Math 550 – Homework 10

For discussion: 4/9 at 5-6 pm in SEO 427.

Problem 1. Let $\pi : P \to M$ be a principal G-bundle and $\Phi : P \to P$ be a gauge transformation with associated map $\eta : P \to G$ (see HW 8 set, last problem). Let $\mathcal{H}$ be a connection on P with connection form ω and curvature Ω. Recall from HW 9:

- $\mathcal{H}^\Phi := \Phi_* \mathcal{H}$ is also a connection.
- Let $\omega^\Phi := (\Phi^{-1})^* \omega$ is the connection form of $\mathcal{H}^\Phi$.

Prove that $\Phi^* \omega = \mathrm{Ad}(\eta^{-1}) \circ \omega + \eta^* \omega_G$, where $\omega_G \in \Omega^1(G; \mathfrak{g})$ is the *Maurer-Cartan form* on G, namely for $v \in T_g G$, set $\omega_G(v) := L_{g^{-1}*} v \in \mathfrak{g}$, where L_g is left-translation by g.

Problem 2. (Classification of real line bundles.)

(a) Let M be a smooth manifold. Let $L \to M$ be a (real) line bundle over M. Prove that L admits an atlas of local trivializations with cocycle valued in $\{\pm \mathrm{Id}\} \subseteq \mathrm{GL}(1, \mathbb{R})$. Conclude that L admits a flat connection.

(b) Hence by the classification of flat bundles, we find that the set $\mathrm{Pic}(M; \mathbb{R})$ of real line bundles over M are in one-to-one correspondence with $\mathrm{Hom}(\pi_1 M, \mathbb{Z}/2\mathbb{Z}) = H^1(M, \mathbb{Z}/2\mathbb{Z})$ via monodromy.

Prove that this is a group isomorphism (where $\mathrm{Pic}(M; \mathbb{R})$ is equipped with the operations of tensor product and dualization).

Remark 1: An independent proof of (a) is that a real line bundle is trivial if and only if it is orientable. One may show that any line bundle $L \to M$ pulls back to an orientable line bundle over some double cover M_L^+ of M. And such double covers are in bijection with $\mathrm{Hom}(\pi_1 M, \mathbb{Z}/2\mathbb{Z})$.

Remark 2: The group isomorphism $\mathrm{Pic}(M; \mathbb{R}) \to H^1(M, \mathbb{Z}/2\mathbb{Z})$ is a characteristic class (with $\mathbb{Z}/2\mathbb{Z}$-coefficients) called the first *Stiefel-Whitney class* w_1.

Problem 3. Let $E_1 \to M$ and $E_2 \to M$ be two complex vector bundles with associated frame bundles $\mathrm{GL}(E_1) \to M$ and $\mathrm{GL}(E_2) \to M$. Fix connection forms ω_1 and ω_2 on $\mathrm{GL}(E_1)$ and $\mathrm{GL}(E_2)$.

(a) Use ω_1 and ω_2 to define a connection form ω on $\mathrm{GL}(E_1 \otimes E_2)$ as follows: Let $n_1 = \mathrm{rk}(E_1)$ and $n_2 = \mathrm{rk}(E_2)$. Consider the principal $\mathrm{GL}(\mathbb{C}^{n_1}) \times \mathrm{GL}(\mathbb{C}^{n_2})$-bundle $\mathrm{GL}(E_1) \times_\Delta \mathrm{GL}(E_2)$ over M that is the pre-image of the diagonal under $\mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \to M \times M$. First define a connection here. Then use the natural representation $\mathrm{GL}(\mathbb{C}^{n_1}) \times \mathrm{GL}(\mathbb{C}^{n_2}) \to \mathrm{GL}(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$.

(b) Compute the curvature of ω in terms of the curvatures of ω_1 and ω_2. Use this to prove

$$c_1(E_1 \otimes E_2) = n_2 c_1(E_1) + n_1 c_1(E_2).$$

Problem 4. Recall that the Chern character is $\mathrm{ch}(E) = \sum_j e^{z_j(E)}$, where $z_j(E)$ are the Chern roots, and the right-hand side is to be understood as a power series.

(a) Let Ω be the curvature of a connection on E. Prove that $\mathrm{ch}(E) = [\mathrm{tr}\exp \frac{i}{2\pi}\Omega]$, where the right-hand side is to be understood as

$$[\mathrm{tr}\exp \frac{i}{2\pi}\Omega] = \sum_{0\le k\le n} \frac{1}{k!}\left[\mathrm{tr}\left(\left(\frac{i}{2\pi}\Omega\right)^k\right)\right].$$

(b) Prove that $\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)$ and $\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1)\mathrm{ch}(E_2)$.

(For the latter, note that the case of line bundles follows from the previous problem (why?).)

Remark 1: The *Picard group* $\mathrm{Pic}(M;\mathbb{C})$ of M is the (abelian) group of complex line bundles over M, where the group multiplication is given by tensor product, and inverse is given by dual ($\cong$ complex conjugate; why are these the same? And why is this the inverse?). The result of Part (b) in particular shows that $c_1 : \mathrm{Pic}(M;\mathbb{C}) \otimes \mathbb{R} \to H^2(M,\mathbb{R})$ is a group isomorphism. This is a complex version of Problem 2.

Remark 2: There is an interesting way to combine c_1 and w_1, namely consider the composition

$$H^1(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{w_1^{-1}} \mathrm{Pic}(M;\mathbb{R}) \xrightarrow{-\otimes_{\mathbb{R}}\mathbb{C}} \mathrm{Pic}(M;\mathbb{C}) \xrightarrow{c_1} H^2(M,\mathbb{Z}),$$

where the middle map is complexification. It turns out that this is the Bockstein homomorphism (for the change of coefficients $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$).

Remark 3: The fact that for line bundles c_1 is an isomorphism on the group of line bundles (see Remark (1)) admits a generalization of sorts to higher rank bundles: The result of Part (b) essentially says that the Chern character is a ring homomorphism from the ring $K^0(M)$ of stable equivalence classes of complex vector bundles over M to the ring $H^\bullet(M)$ of the cohomology of M. Amazingly, this is an isomorphism onto the subring $H^{\mathrm{even}}(M)$ of cohomology with even degrees (Atiyah-Hirzebruch)! This is also true with rational coefficients.

Problem 5.

(a) Let $E \to M$ be a complex vector bundle. Prove that $c_k(E^*) = (-1)^k c_k(E)$.

Hint: Fix a connection form ω on $\mathrm{GL}(E)$. Use this to define a connection form ω^* on $\mathrm{GL}(E^*)$, and relate the curvature of ω^* to the curvature of Ω. Then prove the result.

(b) As an application of the relation $c_k(E^*) = (-1)^k c_k(E)$, prove that if E is the complexification of a real vector bundle over M, then all the odd Chern classes of E vanish.