Fact 6.2.3. Let $(R, \mathfrak{m})$ be a Noetherian local $\mathbf{F}_p$-algebra. For two positive integers e, e', there exists an $\mathfrak{m}$-primary ideal I and an element $u \in (I : \mathfrak{m})$ such that $s_e(R) = p^{-e \dim R} l((I, u)^{[p^e]}/I^{[p^e]})$ and $s_{e'}(R) = p^{-e' \dim R} l((I, u)^{[p^{e'}]}/I^{[p^{e'}]})$. Indeed, by Fact 6.2.1 we may assume R complete, and by Fact 6.2.2 we may assume R F-pure (otherwise take $I = \mathfrak{m}$ and $u = 0$), in particular reduced, so [Pol18, Lemma 5.4] applies.

Proposition 6.2.4. *Let R be a Noetherian ring that satisfies Condition 5.2.2. Let $C = C(R)$ be as in Theorem 5.2.4.*

Then for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $e \le e' \in \mathbf{Z}_{\ge 1}$, $|s_e(R_\mathfrak{p}) - s_{e'}(R_\mathfrak{p})| \le Cp^{-e}$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. If $R_\mathfrak{p}^\wedge$ is not reduced, then $s_e(R_\mathfrak{p}) = s_{e'}(R_\mathfrak{p}) = 0$, see Facts 6.2.2 and 6.2.1. So we only need to show the inequality for those $\mathfrak{p}$ with $R_\mathfrak{p}^\wedge$ reduced. By Fact 6.2.3, we need to show

$$\left| \frac{1}{p^{e \operatorname{ht} \mathfrak{p}}} l_{R_\mathfrak{p}} \left(J^{[p^e]}/I^{[p^e]} \right) - \frac{1}{p^{e' \operatorname{ht} \mathfrak{p}}} l_{R_\mathfrak{p}} \left(J^{[p^{e'}]}/I^{[p^{e'}]} \right) \right| \le Cp^{-e}.$$

where I is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal of $R_\mathfrak{p}$, and $J = (I, u)$ for some $u \in (I : \mathfrak{p}R_\mathfrak{p})$. In particular $l_{R_\mathfrak{p}}(J/I) \le 1$. The inequality now follows from Theorem 5.2.4. □

Lemma 6.2.5. *Let R be a Noetherian ring such that $R/\mathfrak{p}$ is J-0 for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Assume that either R is Gorenstein, or that R is a locally equidimensional quotient of a regular Noetherian ring. Then for all e, the function $\mathfrak{p} \mapsto s_e(R_\mathfrak{p})$ is lower semi-countinuous.

Proof. This is [EY11, Theorems 3.4 and 4.2], except that R is assumed to be excellent there. However, from the proof it is clear that $R/\mathfrak{p}$ being J-0 for all $\mathfrak{p} \in \operatorname{Spec}(R)$ is enough. □

The following result is likely to be well-known; we include it for completeness.

Lemma 6.2.6. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local $\mathbf{F}_p$-algebra. Then the followings hold.*

(i) *If $s(R) > 0$, then R is strongly F-regular.*
(ii) *If R is a G-ring then the converse to (i) holds.*

Proof. Assume $s(R) > 0$. Let $(R', \mathfrak{m}', k')$ be a Noetherian local flat R-algebra with R' complete, $\mathfrak{m}R' = \mathfrak{m}'$, and k' algebraically closed. Fact 6.2.1 shows $s(R') > 0$, and [AL02] shows R' stongly F-regular. Thus R is stongly F-regular by [Has10, Lemma 3.17].

Conversely, assume R is a G-ring and strongly F-regular. Then R is normal [Has10, Corollary 3.7], thus excellent, cf. [Stacks, Tags 0C23 and 0AW6]. By [Has10, Lemma 3.28] the completion $R^\wedge$ is strongly F-regular. By [Has10, Lemma 3.30] there exists a flat local ring map $R^\wedge \to R'$ such that R' is F-finite and strongly F-regular, and that $\mathfrak{m}R'$ is the maximal ideal of R'. By [AL02], $s(R') > 0$, and $s(R) = s(R')$ by Fact 6.2.1. $\square$

Theorem 6.2.7 (cf. [Pol18, Theorem 5.6])**.** *Let R be a Noetherian $\mathbf{F}_p$-algebra that satisfies Condition 5.2.2(i)(ii). Assume that either R is Gorenstein, or that R is a quotient of a regular Noetherian ring. Then the function $\mathfrak{p} \mapsto s(R_\mathfrak{p})$ is lower semi-countinuous.*

Proof. Note that $s(R_\mathfrak{p}) \geq 0$ for all $\mathfrak{p}$. If $s(R_\mathfrak{p}) > 0$ for some $\mathfrak{p}$, then $R_\mathfrak{p}$ is normal by Lemma 6.2.6 and [Has10, Corollary 3.7]. Since R is J-2, the normal locus of R is open, see [EGA IV$_2$, Corollaire 6.13.5]. Thus we may assume R normal, in particular locally equidimensional.

Since a Cohen-Macaulay ring is universally catenary [Stacks, Tag 00NM], R satisfies Condition 5.2.2. By Proposition 6.2.4 the function $\mathfrak{p} \mapsto s(R_\mathfrak{p})$ is the uniform limit of the functions $\mathfrak{p} \mapsto s_e(R_\mathfrak{p})$ which are lower semi-countinuous by Lemma 6.2.5, thus $\mathfrak{p} \mapsto s(R_\mathfrak{p})$ is lower semi-countinuous. $\square$

Corollary 6.2.8. *Let R be a Noetherian quasi-excellent $\mathbf{F}_p$-algebra. Assume that R is either Gorenstein or a quotient of a regular Noetherian ring. Then the locus*

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \mid R_\mathfrak{p} \textit{ is strongly } F\textit{-regular}\}$$

is open.

Proof. For $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p}$ is strongly F-regular if and only if $s(R_\mathfrak{p}) > 0$, see Lemma 6.2.6. $\square$

Remark 6.2.9. A quasi-excellent quotient of a regular ring is always a quotient of a quasi-excellent regular ring. This follows immediately from [KS21].

Remark 6.2.10. Kevin Tucker informed the author that he was able to prove the openness of the strongly F-regular locus for any quotient of a regular $\mathbf{F}_p$-algebra via a different method.

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