# GREENBERG-SHALOM'S COMMENSURATOR HYPOTHESIS AND APPLICATIONS

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Abstract. We discuss many surprising implications of a positive answer to a question raised in some cases by Greenberg in the '70s and more generally by Shalom in the early 2000s. We refer to this positive answer as the Greenberg-Shalom hypothesis. This hypothesis then says that any infinite discrete subgroup of a semisimple Lie group with dense commensurator is a lattice in a product of some factors. For some applications it is natural to extend the hypothesis to cover semisimple algebraic groups over other fields as well.

## Contents



## 1. Introduction

Let $G$ be a real or $p$-adic semisimple Lie group with finite center and without compact factors, or a finite product of such groups. More precisely, we consider $G = \mathbb{G}(k)$ where $k$ is a local field of characteristic zero and $\mathbb{G}$

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is a semisimple algebraic group defined over $k$ and also products of groups of this type. Let $\Gamma \subseteq G$ be a discrete subgroup with commensurator $\Delta$. Borel proved that if $\Gamma$ is an arithmetic lattice, then the commensurator contains the rational points of $G$ and in particular is almost dense in $G$ [Bor66]. Here, 'almost dense' means its closure has finite index in $G$. Greenberg (for $G = \mathrm{SO}(n, 1)$) and Shalom (in general) asked whether almost dense commensurator of a discrete, Zariski-dense subgroup $\Gamma$ detects its arithmeticity. We formulate the positive answer to this question as a hypothesis, since here we are primarily interested in the implications if it is true.

**Hypothesis 1.1** (Greenberg [Gre74], Shalom, see [LLR11])**.** Let $G$ be a semisimple Lie group with finite center and without compact factors. Suppose $\Gamma \subseteq G$ is a discrete, Zariski-dense subgroup of $G$ whose commensurator $\Delta \subseteq G$ is almost dense. Then $\Gamma$ is an arithmetic lattice in $G$.

We will refer to Hypothesis 1.1 as the *Greenberg-Shalom hypothesis*. We remark here that the hypothesis seems quite plausible on first encounter, and Greenberg and Shalom both seem inclined to believe it. On the other hand, the many implications of the hypothesis discussed here may cast some doubt on the likelihood that it is correct. The purpose of this article is to exhibit these connections between the hypothesis and other problems, many of which do not a priori involve commensurators. A short overview of these applications (numbering refers to the corresponding (sub)section):

- (4.1) A question of Benoist on existence of discrete, irreducible free or surface subgroups in products of simple real and $p$-adic Lie groups,
- (4.2) A conjecture of Lyndon-Ullman and Kim-Koberda on groups generated by parabolics,
- (4.3) Existence of elements with integral traces in hyperbolic 3-manifold groups,
- (5) Applications to the rank (i.e. minimal number of generators) of $S$-arithmetic lattices, related invariants, and a question of Lubotzky,
- (6) A question of Serre on coherence of $\mathrm{SL}(2, \mathbb{Z}[1/p])$ and related groups, and a problem of Wise whether coherence is geometric,
- (7) The Margulis-Zimmer conjecture on arithmeticity of commensurated subgroups of $S$-arithmetic lattices.
- (8) A question of Fisher-Larsen-Spatzier-Stover on irreducible surface group acting on products of trees, as well as a related question about the minimal transcendence degree of a surface group representation over a field of characteristic $p$.

To close out this introduction, we discuss the quite limited progress on Greenberg-Shalom's question: First, Margulis proved that non-arithmetic lattices have discrete commensurators, thereby resolving Greenberg-Shalom's question for lattices. At roughly the same time and apparently unaware of Margulis' work, Greenberg proved Hypothesis 1.1 for finitely generated subgroups of $G = \mathrm{SL}(2, \mathbb{R})$ [Gre74]. Building on work by Leininger-Long-Reid [LLR11], Mj proved Hypothesis 1.1 for finitely generated subgroups of $\mathrm{SL}(2, \mathbb{C})$ [Mj11]. For all other cases of $G$, as well as general infinitely generated subgroups of the above, Hypothesis 1.1 is open. Koberda-Mj proved the hypothesis for normal subgroups $\Gamma$ of arithmetic lattices in rank 1 with positive first Betti number [KM19]. Fisher-Mj-Van Limbeek proved the hypothesis for all normal subgroups of lattices [FMvL22]. Greenberg (for $SO(n,1)$) and Mj (in general) also proved that any group satisfying the conditions of the hypothesis has dense limit set [Gre74, Mj11].

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## 2. Notation, standing assumptions, and background on algebraic groups

In this section, we will fix some standing assumptions that will be in place for the rest of this article. Subsequent sections may have further assumptions that will be detailed at the beginning of each section.

**Standing assumptions 2.1.**

- $I$ is a finite set indexing local fields $k_i, i \in I$, of characteristic zero.
- $\mathbb{G}_i$ are connected, absolutely simple, isotropic, adjoint algebraic groups over $k_i$. We set $G_i := \mathbb{G}_i(k_i)$ and $G = \prod_{i \in I} G_i$. When not otherwise specified, we equip $G$ with the analytic topology.
- $\Gamma \subseteq G$ is a discrete subgroup with almost dense commensurator $\Delta \subseteq G$ and with unbounded projection to each $G_i$. Here a subgroup is said to be *almost dense* if its closure has finite index. See Section 2.1 for more information on almost dense subgroups.

**Remark 2.2.** Much (but not all) of our discussion also applies when $k_i$ are local fields of positive characteristic, and the Greenberg-Shalom problem can be formulated, and is open and interesting, in that setting as well.

**Notation 2.3.** Let $G$ be as above. For $i \in I$, we denote by $\mathrm{pr}_i : G \to G_i$ the canonical projection. For $J \subseteq I$, we set $G_J := \prod_{j \in J} G_j$, and $\mathrm{pr}_J : G \to G_J$ denotes the canonical projection.

We introduce the following the following irreducibility assumption:

**Definition 2.4.** Let $G$ be as in the above standing assumptions and let $L \subseteq G$ be a subgroup. We say $L$ is *strongly irreducible* if for every proper subset $J \subsetneq I$, the projection $\mathrm{pr}_J(L) \subseteq G_J$ is almost dense.

From the above definition, it is not at all obvious that problems for general discrete (or closed) groups can be reduced to strongly irreducible discrete (or closed) groups, i.e. if $\Theta \subseteq G$ is a discrete subgroup, it is not clear there exists $J \subseteq I$ such that $\mathrm{pr}_J(\Theta) \subseteq G_J$ is discrete and strongly irreducible. This will be discussed in more detail in Section 3, where strong irreducibility is related to the notion of *irreducibility*, which is easier to reduce to.

2.1. **Preliminaries on simple algebraic groups.** We will now review classical results in the study of simple algebraic groups for later use, and we end by proving that under Standing Assumptions 2.1, the projections of $\Gamma$ are Zariski-dense. We start by fixing some notation:

**Standing assumptions 2.5.** For the rest of this section, $k$ denotes a local field of characteristic zero, and $\mathbb{H}$ denotes a connected semisimple adjoint algebraic group defined over $k$ such that $\mathbb{H}(k)$ does not have compact factors.

We write $H := \mathbb{H}(k)$. Let us start with the following useful definition:

**Definition 2.6.** $H^+$ denotes the group generated by unipotent elements of $H$.

It is not necessary to assume $k$ is a local field of characteristic zero, but the definition of $H^+$ is more difficult in general. To simplify the discussion and since we only need this notion for local fields of characteristic zero, we will only give this definition (but see e.g. [Mar91, Section I.1.5]). We have the following result due to Borel-Tits:

**Theorem 2.7** (Borel-Tits [BT73, 6.7, 6.9, and 6.14])**.**

*(i)* *$H^+ \subseteq H$ is Zariski-dense and (analytically) open. In particular $H^+$ has finite index in $H$.*
*(ii)* *$H^+$ does not contain any proper subgroup of finite index. Hence every finite index subgroup of $H$ contains $H^+$.*

If $k = \mathbb{R}$, then $H^+$ is the connected component of $H$ that contains the identity (see again [BT73, 6.14]). In particular, any open subgroup of $H$ contains $H^+$. A nonarchimedean analogue is the following unpublished result of Tits, with a published proof due to Prasad:

**Theorem 2.8** (Tits-Prasad [Pra82])**.** *Assume in addition that $\mathbb{H}$ is almost simple. Then any noncompact, open subgroup of $H$ contains $H^+$.*

Recall that a subgroup $\Delta \subseteq H$ is almost dense if its closure has finite index in $H$. In view of the above results, $\Delta$ is almost dense if and only if its closure contains $H^+$. Using that $H^+$ is Zariski-dense, it follows that any almost dense group is Zariski-dense.

Now, let us return to the situation of groups with almost dense commensurators and prove the following straightforward lemma:

**Lemma 2.9.** *Suppose $\Theta \subseteq H$ is an infinite subgroup with Zariski-dense commensurator $\Delta$. Then there is a subset of factors $H_J \subseteq H$ containing $\Theta$ and $\Theta \subseteq H_J$ is Zariski-dense.*

**Remark 2.10.** In the above lemma, we do not assume that $\Theta$ is a discrete subgroup of $H$.

*Proof.* Consider the Zariski-closures of finite index subgroups of $\Theta$, partially ordered by inclusion. We say $\Theta' \subseteq \Theta$ is Zariski-dense in $\Theta$ if $\overline{\Theta'}^Z = \overline{\Theta}^Z$. By the Noetherian property, there is a finite index subgroup $\Theta'$ of $\Theta$ such that any finite index subgroup $\Theta'' \subseteq \Theta'$ is Zariski-dense in $\Theta'$. Let $L$ be the Zariski-closure of $\Theta'$ in $H$.

We claim that $\Delta$ normalizes $L$: Indeed, for any $\delta \in \Delta$, there is some finite index subgroup $\Theta'_\delta \subseteq \Theta'$ with $\delta\Theta'_\delta\delta^{-1} \subseteq \Theta'$. Since both $\Theta'_\delta$ and $\delta\Theta'_\delta\delta^{-1}$ are Zariski-dense in $\Theta'$, it follows that $\delta$ normalizes $L$. Since $\Delta$ is Zariski-dense and $L$ is Zariski-closed, it follows that $H$ normalizes $L$. Since $H$ is adjoint, we have $L = H_J$ for some collection of factors $J$.

Finally, we conclude that $\Theta$ is contained in $H_J$: Indeed, the image of $\Theta$ in the remaining factors $H_{J^c}$ is finite and normalized by the projection $\mathrm{pr}_{J^c}(\Delta)$. A finite index subgroup $\Delta'$ of $\Delta$ has projection to $H_{J^c}$ that centralizes the image of $\Theta$ and is still Zariski-dense, so that the image of $\Theta$ is central in $H_{J^c}$. Finally, since $\mathbb{H}$ is adjoint, $H_{J^c}$ is centerless. □

**Remark 2.11.** The proof does not use the analytic topology on $k$, so the statement is still true if $k$ is a number field.

**Remark 2.12.** In particular, if $\Gamma$ and $G$ are as in Standing Assumptions 2.1, then for every $i \in I$, the projection $\mathrm{pr}_i(\Gamma) \subseteq G_i$ is Zariski-dense: Namely, $\mathrm{pr}_i(\Gamma)$ has almost dense (and hence Zariski-dense) commensurator by assumption. Since $\mathrm{pr}_i(\Gamma)$ is unbounded, it is infinite, so that Lemma 2.9 implies it is Zariski-dense.

## 3. Irreducibility

In this section we discuss various notions of irreducibility. Let $G = \prod_i G_i$ be as in Standing Assumptions 2.1, and let $\Gamma \subseteq G$ be a discrete subgroup. We want to define what it means for $\Gamma$ to be *irreducible* in the product $G = \prod_i G_i$. There are two obvious ways one might reduce the study of $\Gamma$ to a simpler scenario, namely by passing to either a quotient of $G$ or a subgroup of $G$. We can pass to a quotient precisely when the projection of

$\Gamma$ to some proper subset of factors is discrete. To see when one may pass to a subgroup is more involved, and we need to introduce some notation.

We set $\mathbb{Q}_\infty := \mathbb{R}$ and if $p$ is a finite prime or $\infty$, we denote by $I_p \subseteq I$ the subset of $\mathbb{Q}_p$-analytic factors of $G$ (so $I_\infty$ denotes the collection of archimedean factors), and we write $G_p := \prod_{i \in I_p} G_i$ for their product. We restrict scalars on all factors to $\mathbb{Q}_p$, and define the $\mathbb{Q}_p$-algebraic group $R_p\mathbb{G} := \prod_{i \in I_p} R_{k_i/\mathbb{Q}_p}\mathbb{G}_i$. We write $R_pG := (R_p\mathbb{G})(\mathbb{Q}_p)$. Of course we have $R_pG \cong G_p$ as $\mathbb{Q}_p$-analytic groups, but $R_pG$ comes equipped with a Zariski-topology as a $\mathbb{Q}_p$-algebraic group. We will refer to this as the $\mathbb{Q}_p$-*Zariski-topology* on $G_p$. Note that given a discrete group $\Gamma \subseteq G$, we could first project to any subproduct $G_J$ such that $\mathrm{pr}_J(\Gamma)$ is discrete, and then pass to the subgroup $H := \prod_p \overline{\mathrm{pr}_{J_p}(\Gamma)}^{\mathbb{Q}_p}$, where the closure is taken with respect to the $\mathbb{Q}_p$-Zariski-topology on $R_pG$. This motivates the following definition:

**Definition 3.1.** A subgroup $\Theta$ is *irreducible* in $G$ if for every proper subset of factors $J \subsetneq I$, the projection of $\Theta$ to $G_J = \prod_{j \in J} G_j$ is not discrete, and for every $p$ (a finite prime or $\infty$), the projection of $\Theta$ to $G_p$ is $\mathbb{Q}_p$-Zariski-dense.

