A characterization of Noetherian local rings

Shiji Lyu

January 2024

Theorem. *Let $(R, \mathfrak{m})$ be a local ring. Then R is Noetherian if and only if the following two conditions are satisfied.*

(*i*) $\mathfrak{m}$ *is finitely generated.*

(*ii*) *Every finitely generated ideal of R is a finite intersection of finitely generated primary ideals.*

Proof. "Only if" is clear. Assume (*ii*) and (*i*). Let $\mathfrak{p}$ be a prime ideal of R. It suffices to show $\mathfrak{p}$ finitely generated. By induction on the number of generators of $\mathfrak{m}$, we may assume that there exists $a \in R$ such that $aR + \mathfrak{p}$ is finitely generated. We may assume $a \notin \mathfrak{p}$. Take a finitely generated ideal $\mathfrak{p}_0 \subseteq \mathfrak{p}$ with $aR + \mathfrak{p}_0 = aR + \mathfrak{p}$, and let $\mathfrak{p}_1$ be an arbitrary finitely generated ideal with $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}$. We shall show that $\mathfrak{p}_0 = \mathfrak{p}_1$, thus $\mathfrak{p} = \mathfrak{p}_0$ is finitely generated.

By (*ii*), we have

$$\mathfrak{p}_1 = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

where $\mathfrak{q}_i$ are finitely generated primary ideals. Fix i with $\mathfrak{q}_i \subseteq \mathfrak{p}$, possible since $\mathfrak{p}_1 \subseteq \mathfrak{p}$. Then $aR + \mathfrak{p}_0 = aR + \mathfrak{q}_i = aR + \mathfrak{p}$, so $\mathfrak{q}_i = aR \cap \mathfrak{q}_i + \mathfrak{p}_0$. Since $\mathfrak{q}_i$ is primary and since $a \notin \mathfrak{p}$, we see $aR \cap \mathfrak{q}_i = a\mathfrak{q}_i$. By Nakayama's Lemma $\mathfrak{q}_i = \mathfrak{p}_0$, so $\mathfrak{p}_1 = \mathfrak{p}_0$, as desired. □