Injective and radicial morphisms of varieties

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Theorem. *Let k be a field, $f : X \to Y$ a k-morphism of finite type k-schemes. Assume that $\dim X > 0$, and that f maps different closed points of X to different closed points of Y. Then there exists a dense open U of X such that $f|_U : U \to Y$ is radicial.*

Proof. We may replace X by a dense open and then its reduction, so we may assume X is a disjoint union of integral schemes. The images of the irreducible components of X are pairwise disjoint since they are constructible and share no closed points. Therefore we may assume X integral. Replace Y by the closure of $f(X)$ we may assume Y is integral and f is dominant.

Since f sends different closed points to different closed points, f is quasi-finite. Upon replacing Y by a nonempty open subscheme and X the preimage, we may therefore find a factorization $X \xrightarrow{g} Z \xrightarrow{h} Y$ of f such that Z is integral, g and h are surjective, h is radicial, and g is finite étale. We may then replace f by g and assume f is finite étale. We must show the degree d of f is 1. Note that for every closed point $x \in X$ we have $[\kappa(x) : \kappa(f(x))] = d$.

Since a finite separable extension is generated by a single element, after replacing Y by a nonempty open we may assume $Y = \mathrm{Spec}(B)$ affine and X is a closed subscheme of $\mathbf{A}^1_Y = \mathrm{Spec}(B[S])$. Fix a closed immersion $Y \to \mathbf{A}^N_k = \mathrm{Spec}(k[T_1, \ldots, T_N])$, such that the image W of Y under the projection $\mathbf{A}^N_k \to \mathrm{Spec}(k[T_N])$ is not a singleton, possible as $\dim Y = \dim X > 0$. Then W is open.

We see X embeds in $\mathbf{A}^{N+1}_k = \mathrm{Spec}(k[T_1, \ldots, T_N, S])$. Let $\pi : \mathbf{A}^{N+1}_k \to \mathrm{Spec}(k[S])$ be the projection. We consider two cases.

Case 1. The set $\pi(X)$ is a singleton. In this case let κ be the residue field of the sole element of $\pi(X)$, κ^s the separable closure of k in κ, and let E/κ^s be a finite separable extension such that E/k is Galois. Since W is open, we can find $t_N \in W(\overline{k})$ such that E embeds in $k(t_N)$, where $\overline{k}$ is an algebraic closure of k. (If k is infinite we may arrange $E = k(t_N)$.) By the definition of W, there exists $t_1, \ldots, t_{N-1} \in \overline{k}$ such that the point $(t_1, \ldots, t_N) \in \mathbf{A}^N(\overline{k})$ is in $Y(\overline{k})$, and we denote the corresponding closed point of Y by y, so $\kappa(y) = k(t_1, \ldots, t_N)$. As f is surjective, there exists $s \in \overline{k}$ such that $(t_1, \ldots, t_N, s) \in \mathbf{A}^{N+1}(\overline{k})$ is in $X(\overline{k})$, and we denote the corresponding closed point of X by x, so $\kappa(x) = k(t_1, \ldots, t_N, s)$. By definition, $\kappa = k(s)$; so $\kappa(x)/\kappa(y)$ is purely inseparable as E embeds in $k(t_N)$. Since f is étale, we see $\kappa(x) = \kappa(y)$, so $d = 1$, as desired.

Case 2. The set $\pi(X)$ is not finite, hence open. Assume k infinite for the moment; the case k is finite is dealt later. Then $\pi(X)$ has a k-rational point. In other words, there exist $(t_1, \ldots, t_N, s) \in \mathbf{A}^{N+1}(\bar{k})$ that is in $X(\bar{k})$ with $s \in k$. Denote the corresponding closed point of X by x, then $\kappa(x) = k(t_1, \ldots, t_N, s)$ and $\kappa(f(x)) = k(t_1, \ldots, t_N)$. But these two fields are equal as $s \in k$, thus $d = 1$.

Finally, we settle the case k finite. We do not actually need much of the reductions above; we continue from the end of the second paragraph of this proof. We have $\dim Y = \dim X > 0$. We may replace Y by a one-dimensional closed subscheme to assume $\dim Y = 1$.

Replacing Y by a nonempty open and k by a finite extension, we may assume Y is smooth and geometrically irreducible over k. If X were not geometrically irreducible over k, then there is a factorization $X \to Y \times_{\operatorname{Spec} k} \operatorname{Spec} k' \to Y$ where k'/k is a nontrivial extension. There exists a point $y \in Y$ with $[\kappa(y) : k]$ and $[k' : k]$ not coprime (by, for example, the Hasse-Weil estimate), so y has more than 1 preimages in $Y \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ and thus more than 1 preimages in X, a contradiction. So X is geometrically irreducible over k. Then, similarly, there exists $x \in X$ such that $[\kappa(x) : k]$ is coprime to d. Since $[\kappa(x) : k] = d[\kappa(f(x)) : k]$ we see that $d = 1$, as desired. $\square$