Most irreducible groups we consider will be discrete, but we do not assume this for the purposes of the above definition.

**Remark 3.2.** If $G$ is simple, then irreducibility is equivalent to $\mathbb{Q}_p$-Zariski-denseness.

**Remark 3.3.** Any discrete irreducible subgroup $\Theta \subseteq G$ has unbounded projections to every simple factor $G_i$: If $G$ itself is simple, this follows from Zariski-denseness of $\Theta$. If there is more than one factor, then for any factor $G_i$, the image of $\Theta$ in $G/G_i$ is indiscrete. For any bounded open set $U \subseteq G/G_i$, the set $\Theta_U$ of elements of $\Theta$ with image in $U$ is infinite and has discrete projection to $G_i$, and therefore is unbounded in $G_i$.

In particular, any discrete, irreducible subgroup of $G$ with almost dense commensurator satisfies Standing Assumptions 2.1.

Recall that for a subset of factors $J \subsetneq I$, we write $\mathrm{pr}_J : G \to G_J$ for the natural projection. Our goal will be to prove that projections of irreducible groups to proper subsets of factors do not merely fail to be discrete, but are in fact almost dense:

**Proposition 3.4.** *Let $\Gamma \subseteq G$ be discrete and irreducible. Then for any proper subset $J \subsetneq I$, the closure $\overline{\mathrm{pr}_J(\Gamma)}$ contains $G_J^+$.*

*Therefore, a discrete subgroup is irreducible if and only if it is strongly irreducible.*

We will often use the following observation without explicit comment:

**Remark 3.5.** For a connected semisimple algebraic group $\mathbb{H}$ defined over a field of characteristic zero $k$, passing from $\mathbb{H}(k)$ to $\mathbb{H}(k)^+$ commutes with restriction of scalars (see [Mar91, Section I.1.7]), so that $(R_pG)^+ = G_p^+$.

**Remark 3.6.** For the statement at the end of the proposition, recall that we defined $\Gamma$ to be *strongly irreducible* if it has the property of almost dense projections. Since $G_J^+ \subseteq G_J$ has finite index (due to Borel-Tits, see Theorem 2.7.(i)), we see that the first statement of the above Proposition 3.4 shows that any irreducible group is strongly irreducible. Conversely, since for all $p$ (a finite prime or $\infty$), $G_p^+$ is $\mathbb{Q}_p$-Zariski-dense in $G_p$ (see Theorem 2.7.(i)), the projection of a strongly irreducible group to $G_p$ is $\mathbb{Q}_p$-Zariski-dense (and indiscrete), and therefore any strongly irreducible group is, as the name suggests, irreducible. Thus to prove Proposition 3.4, it remains to prove its first claim.

The utility of Proposition 3.4 should be clear: irreducibility is much easier to establish than strong irreducibility, since the former only requires $\mathbb{Q}_p$-Zariski-denseness of projections to $R_pG$, whereas the latter requires almost denseness (in the analytic topology!) of all subproduct projections. On the other hand, strong irreducibility is a much more powerful property to use in applications.

We start by proving Proposition 3.4 in the 'pure' case where all factors of $J$ are analytic over the same field. We will actually prove the following stronger statement that will be used in the general case.

**Lemma 3.7.** *Let $\Gamma \subseteq G$ be discrete and irreducible. Let $p$ be a finite prime or $\infty$ and let $H \subseteq G_p$ be closed, nondiscrete subgroup normalized by $\Gamma$. Then there is a subset $I_H \subseteq I_p$ such that $G_{I_H}^+$ is an open subgroup of $H$.*

**Remark 3.8.** In the pure case, the the above lemma indeed shows irreducibility implies strong irreducibility: Letting $\Gamma$ be discrete and irreducible, $J \subsetneq I$, and taking $H := \overline{\mathrm{pr}_J(\Gamma)}$, the above Lemma gives a subset $I_H \subseteq J$ such that $G_{I_H}^+ \subseteq H$ is open. In fact we must have $I_H = J$: Since $H/G_{I_H}^+$ is discrete, if there exists $j \in J \backslash I_H$, then $\mathrm{pr}_j(H)$ and hence $\mathrm{pr}_j(\Gamma)$ would be discrete, but this contradicts irreducibility of $\Gamma$. This shows $G_J^+ \subseteq \overline{\mathrm{pr}_J(\Gamma)}$ and hence the latter has finite index in $G_J$.

*Proof.* Since $H$ is a closed subgroup of the analytic group $G_p$, its Lie algebra $\mathfrak{h}$ is well-defined, and since $H$ is not discrete, $\mathfrak{h}$ is nontrivial. Since $\mathfrak{h}$ is $\mathrm{Ad}(\mathrm{pr}_{I_p}(\Gamma))$-invariant and $\mathrm{pr}_{I_p}(\Gamma)$ is $\mathbb{Q}_p$-Zariski-dense in $G_p$, it follows that $\mathfrak{h}$ is an ideal in the Lie algebra of $G_p$. Hence there is a nonempty subset $I_H \subseteq I_p$ such that $\mathfrak{h} = \oplus_{i \in I_H} \mathfrak{g}_i$. We will show that $H$ contains $G_{I_H}^+$.

To start, note that $H \cap G_{I_H}$ is open and closed in $G_{I_H}$. In the archimedean case, $H$ therefore contains the connected component of identity, which coincides with $G_{I_H}^+$. In the nonarchimedean case, it suffices to prove $H \cap G_i$ is noncompact for every $i \in I_H$, since then the theorem of Tits-Prasad (see Theorem 2.8) shows that $H \cap G_i$ contains $G_i^+$. To establish noncompactness, note that $H \cap G_i$ is normalized by $\mathrm{pr}_i(\Gamma)$, which is unbounded (since $\Gamma$ is irreducible, see Remark 3.3). But any bounded open subgroup has bounded normalizer, so $H \cap G_i$ must be unbounded as well. $\square$

We will complete the proof of Proposition 3.4 in the general, possibly mixed, case.

*Proof.* Write $J = \sqcup_{p \in P_J} J_p$ where $J_p$ consists of the $\mathbb{Q}_p$-analytic factors in $J$ and $P_J$ is a finite subset of primes and possibly $\infty$. Let $H$ denote the closure of the projection of $\Gamma$ to $G_J$. By passing to a finite index subgroup of $\Gamma$, we can assume that $H \subseteq G_J^+$ and we aim to show $H = G_J^+$. First, suppose that the connected component of identity $H^\circ$ of $H$ is nontrivial (so that necessarily $J_\infty \neq \varnothing$). We apply Lemma 3.7 to the subgroup $H^\circ$ and obtain that there is a subset $J_{H,\infty} \subseteq J_\infty$ such that $G^+_{J_{H,\infty}} \subseteq H^\circ$ is open. Therefore we can project to the factors given by $J \backslash J_{H,\infty}$ and assume $J_{H,\infty} = \varnothing$.

We will aim to show that after the above reduction, $P_J$ consists of finite primes and $H$ contains $G^+_{J_p}$ for every $p \in P_J$. Since $H$ is locally compact and totally disconnected, it contains a compact open subgroup $K$. Because $K$ is compact and totally disconnected, its image in $G_{J_\infty}$ is finite, so that by possibly shrinking $K$, we can assume $K$ projects trivially to the archimedean factors $G_{J_\infty}$ and therefore $H$ has discrete projection. However, this is impossible because the projection of $\Gamma \subseteq H$ to $G_{J_\infty}$ is indiscrete, so we must have $J_\infty = \varnothing$.

It remains to show that $H$ contains $G^+_{J_p}$ for every $p \in P_J$. Let $K_p$ be the projection of $K$ to $G_{J_p}$ for $p \in P_J$. By possibly shrinking $K$ further, we can assume that the map $\mu_p : x \mapsto x^p$ is a contraction on $K_p$ for all $p \in P_J$. Then we claim that $K = \prod_{p \in P_J} K_p$. Indeed, let $k = (k_p)_p \in K \subseteq \prod_p K_p$. Fix $p$ and let $d$ be the product of the remaining primes. Since $\mu_d$ is an isometry on $K_p$ but a contraction on $K_\ell$ for all $\ell \neq p$, there is some sequence $n_m \to \infty$ such that $k_p^{d^{n_m}} = \mu_d^{n_m}(k_p) \to k_p$ and hence $k^{d^{n_m}} \to (k_p, e, \ldots, e) \in K_p \times \prod_{\ell \neq p} K_\ell$. Since $K$ is closed, we see that $K_p \subseteq K$ for all $p \in P_J$, as desired.

Further, we claim that $K_p$ is nondiscrete for all $p \in P_J$. Indeed, if the image of $K$ in $G_{J_p}$ were discrete, then so would the image of $H$, but we know this is impossible because $\Gamma$ has nondiscrete image. Since $K_p$ is nondiscrete and $K = \prod_p K_p$ is contained in $H$, it follows that for every $p$, the intersection $H \cap G_{J_p}$ is nondiscrete. We apply Lemma 3.7 to the subgroup $H \cap G_{J_p}$ and obtain that for every $p \in P_J$, there is a subset $J_{H,p} \subseteq J_p$ such that $G^+_{J_{H,p}} \subseteq H$ is open.

Set $J_H := \sqcup_{p \in P_J} J_{H,p}$. To complete the proof, we will argue by contradiction that $J_H = J$: We have shown above that $G^+_{J_H}$ is an open subgroup of $H$, so the image of $H$ in the complementary factors $G_{J \backslash J_H}$ is discrete. This contradicts irreducibility of $\Gamma$. □

## 4. Irreducible groups are lattices

4.1. **Irreducible groups in products of semisimple Lie groups.** We can now give the first application of the Greenberg-Shalom hypothesis.

Aside from its intrinsic interest, this result will provide an important connection to other applications as well.

**Proposition 4.1.** *Assume the Greenberg-Shalom Hypothesis 1.1. Let $G$ be as in Standing assumptions 2.1 with at least two factors, and assume at least one factor is nonarchimedean. Then any discrete and irreducible subgroup $\Gamma \subseteq G$ is an irreducible lattice.*

*Proof.* If $G$ has an archimedean factor, let $G_\infty$ denote the product of all archimedean factors and $G_{\mathrm{na}}$ the product of all nonarchimedean factors. If there is no archimedean factor, let $G_\infty$ be one of the nonarchimedean factors and $G_{\mathrm{na}}$ the product of the remaining factors.

Let $K_{\mathrm{na}} \subseteq G_{\mathrm{na}}^+$ be a maximal compact open subgroup. Let $\Gamma_{\mathrm{na}} \subseteq \Gamma$ denote the subgroup that maps into $K_{\mathrm{na}}$ under projection to $G_{\mathrm{na}}$. Then $\mathrm{pr}_\infty(\Gamma_{\mathrm{na}}) \subseteq G_\infty$ is irreducible, discrete and has commensurator containing the dense subgroup $\mathrm{pr}_\infty(\Gamma) \subseteq G_\infty$. In particular $\mathrm{pr}_\infty(\Gamma_{\mathrm{na}})$ is Zariski-dense in $G_\infty$ (see Lemma 2.9), and hence by the Greenberg-Shalom hypothesis, is a lattice. It then follows that $\Gamma \subseteq G$ is a lattice by an argument similar to the proof of a lemma of Venkataramana's first stated as [LZ01, Lemma 2.3] and generalized in [FMvL22, Lemma 4.1]. The latter applies to subgroups $\Gamma$ of $S$-arithmetic lattices containing a $T$-arithmetic lattice for $T \subseteq S$. Here we need a slightly different version where $\Gamma$ is not given as a subgroup of a lattice, so we include the relevant part of the argument for completeness:

By passing to a finite index subgroup, we can assume without loss of generality that $\Gamma \subseteq G^+$. Recall that $K_{\mathrm{na}} \subseteq G_{\mathrm{na}}^+$ is a maximal compact open subgroup, and that $\mathrm{pr}_{\mathrm{na}}(\Gamma_{\mathrm{na}}) \subseteq K_{\mathrm{na}}$ is dense. Let $F \subseteq G_\infty^+$ be a fundamental domain for the lattice $\mathrm{pr}_\infty(\Gamma_{\mathrm{na}}) \subseteq G_\infty^+$. As in Venkataramana's lemma and [FMvL22, Lemma 4.1], to show $\Gamma \subseteq G$ is a lattice, we will show that $F \times K_{\mathrm{na}}$ is a fundamental domain for $\Gamma \subseteq G^+$.

First we show the $\Gamma$-translates of $F \times K_{\mathrm{na}}$ are disjoint: If $\gamma(F \times K_{\mathrm{na}}) \cap (F \times K_{\mathrm{na}}) \neq \varnothing$, then by projecting to the second factor and using that $K_{\mathrm{na}}$ is a group, we see that $\gamma \in K_{\mathrm{na}}$, and hence $\gamma \in \Gamma_{\mathrm{na}}$. But then by projecting to the first factor, we see that $\gamma F \cap F \neq \varnothing$, and since $F$ is a fundamental domain for $\Gamma_{\mathrm{na}}$, we conclude that $\gamma = e$.

Next, we show that $\Gamma(F \times K_{\mathrm{na}}) = G$. Since $K_{\mathrm{na}}$ is $\mathrm{pr}_{\mathrm{na}}(\Gamma_{\mathrm{na}})$-invariant and $F \subseteq G_\infty$ is a fundamental domain for $\mathrm{pr}_\infty(\Gamma_{\mathrm{na}})$, it suffices to show that $\mathrm{pr}_{\mathrm{na}}(\Gamma)K_{\mathrm{na}} = G$. But this is immediate because $\mathrm{pr}_{\mathrm{na}}(\Gamma) \subseteq G_{\mathrm{na}}^+$ is dense and $K_{\mathrm{na}}$ is open. □

The existence of irreducible surface subgroups in products of $p$-adic groups is a folklore problem that was widely discussed at MSRI in 2015. A variant for actions on products of trees was asked explicitly by Fisher-Larsen-Spatzier-Stover [FLSS18] and will be discussed in detail later (see Section 8). In a conversation in January 2019, Yves Benoist pointed out to the second author that it was also unknown if there are irreducible free groups in products of real and $p$-adic Lie groups or irreducible surface groups in products of real Lie groups. Much earlier, Long and Reid, using some ideas of Magnus

[Mag73], constructed an explicit surface group in $\mathrm{PGL}(2,\mathbb{Q}_2)\times\mathrm{PGL}(2,\mathbb{Q}_3)$ generated by the two matrices:

$$a=\begin{pmatrix}3 & 0\\ 0 & \frac{1}{3}\end{pmatrix},\qquad b=\begin{pmatrix}\frac{1}{8} & 9\\ \frac{1}{32} & \frac{41}{4}\end{pmatrix}$$

This is an orbifold group with an index 4 subgroup that corresponds to a cover by a surface of genus two. Extensive computations by Long-Reid, Agol and Brody suggest that this example is a discrete, irreducible subgroup of the product but according to Proposition 4.1, this would give a negative answer to Greenberg-Shalom's Question 1.1. It is worth comparing with the computer-assisted results of Kim-Koberda [KK22] that are relevant to another application below (see Section 4.2), where groups were shown to be non-free by finding extremely long relators. If Hypothesis 1.1 is correct, then the computational evidence above about the Long-Reid group only indicates that one needs extremely long words in $a$ and $b$ to find elements of integral trace. We summarize the implications of the Greenberg-Shalom hypothesis for these types of questions in the next two results.

**Corollary 4.2.** *Assume the Greenberg-Shalom hypothesis and suppose $G$ has no archimedean factors and $|I|\geq 2$. Then there is no irreducible discrete surface group or irreducible discrete finitely generated free group in $G$. If in addition all simple factors of $G$ have rank one, then there is no discrete surface group in $G$.*

*Proof.* The first statement follows e.g. from the fact that a surface group $\Lambda$ can only be a lattice in a semisimple algebraic group if the group is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ and that a finitely generated free group $F_k$ is only a lattice in a simple algebraic group of rank one over a non-archimedean field or a group locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

For the second point, we have a surface group $\Lambda<G$ where $G=\prod_{i\in I}G_i$ where each $G_i$ is a rank one $p_i$-adic group whose Bruhat-Tits building is a tree. After some initial reductions, we will argue that $\Lambda$ is irreducible. First, by [FLSS18, Theorem 15], there is a subset $J\subset I$ such that the projection of $\Lambda$ to each $G_j$ is faithful and indiscrete where $\Lambda<G_J=\prod_{j\in J}G_j$ is discrete and $|J|\geq 2$. We fix a subset $J$ that is minimal with respect to these properties.

For $j\in J$, the $\mathbb{Q}_{p_j}$-Zariski closure of the projection of $\Lambda$ to $G_j$ is simple: Otherwise it would, up to finite index, fix a point at infinity, but in Section 9, we prove a slight strengthening of [FLSS18, Theorem 15] which shows that the projection of finite index subgroups of $\Lambda$ to each $G_j$ do not fix a vertex at infinity.

We now replace every $G_j$ by the $\mathbb{Q}_{p_j}$-Zariski closure of the projection of $\Lambda$, and argue that $\Lambda\subseteq G_J$ is irreducible. Henceforth we will regard $G_j$ as a $\mathbb{Q}_{p_j}$-algebraic group, and omit the field when referring to its Zariski-topology.

Recall that for each prime $p$, the collection of $p$-adic factors of $G_J$ is indexed by $J_p$, and we write $G_p := \prod_{j \in J_p} G_j$ for the product of the $p$-adic factors of $G_J$. By minimality of $J$, for every $J' \subsetneq J$, the projection of $\Lambda$ to $G_{J'}$ is indiscrete.

Therefore it remains to show that for every prime $p$, the projection of $\Lambda$ to $G_p$ is Zariski-dense. Let $H_p$ be the Zariski-closure of the projection of $\Lambda$, and let $J'_p \subseteq J_p$ be those values of $j$ such that $G_j \subseteq H_p$. We will argue by contradiction that $J'_p = J_p$, and this will complete the proof.

Suppose therefore that $J'_p$ is a proper subset of $J_p$. Consider now the image of $H_p$ in the product of the remaining factors $G_{J_p}/G_{J'_p}$. Since for every $j \in J_p$, the projection of $\Lambda$ to $G_j$ is Zariski-dense, the image of $H_p$ surjects onto (but does not contain) $G_j$ for $j \in J_p \backslash J'_p$. In particular, $G_{J_p}/G_{J'_p}$ consists of at least two factors.

Fix one such factor $G_{j_0}$. Since the kernel of the projection of $H_p$ to $G_{j_0}$ would be normal in the remaining factors, it is a product of some subset of them, but since $H_p$ does not contain any factors of $G_{J_p}/G_{J'_p}$, we conclude $H_p$ projects isomorphically onto $G_{j_0}$.

Therefore the projection

$$H_p \times G_{J'_p} \times G_J/G_{J_p} \to G_{j_0} \times G_{J'_p} \times G_J/G_{J_p}$$

that replaces $H_p$ by $G_{j_0}$ is a topological isomorphism. Since (the projection of) $\Lambda$ is discrete in the former, it is also discrete in the latter. However, since $J_p \backslash J'_p$ consists of at least two factors, this contradicts minimality of $J$. □

We call a surface group $\Lambda$ in $\mathrm{PSL}(2, \mathbb{R})$ *algebraic* if $\Lambda < \mathrm{PSL}(2, \bar{\mathbb{Q}})$. By finite generation of surface groups it follows that $\Lambda < \mathrm{PSL}(2, k)$ for some number field $k$. Let $\mathcal{O}_k$ be the ring of integers of $k$. Then we have:

**Corollary 4.3.** *Let $\Lambda$ be an algebraic surface group and assume the Greenberg-Shalom hypothesis. Then $\Gamma := \Lambda \cap \mathrm{PSL}(2, \mathcal{O}_k)$ is infinite, commensurated by $\Lambda$ and Zariski-dense in $\mathrm{PSL}(2, k)$.*

*Proof.* As $\Lambda$ is finitely generated, there are at most finitely many finite places $S$ of $k$ such that there is an element of $\Lambda$ of norm greater than one in the valuation. Viewing $\Lambda$ as a subgroup of $G := \prod_{s \in S} \mathrm{PSL}(2, k_s)$, we see $\Gamma$ is the intersection of $\Lambda$ with a compact open subgroup in $G$. It follows immediately that $\Lambda$ commensurates $\Gamma$ and that if $\Gamma$ is infinite, it is Zariski dense (see Lemma 2.9), so it remains to show $\Gamma$ is infinite. Corollary 4.2 implies that $\Lambda$ is not discrete in $G$ and so $\Gamma$ is infinite. □

In the above results, we require at least one factor to be totally disconnected (i.e. a $p$-adic Lie group). As mentioned above, the second author learned of variants of this question where both factors are real Lie groups from Yves Benoist in January 2019. The simplest and perhaps most intriguing case is:

**Question 4.4** (Benoist)**.** Is there a free group that acts properly discontinuously and irreducibly on $\mathbb{H}^2 \times \mathbb{H}^2$?

We believe the requirement of a totally disconnected factor in Proposition 4.1 is an artefact of the method rather than a genuine difference. By analogy with Proposition 4.1 we propose the following strengthening of Benoist's question:

**Problem 4.5.** Let $G_1, G_2$ be real semisimple Lie groups with finite center and no compact factors, and suppose $\Delta \subseteq G_1 \times G_2$ is a discrete and irreducible subgroup. Is $\Delta$ an arithmetic lattice?

Just as in Problem 4.1, irreducible groups $\Delta$ as in the above problem give rise to objects in one factor with large commensurator. But since $G_2$ is no longer totally disconnected, they are no longer groups: More precisely, consider $\Gamma := \Delta \cap (G_1 \times U)$ where $U \subseteq G_2$ is an open neighborhood of identity with compact closure. Note that $\Gamma$ is no longer a subgroup, but only an *approximate subgroup*. This approximate subgroup is commensurated by $\Delta$ in the sense of [Mac23b, Definition 2.1.5]. By analogy with Proposition 4.1 we suggest the following variation of Greenberg-Shalom's Question 1.1 as a way of approaching Problem 4.5:

**Question 4.6.** Let $G$ be as in the Standing Assumptions 2.1 and let $\Gamma$ be a discrete, Zariski-dense, approximate subgroup with almost dense commensurator. Is $\Gamma$ an approximate lattice?

We remark here that for approximate subgroups of $G$, there is a well-defined notion of being an approximate lattice and of arithmeticity, and Machado (for real higher rank Lie groups) [Mac23a] and Hrushovski (in general) [Hru20] have simultaneously classified approximate lattices in products of simple algebraic groups defined over local fields: Namely, for any approximate lattice $\Lambda \subseteq G$, we can decompose $G = G_1 \times G_2$ such that $\Lambda$ is commensurable to a product of a lattice in $G_1$ and an arithmetic approximate lattice in $G_2$. Here an arithmetic approximate lattice in $G$ is either an arithmetic lattice or obtained as the intersection of an irreducible lattice in $G \times H$ with $G \times U$, where $U$ is an open neighborhood of identity in $H$. For a thorough account of the state of the art on approximate lattices, see Machado's recent preprint [Mac23b].

4.2. **Groups generated by parabolic elements.** For the remainder of this section, we discuss applications of rigidity of irreducible groups in semisimple Lie groups to problems that do not seemingly involve irreducible groups or commensurators. To state this problem, we let $q \in \mathbb{C}$ be a parameter, and we let $\Delta_q$ be the group generated by

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad b_q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}.$$

The algebraic structure of $\Delta_q$, especially whether it is free or not, has been the subject of much work, starting with Sanov's 1947 theorem that $\Delta_4$ is free [San47], and Brenner's theorem that $\Delta_q$ is free and discrete if $|q| > 4$ [Bre55], which implies $\Delta_q$ is free for transcendental values of $q$. Freeness of $\Delta_q$ for nonzero rational values of $q$ in $(-4, 4)$ is a long-standing open problem first mentioned by Brenner and Hirsch. The question was really first studied in the work of Lyndon-Ullman, and later a negative answer has been conjectured by Kim-Koberda:

**Conjecture 4.7** (Lyndon-Ullman, Kim-Koberda [LU69, KK22])**.** For nonzero rational values of $q$ in $(-4, 4)$, the group $\Delta_q$ is not free.

Kim-Koberda have proved this for denominators up to 24. For $|q| < 4$ (not necessarily rational), Knapp has proved that $\Delta_q$ is discrete if and only if $1 - q/2 = \cos((n-2)\pi/n)$ for some integer $n \geq 3$ [Kna69]. Indeed, note that the element $ab_q^{-1}$ has trace $|2 - q| < 2$, hence it is elliptic. When $q$ is a rational non-integer, this must be an elliptic of infinite order. The case that $\Delta_q$ is discrete has been further studied by Agol, showing in particular that $\Delta_2$ and $\Delta_3$ are not free, see [AOP+20]. Knapp's theorem immediately implies $\Delta_q$ is indiscrete for nonintegral rational values with $|q| < 4$. A complete description of $\Delta_q$ up to finite index, and in particular a positive answer to the above conjecture, follows from Greenberg-Shalom's problem:

**Theorem 4.8.** *Assume the Greenberg-Shalom Hypothesis 1.1. Then for every non-integral rational $q = r/s \in (-4, 4)$, the group $\Delta_q$ has finite index in $\mathrm{SL}(2, \mathbb{Z}[1/s])$. In particular, Conjecture 4.7 is true.*

*Proof.* If $q = r/s$ is written in lowest terms and is not integral, then $\Delta_q$ is a discrete subgroup of $G := \mathrm{SL}(2, \mathbb{R}) \times \prod_{p|s} \mathrm{SL}(2, \mathbb{Q}_p)$. Let $T$ be a minimal subset of factors such that $\Delta_q$ has discrete projection to $G_T$. Note that $T$ consists of at least two factors: The projection of $\Delta_q$ to $\mathrm{SL}(2, \mathbb{R})$ is not discrete by the above-mentioned work of Knapp [Kna69], and its projection to $\mathrm{SL}(2, \mathbb{Q}_p)$ is not discrete because $a \in \mathrm{SL}(2, \mathbb{Z}_p)$ has infinite order. Further the projections are Zariski-dense since their Zariski-closures cannot be solvable, and $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{Q}_p)$ do not contain proper Zariski-closed non-solvable subgroups. Therefore $\mathrm{pr}_T(\Delta_q) \subseteq G_T$ is irreducible, and by Proposition 4.1, $\mathrm{pr}_T(\Delta_q)$ is a lattice in $G_T$. Note that $T$ must contain $\infty$, because $a$ generates an indiscrete subgroup of $\prod_{p|s} \mathrm{SL}(2, \mathbb{Q}_p)$. So $\Delta_q \subseteq \mathrm{SL}(2, \mathbb{Z}[1/s])$ is $T$-arithmetic, and is therefore commensurable with $\mathrm{SL}(2, \mathbb{Z}[1/t])$, where $t$ denotes the product of the finite primes in $T$.

To see that $t = s$, it suffices to show that $\Delta_q$ has unbounded projection to $\mathrm{SL}(2, \mathbb{Q}_p)$ for all $p \mid s$. Indeed, $ab$ has trace $2 - q = 2 - r/s$, and is therefore not elliptic in $\mathrm{SL}(2, \mathbb{Q}_p)$. In particular, $ab$ generates an unbounded subgroup. □

In fact, this strategy can handle not just rational, but all algebraic values of $q$:

**Theorem 4.9.** *Assume the Greenberg-Shalom Hypothesis 1.1. Let $q \in \overline{\mathbb{Q}}$ be an algebraic number that is not an algebraic integer, and let $\Delta_q$ denote the corresponding Lyndon-Ullman group.*

*Then* $\Delta_q$ *is free if and only if there is a Galois automorphism* $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *such that* $\Delta_{\sigma(q)}$ *is free and discrete.*

*Proof of Theorem 4.9.* The proof is similar to the previous one. Let $k = \mathbb{Q}(q)$. Since $q \notin \mathcal{O}_k$, there exists a valuation $\nu$ of $k$ such that $|q|_\nu > 1$. Let $p$ be the prime such that $k_\nu$ is a finite extension of $\mathbb{Q}_p$ and consider $\Delta_q \subseteq \mathrm{PSL}(2, k_\nu)$. Note that $\Delta_q$ is not discrete since $a \in \Delta_q \cap \mathrm{PSL}(2, \mathcal{O}_\nu)$ has infinite order. Therefore to show $\Delta_q$ is almost-dense in a semisimple $p$-adic Lie group, it suffices to show its $\mathbb{Q}_p$-Zariski-closure is semisimple. To see this, it suffices to show the Lie algebra $\mathfrak{g}_q$ of the $\mathbb{Q}_p$-Zariski-closure of $\Delta_q$ is semisimple.

Note that $\Delta_q$ is $k_\nu$-Zariski-dense in $\mathrm{PSL}(2, k_\nu)$ because it is not virtually solvable. This implies the $k_\nu$-span of $\mathfrak{g}_q$ is all of $\mathfrak{sl}(2, k_\nu)$ (because it is a $\Delta_q$-invariant Lie subalgebra), and therefore $\mathfrak{g}_q$ must itself be semisimple: The $k_\nu$-span of its solvable radical would be an ideal in the $k_\nu$-span of $\mathfrak{g}_q$.

By the Greenberg-Shalom hypothesis, $\Delta_q$ is an arithmetic lattice in a semisimple $p$-adic Lie group $H$. Note that $H$ has at least two factors because $\Delta_q$ is not discrete in its $\mathbb{Q}_p$-Zariski-closure inside $\mathrm{PSL}(2, k_\nu)$. In particular, $\Delta_q$ is a higher rank arithmetic lattice and hence not free. □

4.3. **Arithmetic of hyperbolic 3-manifolds.** In this section we let $G = \mathrm{PO}(3,1) = \mathrm{Isom}(\mathbb{H}^3)$ and discuss an application to hyperbolic 3-manifolds, i.e. to manifolds of the form $K\backslash G/\Lambda$ where $\Lambda$ is a lattice. It follows from work of Selberg, Calabi, Raghunathan, and Garland [Sel60, Cal61, Rag67, Gar66] that there is a number field $k$ such that $\Lambda < G(k)$. One can take $k$ to be the trace field of $\Lambda$ but this is not important for this application. It is a natural and reasonably well-known question to ask whether $\Lambda$ necessarily contains any integral matrices. We let $\mathcal{O}_k$ denote the ring of $k$-integers. As an application of our methods, we prove:

**Theorem 4.10.** *Assume the Greenberg-Shalom Hypothesis 1.1. For any finite volume hyperbolic 3-manifold* $M = \mathbb{H}^3/\Lambda$*, the subgroup* $\Gamma := \Lambda \cap G(\mathcal{O}_k)$ *is infinite, commensurated by* $\Lambda$*, and Zariski-dense in* $G(k)$*.*

We remark here that Theorem 4.10 is known unconditionally and more generally for non-uniform lattices by exploiting simple properties of unipotent elements. We formulate and prove this below for completeness. One can take this as evidence that it is reasonable to expect Theorem 4.10 to be true.

*Proof.* As discussed above, there exists a number field $k$ and an algebraic group $\mathbb{G}$ defined over $k$ such that $\Lambda \subseteq \mathbb{G}(k)$. Here $\Lambda$ is a lattice in $\mathbb{G}(\mathbb{R}) = \mathrm{PO}(3,1)$, so $\mathbb{G}$ is a form of $\mathrm{PO}(4)$. Since $\Gamma$ is finitely generated, the set of places $S$ of $k$ such that there exists a matrix entry of a generator of $\Lambda$ with valuation $> 1$, is finite. It follows immediately that $\Lambda$ commensurates $\Gamma$. Since the commensurator of $\Gamma$ is Zariski-dense, as soon as $\Gamma$ is infinite, $\Gamma$ itself is Zariski-dense (see Lemma 2.9 and Remark 2.11). It remains to prove that $\Gamma$ is infinite.

At any place $s \notin S$, we have $\Lambda \subseteq \mathbb{G}(\mathcal{O}_s)$. In particular, if $S = \varnothing$, then $\Gamma = \Lambda$ and we are done. Now assume $S \neq \varnothing$. If $\Lambda \subseteq G_S$ is indiscrete, then $\Gamma = \Lambda \cap \mathbb{G}(\mathcal{O}_S)$ is infinite and we are done.

Henceforth assume that $\Lambda \subseteq G_S$ is discrete. Since PO(4) is of type $D_2$ and any split group of type $D_2$ is a product of groups of type $A_1$, the group $G_S := \prod_{s\in S} \mathbb{G}(k_s)$ splits as a product of rank one factors, say $G_S = \prod_{i\in I} G_i$. Choose a maximal subset $J \subseteq I$ (possibly $J = I$) such that $\mathrm{pr}_J(\Lambda) \subseteq G_J$ is discrete. We note that $J$ consists of at least two factors: Since $G_i$ has rank 1, its Bruhat-Tits building is a tree, and since $\Lambda$ is not virtually free, it cannot act properly discontinuously on a bounded degree tree.

We claim that $\Lambda$ is irreducible in $G_J$: By maximality of $J$, the projection of $\Lambda$ to any proper subset of factors is indiscrete, so it remains to show that $\Lambda$ is $\mathbb{Q}_p$-Zariski-dense in $G_{J_p}$ for every prime $p$. Here $G_{J_p}$ denotes the product of the $p$-adic factors of $G_J$, and $G_{J_p}$ is a $\mathbb{Q}_p$-algebraic group by restriction of scalars as in Section 3.

Since $\Lambda \subseteq \mathbb{G}(k)$, its $\mathbb{R}$-Zariski-closure in $\mathbb{G}(\mathbb{R})$ is defined over $k$ (see e.g. [Zim84, Proposition 3.1.8]). Since $\Lambda \subseteq \mathbb{G}(\mathbb{R})$ is a lattice, it is $\mathbb{R}$-Zariski-dense. Combining these two observations, we see that $\Lambda$ is $k$-Zariski-dense in $\mathbb{G}(k)$. Restricting scalars from $k$ to $\mathbb{Q}$, we conclude that $\Lambda$ is $\mathbb{Q}$-Zariski-dense in $(R_{k/\mathbb{Q}}\mathbb{G})(\mathbb{Q})$. Now write $S_p$ for the set of $p$-adic places of $S$. We have

$$G_{S_p} = \prod_{s\in S_p} \mathbb{G}(k_s) = \prod_{s\in S_p} (R_{k/\mathbb{Q}}\mathbb{G})(\mathbb{Q}_p).$$

We regard $G_{S_p}$ as a $\mathbb{Q}_p$-algebraic group using the product structure given by the right-hand side. Since the diagonal embedding

$$R_{k/\mathbb{Q}}\mathbb{G} \hookrightarrow \prod_{s\in S_p} R_{k/\mathbb{Q}}\mathbb{G}$$

is defined over $\mathbb{Q}$ and $\Lambda$ is $\mathbb{Q}$-Zariski-dense in $(R_{k/\mathbb{Q}}\mathbb{G})(\mathbb{Q})$, the $\mathbb{Q}$-Zariski-closure of $\Lambda$ in $\prod_{s\in S_p}(R_{k/\mathbb{Q}}\mathbb{G})(\mathbb{Q})$ is exactly the diagonal. So the $\mathbb{Q}_p$-Zariski-closure of $\Lambda$ in $G_{S_p}$ contains the diagonally embedded copy $\mathbb{G}(k)$. However, since $\mathbb{G}$ is adjoint, it satisfies weak approximation, so $\mathbb{G}(k)$ is (analytically) dense in $G_{S_p}$. So the $\mathbb{Q}_p$-Zariski-closure of $\Lambda$ in $G_{S_p}$ (which is of course analytically closed) contains a dense set and therefore must be all of $G_{S_p}$. This shows that $\Lambda$ is $\mathbb{Q}_p$-Zariski-dense in $G_{S_p}$.

Since every factor of $G_{S_p}$ is defined over $\mathbb{Q}_p$, the projection $G_{S_p} \to G_{J_p}$ is also defined over $\mathbb{Q}_p$. It follows that the image of $\Lambda$ in $G_{J_p}$ is also $\mathbb{Q}_p$-Zariski-dense. □

We now formulate and prove the known version of Theorem 4.10 for non-uniform lattices. We thank Alan Reid for drawing this fact and its proof to our attention. We give a statement for non-uniform lattices in all rank one Lie groups, even though except in the case of $\mathrm{SO}(n,1)$, stronger results are known. If $G$ is a rank one Lie group not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ then it follows as above that any lattice $\mathbb{G} < G$ is conjugate into $G(k)$ for some number field $k$.

**Theorem 4.11.** *Let $G$ be a rank one Lie group not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ and $\Gamma < G$ a non-uniform lattice, then subgroup $\Gamma := \Lambda \cap G(\mathcal{O}_k)$ is infinite, commensurated by $\Lambda$, and Zariski-dense in $G(k)$.*

*Proof.* By results of Garland-Ragunathan [GR70], we know that for some choice of parabolic $P$, $\Gamma$ intersects the unipotent radical $U$ of $P$ in a lattice. Furthermore, we know $\Gamma \cap Z(U)$ is also non-trivial where $Z(U)$ is the center of $U$. If $\gamma \in \mathbb{Z}(U)(k)$ then it is elementary that for some $n$, we have $\gamma^n \in Z(U)(\mathcal{O}_k)$ since on the center of a unipotent group, multiplication is just addition of matrix coefficients. This suffices to prove the theorem by the discussion the proof of the last theorem. □

## 5. Rank gradient

For a finitely generated group $\Gamma$, the *rank* $\mathrm{rk}(\Gamma)$ is the minimal number of generators in a generating set for $\Gamma$. It is interesting to consider ranks of finite index subgroups of $\Gamma$. Writing $\Gamma$ as a quotient of a free group $F_r$ of rank $r := \mathrm{rk}(\Gamma)$, the pre-image of a finite index subgroup $\Theta \subseteq \Gamma$ in $F_r$ is a free subgroup of index $[\Gamma : \Theta]$, and therefore has rank $[\Gamma : \Theta](\mathrm{rk}(\Gamma) - 1) + 1$. In particular, $\mathrm{rk}(\Theta)$ grows at most linearly in $[\Gamma : \Theta]$. Given a chain of finite index subgroups

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots,$$

the *rank gradient* of $\Gamma$ with respect to the chain $(\Gamma_n)_n$ is

$$\mathrm{rg}(\Gamma, (\Gamma_n)_n) := \lim_{n\to\infty} \frac{\mathrm{rk}(\Gamma_n) - 1}{[\Gamma : \Gamma_n]}.$$

Lackenby introduced rank gradient as a generalization of Heegaard gradient, which is an invariant for 3-manifolds [Lac05]. Abert-Nikolov proved that if the chain $(\Gamma_n)_n$ is *Farber* (e.g. if $\Gamma_n$ are contained in a chain of normal subgroups of $\Gamma$ that has trivial intersection), then the rank gradient computes the *cost* of the action of $\Gamma$ on the boundary of the coset tree associated to the chain, i.e. the profinite space $\varprojlim \Gamma/\Gamma_n$ [AN12]. In particular, the rank gradient is stable under refining the chain $(\Gamma_n)_n$. Cost is more generally defined for probability measure-preserving Borel actions of $\Gamma$, and the *fixed price problem* asks whether the cost of an action only depends on $\Gamma$ and not on the action itself. A positive answer for many groups, including higher rank nonuniform irreducible lattices in real Lie groups has been given by Gaboriau [Gab00]. Abert-Gelander-Nikolov have given a positive answer for right-angled lattices in such Lie groups, which includes the first uniform examples [AGN17]. Recently, Fraczyk-Mellick-Wilkens have positively answered the question for all lattices in higher rank real Lie groups [FMW23]. In all of these results, the cost is 1, which implies vanishing of the rank gradient for any Farber sequence.

In the context of lattices in semisimple Lie groups, oftentimes one can find many finite index subgroups with a uniformly bounded number of generators. This is not the case for lattices in $\mathrm{SL}(2,\mathbb{R})$, since those surject onto free groups, and so do some lattices in other rank 1 groups. However,by combining work of Raghunathan, Tits, and Venkataramana, one can prove that for a nonuniform higher rank arithmetic lattice $\Gamma$ in a $\mathbb{Q}$-simple real Lie group $G$, there exists $k \geq 2$ such that $\Gamma$ has arbitrarily small $k$-generated finite index subgroups in the sense that any finite index subgroup of $\Gamma$ contains a further finite index $k$-generated subgroup (see [SV05, Remark 3]). In particular, the cost of Farber chains is always 1. Sarma-Venkataramana have shown that one can take $k = 3$ [SV05]. Sarma has studied the same problem for $S$-arithmetic lattices, and in particular has proven that for an $S$-arithmetic lattice in $\mathrm{SL}(2, K)$, where $K$ is a number field and $|S| \geq 2$, one can take $k = 3$ [Sar06]. For $\mathrm{SL}(n,\mathbb{Z}), n \geq 3$, Lubotzky has asked whether one can take $k = 2$, i.e. whether $\mathrm{SL}(n,\mathbb{Z}), n \geq 3$, has "arbitrarily small 2-generator finite index subgroups" [Lub86], and this was proven by Meiri [Mei17]. The analogue of Lubotzky's question is open for all other commensurability classes of higher rank lattices, including $S$-arithmetic ones.

Heuristically, under the Greenberg-Shalom hypothesis, for irreducible lattices $\Gamma \subseteq G$ (where $G$ consists of at least 2 factors) one should indeed have $k = 2$, and in particular, the cost of Farber chains is always 1: Namely, it is plausible that that a generic pair of elements generates an irreducible group, which would have finite index by Proposition 4.1. The following result shows that one can indeed make such an argument (with $k$ given by the number of factors):

**Theorem 5.1.** *Assume the Greenberg-Shalom hypothesis. Let $G = \prod_i G_i$ be a product of $r \geq 2$ factors as in Standing assumptions 2.1, and assume $G_i$ correspond to different places $p_i$ of $\mathbb{Q}$. Here we allow $p = \infty$ as a place. Then any irreducible lattice $\Gamma \subseteq G$ contains arbitrarily small $r$-generated finite index subgroups.*

*In particular, $\Gamma = \mathrm{SL}(n, \mathbb{Z}[1/p])$ contains arbitrarily small 2-generated finite index subgroups.*

**Question 5.2.** Let $G$ and $\Gamma$ be as in the above theorem, and assume $r > 2$. Does $\Gamma$ even contain a single $k$-generated finite index subgroup for some $k < r$? Does $\Gamma$ admit arbitrarily small $k$-generated subgroups for some $k < r$? E.g. for $k = 3$ as in the results of Sarma-Venkataramana and Sarma mentioned above? Or even $k = 2$ as in Lubotzky's question and Meiri's theorem?

**Remark 5.3.** For $N \geq 2$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z}[1/N])$, the answer to the above question is positive under the assumption of the Greenberg-Shalom Hypothesis 1.1: Namely, then the Lyndon-Ullman group $\Delta_{1/N}$ is a lattice in $\mathrm{SL}(2,\mathbb{R}) \times \prod_{p|N} \mathrm{SL}(2, \mathbb{Q}_p)$ (see Theorem 4.8), and therefore is a finite index 2-generated subgroup of $\mathrm{SL}(2, \mathbb{Z}[1/N])$. This shows that Greenberg-Shalom in at least some cases implies $k = 2$ even when $r > 2$.

Since the rank gradient is stable under refining the chain of subgroups, we have the following application of Theorem 5.1:

**Corollary 5.4.** *Assume the Greenberg-Shalom hypothesis, and let $G$ be as in Standing Assumptions 2.1 with simple factors corresponding to different places of $\mathbb{Q}$, and $\Gamma \subseteq G$ an irreducible lattice. Then the rank gradient of $\Gamma$ with respect to any Farber chain of finite index subgroups vanishes.*

As mentioned above, Fraczyk-Mellick-Wilkens have proven this result unconditionally for higher rank lattices in real Lie groups. It seems plausible (but we do not know) that their methods extend to the setting of lattices as in the corollary as well.

To prove Theorem 5.1, we start with some preliminary arguments. First, we recall some facts about Jordan projections and loxodromic elements (see e.g. [BQ16, Chapter 6]). For a connected simple algebraic group $H$ over a local field of characteristic 0, every nontrivial element admits a Jordan decomposition $g = g_e g_h g_u$ as a product of a commuting triple consisting of an elliptic element $g_e$, a hyperbolic element $g_h$, and a unipotent element $g_u$. An element $g \in H$ is called *loxodromic* if $g_h$ is regular, i.e. conjugate into $\exp \mathfrak{a}^+$, where $\mathfrak{a}^+$ is the (open) positive Weyl chamber. Any loxodromic element $g$ is semisimple, i.e. $g_u = e$, and is therefore conjugate to an element of the form $m \exp(X)$ where $X \in \mathfrak{a}^+$ and $m \in Z_K(A)$ centralizes the maximal torus $A = \exp \mathfrak{a}$. If $H$ is real, then for any Zariski-dense subgroup $\Theta \subseteq H$, the set $\Theta_{\text{lox}}$ of loxodromic elements is also Zariski-dense (and, in particular, nonempty) in $H$ (see e.g. [BQ16, Theorem 6.36]). If $H$ is $p$-adic, this is true under the additional assumption that every simple root of $H$ is unbounded on the set of Cartan projections of $\Theta$ (see [BQ16, Lemma 9.2]). These existence results will be useful to us because, by the following lemma, a loxodromic element and a generic element generate a Zariski-dense subgroup:

**Lemma 5.5.** *Let $H$ be a connected simple algebraic group over a local field of characteristic 0, and let $h \in H$ be nontrivial. Assume that either*

*(i) $h$ is loxodromic, or*
*(ii) $H$ is nonarchimedean and $h$ belongs to a compact, open, torsion-free subgroup that is contained in the image of the exponential map.*

*Then there exists a nonempty Zariski-open (and in particular, open and dense) set $S \subseteq H$ such that for $g \in S$, the group $\langle g, h\rangle$ is Zariski-dense in $H$.*

*Proof.* The idea is to consider the set of elements $g$ such that $\mathrm{Ad}(g)$ and $\mathrm{Ad}(h)$ have no joint invariant subspace in the Lie algebra $\mathfrak{h}$ of $H$. Then for such $g$, as long as the Lie algebra of $\overline{\langle g, h\rangle}^Z$ is nontrivial, it must be all of $\mathfrak{h}$, and this will imply $\langle g, h\rangle$ is Zariski-dense. However, we will actually work with a smaller set of $g$ that we can prove is Zariski-open.

To define this set, for $g \in H$ and $0 < d < \dim \mathfrak{h}$, let $I_g(d)$ denote the collection $d$-dimensional $\mathrm{Ad}(g)$-invariant subspaces of $\mathfrak{h}$, and let $I_g := \cup_d I_g(d)$

denote the collection of all proper $\mathrm{Ad}(g)$-invariant subspaces of $\mathfrak{h}$. Note that $I_g(d)$ is a Zariski-closed subset of the Grassmannian $\mathrm{Gr}_d(\mathfrak{h})$. Let $S_d$ be the set of all elements $g \in H$ such that $\mathrm{Ad}(g)I_h(d) \cap I_h(d) = \varnothing$, so the complement of $S_d$ consists of those $g \in H$ such that there exists $x \in I_h(d)$ with $gx \in I_h(d)$. Note also that, as promised, we indeed have $I_g(d) \cap I_h(d) = \varnothing$ for all $g \in S_d$.

Let

$$\pi_H : H \times \mathrm{Gr}_d(\mathfrak{h}) \times \mathrm{Gr}_d(\mathfrak{h}) \to H$$

be the projection onto the first factor. In terms of this map, we have

$$S_d^c = \pi_H(\{(g, x, gx) \in H \times \mathrm{Gr}_d(\mathfrak{h}) \times \mathrm{Gr}_d(\mathfrak{h}) \mid x, gx \in I_h(d)\}).$$

Let $\alpha : H \times \mathrm{Gr}_d(\mathfrak{h}) \to \mathrm{Gr}_d(\mathfrak{h})$ denote the action map $\alpha(g, x) := gx$. Then we have

$$S_d^c = \pi_H(\mathrm{Graph}(\alpha) \cap (H \times I_d(h) \times I_d(h))).$$

Using this description, we will show that $S_d^c$ is Zariski-closed. A projection $X \times Y \to X$ of varieties is a closed map (with respect to the Zariski-topology on $X \times Y$) if $Y$ is a projective variety, so $\pi_H$ is a closed map. Of course $H \times I_d(h) \times I_d(h)$ is Zariski-closed, so it remains to show $\mathrm{Graph}(\alpha)$ is Zariski-closed. This is a special case of the general fact that the graph of an algebraic map $F : X \to Y$ to a projective variety $Y$ is Zariski-closed in $X \times Y$. We have shown $S_d$ is Zariski-open, and therefore so is $S = \cap_d S_d$.

Next, we show that $S \neq \varnothing$. For this, we use the following property of $h$ that holds under the assumption that either $h$ is loxodromic or sufficiently close to identity: Namely, there exists $X \in \mathfrak{h}$ such that $I_h$ is contained in the collection $I_X$ of $\mathrm{ad}(X)$-invariant proper subspaces of $\mathfrak{h}$. We prove this by considering the cases that (i) $h$ is loxodromic, and (ii) $h$ is contained in a compact open torsion-free subgroup.

In Case (i), $h$ is conjugate by some $c \in H$ to an element of the form $m \exp(X)$, where $m$ is elliptic and centralizes $A$, and $X \in \mathfrak{a}^+$ is regular. By inspecting the action of $\mathrm{Ad}(m)$ and $\mathrm{Ad}(\exp X)$ relative to the root space decomposition of $\mathfrak{h}$, we see that $I_{m\exp(X)} \subseteq I_{\exp X}$ and also that $I_{\exp X} = I_X$. Therefore $I_h \subseteq I_{c^{-1}\exp(X)c}$. We can write $c^{-1}\exp(X)c = \exp(\mathrm{Ad}(c)^{-1}X)$, and therefore $I_h \subseteq I_{\mathrm{Ad}(c)^{-1}X}$. This completes Case (i).

In Case (ii), we can write $h = \exp(X)$. Then of course any $\mathrm{ad}(X)$-invariant subspace is $\mathrm{Ad}(h)$-invariant. For the converse, note that if $H$ is $p$-adic, we have $h^{p^n} = \exp(p^n X) \overset{n\to\infty}{\longrightarrow} \mathrm{Id}$, and

$$\mathrm{ad}(X)v = \lim_{n\to\infty} \frac{1}{p^n} \left(\mathrm{Ad}(\exp(p^n X)) - \mathrm{Id}\right) v.$$

Now let $V$ be an $\mathrm{Ad}(h)$-invariant subspace. Then $V$ is also $\mathrm{Ad}(h^{p^n})$-invariant for any $n$, and by the above formula, is also $\mathrm{ad}(X)$-invariant.

Finally, by a theorem of Bois [Boi09], the Lie algebra $\mathfrak{h}$ is 1.5-generated, i.e. for every $0 \neq A \in \mathfrak{h}$ there exists $B \in \mathfrak{h}$ such that $\{A, B\}$ generates $\mathfrak{h}$ as a Lie algebra. Further, the set of $B$ with this property is Zariski-open (see [Boi09, Proposition 1.1.3]). Choose $Y$ such that $\{X, Y\}$ generates $\mathfrak{h}$, so that

$I_X \cap I_Y = \varnothing$. Since the set of possible choices of $Y$ is Zariski-open, we can choose such $Y$ that is loxodromic, i.e. $Y = E + A$ where $A$ is regular and $E$ is elliptic centralizing the maximal torus containing $A$. As we commented before, for such elements we have $I_{\exp Y} \subseteq I_A$. Since $\exp(Y)$ is loxodromic, $\mathrm{Ad} \exp Y$ contracts towards $I_Y$, so that after possibly replacing $Y$ by a large scalar multiple $\mathrm{Ad}(\exp Y) I_X$ is contained in a neighborhood of $I_Y$ disjoint from $I_X$, so $\exp Y \in S$.

It remains to show that for $g \in S$, the group $\langle g, h \rangle \subseteq H$ is Zariski-dense. Let $g \in S$ and note that $\langle g, h \rangle$ is infinite (because $h$ has infinite order), and hence its Zariski-closure has nontrivial Lie algebra, which is of course $\langle \mathrm{Ad}(g), \mathrm{Ad}(h) \rangle$-invariant. It follows that the Lie algebra of $\overline{\langle g, h \rangle}^Z$ is all of $\mathfrak{h}$, i.e. $\overline{\langle g, h \rangle}^Z$ is open. However, any proper Zariski-closed subgroup is nowhere dense, so we must have $\overline{\langle g, h \rangle}^Z = H$. □

Henceforth we will refer to the $\mathbb{Q}_{p_i}$-Zariski-topology on $G_i$ simply as the Zariski-topology, and likewise for the product Zariski-topology on $G = \prod_i G_i$.

To find elements in the Zariski-open sets given by the above lemma, we need to establish Zariski-denseness of suitable subsets of $\Gamma$. This is accomplished by the following lemma:

**Lemma 5.6.** *Assume the notation of Theorem 5.1. Fix $j \in I$ and let $U \subseteq G/G_j$ be an open neighborhood of identity. Denote by $\Gamma_j \subseteq \Gamma$ the subset consisting of elements $\gamma \in \Gamma$ whose projection mod $G_j$ lies in $U$.*

*Then*

*(i)* $\mathrm{pr}_j(\Gamma_j)$ *is Zariski-dense in $G_j$, and*

*(ii) If $G_j$ is $p$-adic for some $p$, then every simple root of $G_j$ is unbounded on the Cartan projections of* $\mathrm{pr}_j(\Gamma_j)$.

*Proof.* Proof of (i): As $U$ decreases, so does the Zariski-closure of $\mathrm{pr}_j(\Gamma_j)$ in $G_j$. By the Noetherian property of $G_j$, these Zariski-closures eventually stabilize, so we can assume stabilization has occurred at $U$, and denote the corresponding Zariski-closure by $H_j \subseteq G_j$. Then choosing $V \subseteq U$ symmetric such that $V^2 \subseteq U$, it is easy to see that $H_j^2 \subseteq H_j$ and $H_j^{-1} = H_j$, i.e. $H_j$ is a group. Further, if $\gamma \in \Gamma$ is fixed, we can choose $V_\gamma \subseteq U$ such that $\gamma V_\gamma \gamma^{-1} \subseteq U$. It follows that $H_j$ is normalized by $\mathrm{pr}_j(\gamma)$. Since $\gamma \in \Gamma$ was arbitrary, we conclude that $H_j$ is normalized by $\mathrm{pr}_j(\Gamma)$, which is Zariski-dense in $G_j$. Since $H_j$ is Zariski-closed, this implies that $H_j$ is normal in $G_j$, and since $H_j$ is infinite, we must have $G_j^+ \subseteq H_j$. Finally, since $G_j^+$ is Zariski-dense (see Theorem 2.7.(i)), we conclude that so is $H_j$.

Proof of (ii): Let $\kappa_j(\Gamma_j)$ be the set of Cartan projections of $\mathrm{pr}_j(\Gamma_j)$, and let $\alpha$ be any simple root of $G_j$. We need to show that $\alpha$ is unbounded on $\kappa_j(\Gamma_j)$. First note that $\mathrm{pr}_j(\Gamma_j)$ is unbounded (since any sequence of elements whose projections converge to identity in $G/G_j$ must diverge in $G_j$). Therefore $\kappa_j(\Gamma_j)$ is unbounded as well, and since the Weyl group orbit of $\alpha$ spans the dual $\mathfrak{a}^*$, there exists a Weyl group element $w \in W$ such that $w\alpha$ is unbounded on $\kappa_j(\Gamma_j)$.

By a result of Tits [Tit66], there exists a finite extension $\widetilde{W}$ of $W$ contained in $G$ such that if $\widetilde{w} \in \widetilde{W}$ has image $w \in W$, then $\mathrm{Ad}(\widetilde{w})$ coincides with the Weyl group action of $w$ on $\mathfrak{a}$. After possibly conjugating by an element of $A$, we can assume $\widetilde{W}$ is contained in the maximal compact subgroup $K_j$ of $G_j$. Since $\Gamma$ has dense projection to $G_j$ and $K_j$ is open, we can choose $\gamma \in \Gamma$ such that $\mathrm{pr}_j(\gamma) \in \widetilde{w}^{-1}K_j$, say $\mathrm{pr}_j(\gamma) = \widetilde{w}^{-1}k$ for some $k \in K_j$. Now choose $\gamma_n \in \Gamma_j$ converging to identity in $G_j$ and such that $(w\alpha)(\kappa_j(\gamma_n)) \to \infty$. For $n \gg 1$, we have $q_j(\gamma\gamma_n\gamma^{-1}) \in U$, so $\gamma\gamma_n\gamma^{-1} \in \Gamma_j$. Let

$$\mathrm{pr}_j(\gamma_n) = l_n a_n l_n'$$

be the Cartan decomposition of $\mathrm{pr}_j(\gamma_n)$. Then we simply compute

$$\mathrm{pr}_j(\gamma\gamma_n\gamma^{-1}) = \widetilde{w}^{-1} k l_n a_n l_n' k^{-1} \widetilde{w}.$$

Using that $\widetilde{W} \subseteq K_j$, we see that the Cartan projection of $\mathrm{pr}_j(\gamma\gamma_n\gamma^{-1})$ is given by

$$\kappa_j(\gamma\gamma_n\gamma^{-1}) = \widetilde{w}^{-1} a_n \widetilde{w}.$$

and

$$\alpha(\widetilde{w}^{-1} a_n \widetilde{w}) = (w\alpha)(\gamma_n) \to \infty.$$

□

With these preliminary arguments completed, we can start the proof of Theorem 5.1 proper:

*Proof of Theorem 5.1.* Index the factors by a set of primes and possibly $\infty$. If there is no archimedean factor, relabel one of the finite primes to be $\infty$. Then for $p < \infty$, let $K_p \subseteq G_p$ be a torsion-free compact open subgroup contained in the image of the exponential map on $G_p$ and set $K := \prod_{p<\infty} K_p$. For any $p$, let $q_p : G \to G/G_p \cong \prod_{\ell \neq p} G_\ell$ be the quotient mod $G_p$, and define the subgroup $\Gamma_K := \Gamma \cap q_\infty^{-1}(K)$ consisting of those elements of $\Gamma$ whose projection mod $G_\infty$ lies in $K$. By Lemma 5.6, $\mathrm{pr}_\infty(\Gamma_K)$ is Zariski-dense in $G_\infty$, and hence contains a loxodromic element. Let $a \in \Gamma_K$ be such that $\mathrm{pr}_\infty(a)$ is loxodromic.

We will now show that for every $p < \infty$, there exists $b_p \in \Gamma$ such that

(i) $\langle a, b_p \rangle$ is indiscrete mod $G_p$, and
(ii) $\langle a, b_p \rangle$ has Zariski-dense projections to $G_\infty$ and to $G_p$.

Assuming existence of such $b_p$, it is easy to see $\langle a, b_p \mid p < \infty \rangle$ is irreducible in $G$. Indeed, the projections mod $G_p$ are indiscrete for $p < \infty$ by Property (i). The projection mod $G_\infty$ is indiscrete because $q_\infty(a)$ belongs to the compact torsion-free subgroup $K$. Finally, projections to any factor are Zariski-dense by Property (ii).

So it remains to establish existence of elements $b_p \in \Gamma$ with the above Properties (i) and (ii). By Lemma 5.5 above, for every factor $G_\ell$ (including $G_\infty$) there exists an open dense set $S_\ell \subseteq G_\ell$ such that for any $g_\ell \in S_\ell$, the group $\langle g_\ell, \mathrm{pr}_\ell(a)\rangle$ is Zariski-dense in $G_\ell$. For every $p < \infty$, the set $\prod_{\ell \neq p} S_\ell \subseteq \prod_{\ell \neq p} G_\ell$ is open and dense, so we can choose $g_p \in \prod_{\ell \neq p} S_\ell$ sufficiently close to identity so that iterated commutators of $g_p$ and $q_p(a)$ converge to identity in $G/G_p$. Since the subsets $S_\ell \subseteq G_\ell, \ell \neq p$, are open, there is an open neighborhood $V_p$ of $g_p \in \prod_{\ell \neq p} G_\ell$ such that these properties hold for any element of $V_p$. To summarize, we have that for any $g \in V_p$, the group $\langle g, q_p(a)\rangle$ is indiscrete in $G/G_p$, and has Zariski-dense projection to $G_\ell$ for all $\ell \neq p$.

Since $\Gamma$ has dense projection to $\prod_{\ell \neq p} G_\ell$, we can choose choose $\gamma_p \in \Gamma$ such that $q_p(\gamma_p) \in V_p$. Further choose $W_p \ni e$ an open neighborhood of identity in $G/G_p$ such that $q_p(\gamma_p)W_p \subseteq V_p$. Let $\Gamma_p := \Gamma \cap q_p^{-1}(W_p)$ consist of those elements of $\Gamma$ whose projection mod $G_p$ lies in $W_p$.

We now consider elements $b_p$ of the form $\gamma_p\eta_p$, where $\eta_p \in \Gamma_p$. For such elements, the projection to $G_\infty$ lies in $S_\infty$, so $\langle a, b_p\rangle$ has Zariski-dense projection to $G_\infty$. Further, iterated commutators of $q_p(b_p)$ and $q_p(a)$ converge to identity in $G/G_p$, and they cannot terminate since then $\langle a, b_p\rangle$ would be nilpotent and in particular not Zariski-dense in $G_\infty$. This establishes both claimed Properties (i) and (ii) of $b_p$ except the Zariski-denseness of the projection to $G_p$, so we will now show we can choose $b_p$ to guarantee this as well.

This last property exactly means that we require $\mathrm{pr}_p(b_p) \in S_p$. On the other hand, the possible choices of $\mathrm{pr}_p(b_p)$ are among $\mathrm{pr}_p(\gamma_p)\,\mathrm{pr}_p(W_p)$. This latter set is Zariski-dense by Lemma 5.6 (applied with $U = W_p$), and since $S_p$ is nonempty and Zariski-open, we have $S_p \cap \mathrm{pr}_p(\gamma_p)\,\mathrm{pr}_p(W_p) \neq \varnothing$, so that there exists a satisfactory choice of $b_p$. □

## 6. Coherence

Recall the following definition of coherent groups:

**Definition 6.1.** A finitely presented group $\Gamma$ is *coherent* if any finitely generated subgroup of $\Gamma$ is finitely presented.

Free groups and surface groups are coherent, as are abelian and, more generally, polycyclic groups. Scott has shown 3-manifold groups are coherent [Sco73], and since then, coherence has emerged as one of their salient properties. On the other hand, $F_2 \times F_2$ is incoherent: Baumslag-Roseblade have proved a subgroup of $F_2 \times F_2$ is finitely presented if and only if it is a finite extension of a finite product of finite rank free groups, and that there are many finitely generated subgroups that are not of this form [BR84].

See [Wis20] for a survey of coherent groups, as well as many open problems. The class of coherent groups is not well-understood: For example, Serre famously asked whether $\mathrm{SL}(2, \mathbb{Z}[1/p])$, and more generally $\mathrm{GL}(2, \mathbb{Q})$, are coherent [Ser74]. Below we give a positive answer for the former and indeed for $\mathrm{SL}(2, \mathbb{Q})$ assuming the Greenberg-Shalom hypothesis. (Note however that $\mathrm{SL}(4, \mathbb{Z})$ is incoherent as it contains the incoherent group $F_2 \times F_2$. It is unknown whether $\mathrm{SL}(3, \mathbb{Z})$ is coherent.)

The relevance of the Greenberg-Shalom hypothesis to coherence of lattices is that by Proposition 4.1, irreducible lattices in products do not have many subgroups: Namely either a subgroup is not irreducible or it is also a lattice and therefore has finite index. In the following case, we have sufficient control over reducible groups that we can conclude they are finitely presented:

**Theorem 6.2.** *Assume the Greenberg-Shalom Hypothesis 1.1. Let $S$ be a (nonempty) finite set of places of $\mathbb{Q}$ and either assume all places are finite or that there is one infinite and finite place in $S$. Set $G_s := \mathrm{PGL}(2, \mathbb{Q}_s)$ and $G := G_S$. Then any $S$-arithmetic lattice $\Lambda \subseteq G$ is coherent.*

Now we prove Theorem 6.2:

*Proof.* Let $\Gamma \subseteq \Lambda$ be finitely generated. Let $H$ denote the $\mathbb{Q}$-Zariski-closure of $\Gamma$. If $H$ is a proper subgroup then it is solvable. In the case when we have one real and one finite place, this implies that $\Gamma$ is either virtually abelian or virtually contained in a Baumslag-Solitar group $BS(1,p)$. In either case, $\Gamma$ is finitely presented and we are done. So we can assume $\Gamma$ is $\mathbb{Q}$ Zariski dense and so all projections are Zariski dense.

Choose a minimal subset $T \subseteq S$ (possibly $T = S$) such that $\mathrm{pr}_T(\Gamma) \subseteq G_T$ is discrete. Then $\mathrm{pr}_J(\Gamma)$ is irreducible in $G_J$. If $|T| \geq 2$, then by Proposition 4.1, $\Gamma$ is an irreducible lattice in $G_T$ and hence is finitely presented, and the proof is complete.

It remains to consider the case that $\Gamma$ projects discretely to a factor $G_t$, which is either $\mathrm{PGL}(2, \mathbb{R})$ or $\mathrm{PGL}(2, \mathbb{Q}_p)$. In either case, all discrete, finitely generated subgroups are finitely presented: In $\mathrm{PGL}(2, \mathbb{R})$, any such subgroup is a surface group or is virtually free. In $\mathrm{PGL}(2, \mathbb{Q}_p)$, every such group is virtually free. □

**Remark 6.3.** It is clear from the proof that something more general can be proven by the same argument. For example, we can consider number fields other than $\mathbb{Q}$ if we assume all places are finite. A similar argument may work for (some) other number fields if we allow one infinite place and one finite place. However, that restriction is really needed as it is known that $\mathrm{SL}(2, \mathbb{Z}[1/n])$ is not coherent if $n$ is composite [Ser74].

This also has implications for a fundamental question about coherence, to our knowledge first explicitly posed by Wise [Wis20, Problem 9.16], namely whether coherence is a geometric (i.e. quasi-isometry invariant) property? We note that Wise already explicitly hedges against this.

**Corollary 6.4.** *Assume the Greenberg-Shalom hypothesis. Then coherence is not a quasi-isometry invariant.*

*Proof.* Let $p, q$ be distinct primes and consider lattices in $G := \mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$. All such lattices are cocompact and hence quasi-isometric. By Theorem 6.2, the irreducible lattices in $G$ are coherent. On the other hand, any lattice in $\mathrm{PGL}(2, \mathbb{Q}_p)$ or $\mathrm{PGL}(2, \mathbb{Q}_q)$ is virtually free (of rank $> 1$), so $G$ admits reducible lattices that are products $F_m \times F_n$ of free groups (with $m, n > 1$). These are incoherent by the above-mentioned result of Baumslag-Roseblade [BR84]. □

## 7. Margulis-Zimmer conjecture

A major motivation for Greenberg-Shalom's question is the following conjecture advertised by Margulis-Zimmer in the late '70s, seeking to classify commensurated subgroups of higher rank lattices $\Lambda$. Here we say $\Gamma \subseteq \Lambda$ is *commensurated* if $\Lambda \subseteq \mathrm{Comm}_G(\Gamma)$.

**Conjecture 7.1** (Margulis-Zimmer, see [SW13])**.** Let $\mathbb{G}$ be a semisimple algebraic group defined over a number field $k$ and $S$ a finite set of valuations of $k$. Assume $\mathbb{G}$ has higher $S$-rank. Assume $\Lambda = \mathbb{G}(\mathcal{O}_S)$ is an $S$-arithmetic lattice in $\mathbb{G}$, then any commensurated subgroup of $\Lambda$ is either finite or $S'$-arithmetic for some $S' \subseteq S$.

Here, the *$S$-rank* of $\mathbb{G}$ is the sum of the $k_\nu$-ranks over all valuations $\nu \in S$, and $\mathbb{G}$ is said to have *higher $S$-rank* if its $S$-rank is at least 2.

**Remark 7.2.**

(1) For example, if $\Gamma$ is a commensurated subgroup of $\mathrm{SL}(n, \mathbb{Z}[1/p])$ (where $n \geq 2$), then $\Gamma$ is predicted to be either finite, finite-index, or commensurable to $\mathrm{SL}(n, \mathbb{Z})$.
(2) Conjecture 7.1 is motivated by and strengthens Margulis' Normal Subgroup Theorem.
(3) Question 1.1 and Conjecture 7.1 are closely related: For example, if $\Gamma \subseteq \mathbb{G}(\mathcal{O}_S)$ is a commensurated subgroup and $\Gamma \cap \mathbb{G}(\mathcal{O})$ is infinite, then a positive answer to Question 1.1 proves $\Gamma$ intersects $\mathbb{G}(\mathcal{O})$ in a lattice. $S'$-arithmeticity of $\Gamma$ then follows from Venkataramana's result (see [LZ01, Proposition 2.3]) that the only intermediate subgroups between $\mathbb{G}(\mathcal{O})$ and $\mathbb{G}(\mathcal{O}_S)$ are $S'$-arithmetic for some $S' \subseteq S$, which proves Conjecture 7.1 for such groups.

Using well-chosen unipotent generating sets, Venkataramana has proven the Margulis-Zimmer Conjecture for arithmetic lattices $\Gamma = \mathbb{G}(\mathbb{Z})$ in simple groups defined over $\mathbb{Q}$ [Ven87]. Shalom-Willis have proved Conjecture 7.1 in more instances, including the first that are not simple [SW13]. Their proof crucially relies on fine arithmetic properties for lattices in these groups, namely bounded generation by unipotents. As it is now known that bounded generation is not a common property for higher rank lattices [CRRZ22], different approaches are needed. The results in [FMvL22] yield partial results on this conjecture and the first that do not depend on unipotent elements, see Corollary 1.5 in that paper. This is the only prior work on Conjecture 7.1.

We now deduce a special case of Conjecture 7.1 assuming the Greenberg-Shalom hypothesis. Namely we deduce the case where $G$ has at least two factors, at least one of which is non-archimedean. The deduction of this case of the Margulis-Zimmer conjecture from the Greenberg-Shalom hypothesis is quite simple modulo arguments we have already made. The existence of at least two factors is essential to the argument, but the requirement of a non-archimedean one can be removed if the Greenberg-Shalom hypothesis is strengthened to cover approximate groups (see Question 4.6), but we do not pursue this here.

Let $G = \prod_I G_i$ is as in 2.1. Define the *rank* of $G$ by $\mathrm{rk}(G) = \sum_{i \in I} k_i\text{-}\mathrm{rk}(G_i(k_i))$ and assume that $\mathrm{rk}(G) \geq 2$. Let $G_{\mathrm{na}}$ denote the product of $G_i$'s over all non-archimedean factors. We establish the following case of Conjecture 7.1.

**Theorem 7.3.** *Assume the Greenberg-Shalom hypothesis and suppose $G_{na}$ is non-trivial and $|I| \geq 2$. Let $\Lambda$ denote an irreducible lattice in $G$, and $\Gamma \subset \Lambda$ be an infinite subgroup commensurated by $\Lambda$. Then there exists $J \subset I$ such that $\mathrm{pr}_J(\Gamma)$ (isomorphic to $\Gamma$ under $\mathrm{pr}_J$) is a lattice in $G_J$.*

*Proof.* If $\Gamma$ is irreducible, it is a lattice by Proposition 4.1.

Assume therefore that $\Gamma$ is reducible. Hence there exists a minimal subset $J \subsetneq I$ such that $\mathrm{pr}_J(\Gamma)$ is discrete. Irreducibility of $\Lambda$ implies that $\mathrm{pr}_J(\Lambda)$ is dense in $G_J$. Note also that $\mathrm{pr}_J$ is injective on $\Lambda$, and hence on $\Gamma$. Thus, $\mathrm{pr}_J(\Gamma) \subset G_J$ is an infinite discrete subgroup commensurated by $\mathrm{pr}_J(\Lambda)$, where the latter is almost dense. Then the Greenberg-Shalom hypothesis implies that $\mathrm{pr}_J(\Gamma) \subset \mathrm{pr}_J(G)$ is a lattice. $\square$

One might want to reverse this implication, but it is not immediate. If one assumes that $\Gamma < G$ as in Greenberg-Shalom's Question 1.1 is contained in an arithmetic lattice $\mathbb{G}(\mathcal{O}) \subseteq G$ (where $\mathcal{O}$ is the ring of integers of a number field $k$), then it follows from an old argument of Borel that the commensurator $\Lambda := \mathrm{Comm}_G(\Gamma)$ is contained in the $\mathbb{G}(k)$ see [Bor66] or [Zim84, Prop. 6.2.2]. (In Zimmer this is stated for arithmetic lattices, but the only property used is that $\Gamma$ is a Zariski-dense subgroup of an arithmetic lattice). Then $\Lambda$ contains finitely generated subgroups that are dense in $G$ and contained in $S$-arithmetic lattices $\mathbb{G}(\mathcal{O}_S)$, where $S$ is a finite set of places of $k$. It is, however, not at all clear in general how to force $\Lambda$ to contain an $S$-arithmetic lattice without already proving $\Gamma$ is a lattice. In this sense, the Greenberg-Shalom hypothesis is a strengthening of a special case of the Margulis-Zimmer conjecture.

## 8. Positive characteristic and automorphism groups of trees

Going beyond semisimple Lie groups, one can study discrete irreducible subgroups of automorphism groups of trees. Their irreducible lattices are known to exhibit rigidity by the work of Burger-Mozes [BM00a, BM00b]. Let $\Gamma$ be the fundamental group of a surface of genus at least 2 and let $T_i$ for $i = 1, \ldots, k$ be a bounded valence simplicial tree.

In this setting we have the following questions due to Fisher-Larsen-Spatzier-Stover:

**Question 8.1** (Fisher-Larsen-Spatzier-Stover [FLSS18])**.**

(1) Does there exist $\rho : \Gamma \to \mathrm{Aut}(T_1 \times \cdots \times T_k)$ with discrete image?
(2) Does there exist $\rho$ as in (1) where $\rho$ takes values in a product $G_1 \times \cdots \times G_k$ where each $G_i$ is a rank one simple algebraic group over a non-archimedean local field?
(3) Can $\Gamma$ be faithfully represented into $\mathrm{PGL}(2, K)$ for some global field $K$ of positive characteristic?

If there is an action as in (1), then on a subgroup of finite index, $\rho$ is the diagonal embedding from homomorphisms $\rho_i : \Gamma \to \mathrm{Aut}(T_i)$. Question 8.1(2) is only implicit in [FLSS18]. In [FLSS18], it is shown a positive answer to the third question gives a positive answer to the first and second. Questions 8.1(1) and (2) are clearly related to Corollary 4.2. We remark here that a negative answer to the analogue of Greenberg-Shalom's question for automorphism groups of trees is given by Burger and Mozes in [BM96, Proposition 8.1], and that one can also construct irreducible subgroups in products of trees as done by e.g. the third author and Huang in [HM24]. However, Question 8.1(1) is still open as is the corresponding question for finitely generated free groups.

We show that the Greenberg-Shalom hypothesis for groups defined over local fields of positive characteristic implies a negative answer to (3). This does not completely resolve (2). The main result of [FLSS18] produces a surface group in $PSL(2, F)$ where $F$ is a positive characteristic field of transcendence degree 2. If the answer to (3) is negative, then that result is sharp in that there is no representation into a positive characteristic field of transcendence degree one.

**Proposition 8.2.** *If the Greenberg-Shalom hypothesis is correct for $G = \mathrm{PGL}(2, k)$ where $k$ is a local field of positive characteristic (replacing the notion of almost denseness by containing the cocompact group $G^+$), then Fisher-Larsen-Spatzier-Stover's Question 8.1*(3) *has a negative answer.*

We first give some indication of the proof strategy and introduce necessary background to carry out this strategy. Suppose there exists a surface group $\Delta \subseteq \mathrm{PGL}(2, K)$ for some global field $K$ of positive characteristic. Set $\mathbb{G} := \mathrm{PGL}(2)$. Since $\Delta$ is finitely generated, there exists a finite set of places $S$ of $K$ such that $\Delta \subseteq G_S$ is discrete. Since $\mathrm{PGL}(2, K_s)$ cannot contain a discrete surface group, we can assume that $|S| \geq 2$ and that the projection of $\Delta$ to any collection of subfactors is not discrete.

We will now briefly indicate the idea for the rest of the proof. After some initial reductions, we can assume $\Delta$ is Zariski-dense. By fixing a place $s$ and a compact open subgroup $K_s \subseteq G_s$, we can define $\Gamma_s := \Delta \cap K_s$, which is commensurated by $\Delta$ and is discrete and Zariski-dense in $G_{S \setminus s}$. If the closure of $\Delta$ in $G_{S \setminus s}$ contains $G^+_{S \setminus s}$, then by the positive characteristic analogue of the Greenberg-Shalom hypothesis, $\Gamma_s$ projects to a lattice in $G_{S \setminus s}$, so that by the same variation of Venkataramana's lemma as in the proof of Proposition 4.1, $\Delta$ is a lattice in $G_S$, but this is impossible as surface groups are not lattices in totally disconnected groups.

To prove $\overline{\Delta} \subseteq G_{S\setminus s}$ contains $G^+_{S\setminus s}$, one needs a strong approximation result in positive characteristic. Strong approximation in this setting is due to Pink [Pin98, Pin00] (strengthening earlier work by Weisfeiler [Wei84]), but unlike in characteristic zero, one does not merely need to pass to universal covers, but instead to so-called 'minimal quasi-models'. We will now introduce this terminology (in the simplified setting of simple groups, see also [LS03, pp. 416-17]).

Let $\mathbb{G}$ be a connected, absolutely simple, adjoint linear algebraic group over a global field $L$, and write $G := \mathbb{G}(L)$. Let $\Gamma \subseteq G$ be finitely generated and Zariski-dense. Suppose now $K \subseteq L$ is another global field, and $\mathbb{H}$ is a connected, absolutely simple, adjoint linear algebraic group over $K$, and $\varphi : \mathbb{H} \times_K L \to \mathbb{G}$ is an isogeny such that $\Gamma \subseteq \varphi(\mathbb{H}(K))$. Here the notation $\mathbb{H} \times_K L$ means that we extend scalars from $K$ to $L$, i.e. we view the $K$-algebraic group $\mathbb{H}$ as an $L$-algebraic group. Then $(K, \mathbb{H}, \varphi)$ is called a *weak quasi-model* of $(L, \mathbb{G}, \Gamma)$. If for any weak quasi-model $(K, \mathbb{H}, \varphi)$ of $(L, \mathbb{G}, \Gamma)$, we have $K = L$ and $\varphi$ is an isomorphism, then $(L, \mathbb{G}, \Gamma)$ is called *minimal*. For any weak quasi-model, the map $\varphi : H \to G$ is injective, so that $\Gamma$ can be identified with its pre-image in $H$. It follows that we can replace $(L, \mathbb{G}, \Gamma)$ by $(K, \mathbb{H}, \Gamma)$. Pink proves this process has to terminate and therefore that minimal weak quasi-models always exist:

**Theorem 8.3** (Pink [Pin98, Theorem 3.6])**.** *Any triple $(L, \mathbb{G}, \Gamma)$ as above admits a minimal weak quasi-model $(K, \mathbb{H}, \varphi)$. In addition, $K$ is unique, and $\mathbb{H}$ and $\varphi$ are unique up to unique isomorphism.*

We will also have occasion to have the following result for minimal quasi-models for commensurated subgroups:

**Proposition 8.4** (Pink [Pin00, Proposition 3.10])**.** *Let $(L, \mathbb{G}, \Gamma)$ be as above and minimal. Let $\Delta \subseteq \Gamma$ be a commensurated subgroup that is Zariski-dense in $G$. Then $(L, \mathbb{G}, \Delta)$ is also minimal.*

Now we turn our attention towards strong approximation. Let $(L, \mathbb{G}, \Gamma)$ be as above and minimal. Let $S$ be the set of places of $s$ of $L$ such that the image of $\Gamma$ in $G_s := \mathbb{G}(L_s)$ is unbounded. Denote by $\mathbb{A}^S_L$ the ring of adèles of $L$ away from $S$, and let $\pi : \widetilde{\mathbb{G}} \to \mathbb{G}$ denote the universal cover. Since $\ker \pi$ is central, the commutator map on $\widetilde{\mathbb{G}}$ descends to $\mathbb{G} \times \mathbb{G} \to \widetilde{\mathbb{G}}$. Let $\Gamma'$ be the subgroup of $\widetilde{G} := \widetilde{\mathbb{G}}(L)$ generated by the image of $\Gamma$ under this commutator map. We are now in position to state Pink's strong approximation result:

**Theorem 8.5** (Pink [Pin00, Theorem 0.2])**.** *Let $(L, \mathbb{G}, \Gamma)$ be as above and minimal. Then the closure of $\Gamma'$ is open in $\widetilde{\mathbb{G}}(\mathbb{A}^S_L)$.*

We can now prove Proposition 8.2:

*Proof.* Suppose there exists a discrete surface group $\Gamma \subseteq \mathrm{PGL}(2, K)$ for some global field $K$ of positive characteristic, which is necessarily Zariski-dense (since otherwise its Zariski-closure would be virtually solvable). By Pink's Theorem 8.3, there exists a minimal weak quasi-model $(F, \mathbb{G}, \Delta)$ where $F \subseteq K$ is a subfield and $\mathbb{G}$ is an absolutely simple adjoint group over $F$, and there is a $K$-isogeny $\varphi : \mathbb{G} \times_F K \to \mathrm{PGL}(2)$ with $\Delta \subseteq \varphi(\mathbb{G}(F))$. As above, consider the set $S$ of places $s$ of $F$ such that $\Gamma$ is unbounded in $G_s := \mathbb{G}(F_s)$. Since $\Gamma$ is finitely generated, $S$ is finite, and $\Gamma$ is discrete in $G_S := \prod_{s \in S} G_s$.

We claim that $|S| \geq 2$. We must have $S \neq \varnothing$, simply because a surface group is infinite and therefore cannot be discrete in a compact group. Fix $s \in S$, and extend $s$ to a valuation on $K$, so that $\varphi(\mathbb{G}(F_s)) \subseteq \mathrm{PGL}(2, K_s)$. Then $G_s := \mathbb{G}(F_s)$ has $F_s$-rank 1: If $A \cong (F_s^\times)^d \subseteq \mathbb{G}(F_s)$ is a maximal $F_s$-split torus, then $A \times_{F_s} K_s \cong (K_s^*)^d$ is a $K_s$-split torus in $\mathbb{G}(K_s)$. Since $\mathbb{G}(K_s)$ is isogenous to $\mathrm{PGL}(2, K_s)$, we must have $d \leq 1$. Since $G_s$ has $F_s$-rank 1, its Bruhat-Tits building is a tree. Since a surface group cannot act properly on a tree, $\Gamma \subseteq G_s$ cannot be discrete. This shows that there must be at least one additional place $s' \in S$, i.e. $|S| \geq 2$.

Now fix $s \in S$ and compact open subgroups $U_s \subseteq G_s$ and $U_{S \setminus s} \subseteq G_{S \setminus s}$. Let $\Gamma_s := \Gamma \cap (U_s \times G_{S \setminus s})$ and $\Gamma_{S \setminus s} := \Gamma \cap (G_s \times U_{S \setminus s})$. Both $\Gamma_s$ and $\Gamma_{S \setminus s}$ are commensurated by $\Gamma$. It is easy to see both $\Gamma_s$ and $\Gamma_{S \setminus s}$ are Zariski-dense in $G$: Since the argument is entirely the same for both, we will only include it for $\Gamma_s$. Since $G = \mathbb{G}(F)$ is Noetherian over $F$, the Zariski-closure of finite index subgroups of $\Gamma_s$ stabilizes as the subgroups decrease. By possibly shrinking $U_s$, without loss of generality we can assume the Zariski-closure $\overline{\Gamma_s}^Z \subseteq G$ is invariant under passing to finite index subgroups of $\Gamma_s$. For $\gamma \in \Gamma$, we can choose a finite index subgroup $\Gamma_\gamma \subseteq \Gamma_s$ such that $\gamma \Gamma_\gamma \gamma^{-1} \subseteq \Gamma_s$. It follows that $\overline{\Gamma_s}^Z$ is normalized by $\gamma$. Since $\gamma \in \Gamma$ was arbitrary and $\Gamma \subseteq G$ is Zariski-dense, $\overline{\Gamma_s}^Z$ is normal in $G$. But since $G$ is simple, we must have $\overline{\Gamma_s}^Z = G$.

Since $\Gamma_{S \setminus s} \subseteq G$ is Zariski-dense, and minimality of $(F, \mathbb{G}, \Gamma)$ passes to commensurated Zariski-dense subgroups of $\Gamma$ (see Pink's Proposition 8.4), Pink's Strong Approximation Theorem 8.5 applies to $(F, \mathbb{G}, \Gamma_{S \setminus s})$. Since $s$ is the only place of $F$ such that $\Gamma_{S \setminus s}$ has unbounded projection to $G_s$, strong approximation shows that the closure of $\Gamma'_{S \setminus s}$ in $\widetilde{\mathbb{G}}(\mathbb{A}_F^s)$ is open. The natural projection $\widetilde{\mathbb{G}}(\mathbb{A}_F^s) \to G_{S \setminus s}$ is open, so the closure of $\Gamma'_{S \setminus s}$ in the latter is also open, and hence so is the closure of $\Gamma'$. Then $\overline{\Gamma'}$ is open and unbounded in $\widetilde{G}_{S \setminus s}$, so by the Tits-Prasad Theorem 2.8, $\overline{\Gamma'} = \widetilde{G}_{S \setminus s}$. (We note here that in Section 2, we considered only fields of characteristic zero, but the Tits-Prasad theorem is valid in arbitrary characteristic). By applying the universal covering map and using that the image of the universal covering map is precisely $G_{S \setminus s}^+$, we see that $\overline{\Gamma}$ contains $G_{S \setminus s}^+$.

Since $\Gamma_s$ is discrete and Zariski-dense in $G_{S \setminus s}$ and commensurated by the group $\Gamma$ whose closure contains $G_{S \setminus s}^+$, the positive characteristic analogue of the Greenberg-Shalom hypothesis implies $\Gamma_s \subseteq G_{S \setminus s}$ is a lattice. The variation on Venkataramana's lemma given in the latter half of the proof of Proposition 4.1 then applies and shows $\Gamma \subseteq G = G_s \times G_{S\setminus s}$ is a lattice. But surface groups can only be lattices in groups isogenous to $\mathrm{SL}(2,\mathbb{R})$, and in particular, not in a product of algebraic groups defined over local fields of positive characteristic. This is a contradiction. □

## 9. Automorphism groups of trees

In this section we prove an improvement on [FLSS18, Theorem 15] that we used in the proof of Corollary 4.2. The improvement is minor and the main ideas are present in [FLSS18] but we need a stronger statement than given there and expect it to be needed in other applications.

**Theorem 9.1.** *Suppose that $\Lambda$ is a torsion free hyperbolic group that is not free. Let*

$$X = T_1 \times \cdots \times T_n$$

*be a product of finite-valence trees, set $G_i := \mathrm{Aut}(T_i)$, and $G := \prod G_i$. Let $\mathrm{pr}_i$ denote the projection of $G$ onto $G_i$. If $\rho : \Lambda \to G$ is a discrete and faithful representation, then there are at least two values of $i$ such that $\rho_i := \mathrm{pr}_i \circ \rho$ is faithful and has indiscrete image. Moreover, suppose $\rho_1, \ldots, \rho_r$ are faithful representations and the other $\rho_i$ are not. Then the representation*

$$\rho_1 \times \cdots \times \rho_r : \Lambda \to G_1 \times \cdots \times G_r$$

*is discrete and faithful. If we further assume $\rho_1 \times \cdots \times \rho_r$ is minimal for the property of having discrete image, then for all $i$, $\rho_i(\Lambda)$ does not fix a point at infinity on $T_i$.*

*Proof.* The only new statement is the last one concerning no fixed points at infinity. Assume the action on one tree $T_1$ fixes a point at infinity, that all $\rho_i$ are faithful, that $r > 1$ and that $(\rho_2 \times \cdots \times \rho_r)(\Lambda)$ is not discrete. If we fix a point $\eta$ at infinity on $T_1$, we have a height, or Busemann, function $b_\eta : T_1 \to \mathbb{Z}$. It is easy and standard that if $\rho_1(\Lambda)$ fixes $\eta$ and $x_0 \in T_1$ is chosen such that $b_\eta(x_0) = 0$, then the map

$$\begin{aligned} B_\eta :\Lambda &\longrightarrow \mathbb{Z} \\ \lambda &\longmapsto b_\eta(\lambda x_0) \end{aligned}$$

is a homomorphism.

Let $K = \ker(B_\eta)$. Observe that since $G_1$ has no parabolic elements, any $\lambda \in K$ fixes a point in $T_1$. Now fix a vertex $v$ in $T_2 \times \cdots \times T_r$ and let $\Delta$ be the stabilizer of $v$ in $\Lambda$ under the action defined by $\rho_2 \times \cdots \times \rho_r$. Note $\Delta$ is non-trivial by hypothesis.

It suffices to show that $K \cap \Delta$ is nonempty and this is done almost exactly as in [FLSS18]. Consider $x \in K$ and $y \in \Delta$. As $K$ is normal in $\Lambda$, the commutators $[x, y^n] = x(y^n x^{-n} y^{-n})$ belong to $K$ for any $n \in \mathbb{Z}$. It suffices to find values of $n$ such that the commutator also belongs to $\Delta$. As in [FLSS18], we see that since $y$ fixes $v$, the points $y^n x^{-1} y^{-n} \cdot v = y^n x^{-1} \cdot v$ all lie in a ball centered at $v$ of radius $d(v, x^{-1} \cdot v)$. Since all trees are finite valence, this ball is a finite set and so there are $n_1 \neq n_2 \in \mathbb{Z}$ such that $y^{n_1}x^{-1}\cdot v = y^{n_2}x^{-1}\cdot v$ this implies $xy^{n_1-n_2}x^{-1}v = v$, so that $[x, y^{n_1-n_2}] \in \Delta$. This contradicts discreteness of $(\rho_1 \times \cdots \times \rho_r)(\Lambda)$. □

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