Math 550 – Homework 11

For discussion: 4/23 at 5-6 pm in SEO 427.

Problem 1: Using that any bundle admits a Hermitian metric, prove that the Chern classes lie in $H^\bullet(M,\mathbb{R})$.

Problem 2. Let $E_1 \to M$ and $E_2 \to M$ be two complex vector bundles with associated frame bundles $\mathrm{GL}(E_1) \to M$ and $\mathrm{GL}(E_2) \to M$. Fix connection forms ω_1 and ω_2 on $\mathrm{GL}(E_1)$ and $\mathrm{GL}(E_2)$.

(a) Use ω_1 and ω_2 to define a connection form ω on $\mathrm{GL}(E_1 \otimes E_2)$ as follows: Let $n_1 = \mathrm{rk}(E_1)$ and $n_2 = \mathrm{rk}(E_2)$. Consider the principal $\mathrm{GL}(\mathbb{C}^{n_1}) \times \mathrm{GL}(\mathbb{C}^{n_2})$-bundle $\mathrm{GL}(E_1) \times_\Delta \mathrm{GL}(E_2)$ over M that is the pre-image of the diagonal under $\mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \to M \times M$. First define a connection here. Then use the natural representation $\mathrm{GL}(\mathbb{C}^{n_1}) \times \mathrm{GL}(\mathbb{C}^{n_2}) \to \mathrm{GL}(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$.

(b) Compute the curvature of ω in terms of the curvatures of ω_1 and ω_2. Use this to prove

$$c_1(E_1 \otimes E_2) = n_2 c_1(E_1) + n_1 c_1(E_2).$$

Problem 3.

(a) Let $L \to M$ be a complex line bundle. Prove that $L \otimes L^* \to M$ is trivial.

(b) Let $E \to M$ be a complex vector bundle. Use the previous part and the splitting principle to prove that $c_k(E^*) = (-1)^k c_k(E)$.

Problem 4: The goal of this problem is to prove that a line bundle with trivial Chern class is trivial. A topological proof using the classifying space $\mathbb{C}P^\infty$ of line bundles was given in lecture, but here we will give a geometric proof.

Let $L \to M$ be a complex line bundle with frame bundle $P \to M$ and assume $c_1(L) = 0$.

(a) Prove that P admits a flat connection. *Hint:* First choose any connection ω. Use vanishing of c_1 to show you can adjust ω to be flat.

(b) Assume now $H^1(M,\mathbb{Z})$ is torsion-free. Prove that L is trivial. *Hint:* Note that a flat bundle with connection is given by (a conjugacy class of) a representation of $\pi_1(M)$ to the structure group.

(c) Now suppose $H^1(M,\mathbb{Z})$ contains torsion. Conclude from the previous parts that there exists a finite cover $p : \overline{M} \to M$ such that $p^*L \to \overline{M}$ is trivial.

Problem 5 (Meaning of top Chern class). Let $E \to M$ be a rank r complex vector bundle. Use stability of Chern classes to prove that if E admits a nowhere vanishing section, then $c_r(E) = 0$.

Remark: The converse is also true. Soon, you will be able to prove this using the Euler class.

Problem 6. Let $E \to M$ be a rank n complex vector bundle. Recall that the *determinant bundle* is the top exterior power $\det E = \Lambda^n E$.

(a) Use the splitting principle to prove that $c_1(E) = c_1(\det E)$. *Hint:* Recall that $\Lambda(V \oplus W) = \Lambda(V) \otimes \Lambda(W)$.

(b) Prove that $c_1(E) = 0$ if and only if E admits a connection with holonomy contained in $\mathrm{SU}(n)$ (i.e. E can be obtained as an associated bundle of a principal $\mathrm{SU}(n)$-bundle).

Problem 7. Recall that the Chern character is $\mathrm{ch}(E) = \sum_j e^{z_j(E)}$, where $z_j(E)$ are the Chern roots, and the right-hand side is to be understood as a power series.

(a) Let Ω be the curvature of a connection on E. Prove that $\mathrm{ch}(E) = [\mathrm{tr}\exp \frac{i}{2\pi}\Omega]$, where the right-hand side is to be understood as

$$[\mathrm{tr}\exp \frac{i}{2\pi}\Omega] = \sum_{0\le k\le n} \frac{1}{k!}\left[\mathrm{tr}\left(\left(\frac{i}{2\pi}\Omega\right)^k\right)\right]$$

and $\mathrm{tr}\left(\left(\frac{i}{2\pi}\Omega\right)^k\right)$ is interpreted as a form using the isomorphism of invariant polynomials and symmetric forms (note that $X \mapsto \mathrm{tr}(X^k)$ is an invariant polynomial on the space of matrices).

(b) Prove that $\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)$ and $\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1)\mathrm{ch}(E_2)$.

(For the latter, note that the case of line bundles follows from the previous problem (why?).)

Remark: In the special case of line bundles, Problem 2 showed that the first Chern class (which coincides with the Chern character for those bundles) is a homomorphism $\mathrm{Pic}(M, \mathbb{C}) \to H^2(M, \mathbb{C})$. In fact, in lecture we showed it is an isomorphism if one takes $\mathbb{Z}$ coefficients (cf. Problem 4).

This problem generalizes this to higher rank bundles: The result of Part (b) shows that the Chern character is a homomorphism from the ring $K^0(M)$ of stable equivalence classes of complex vector bundles over M to the ring $H^\bullet(M)$ of the cohomology of M. Amazingly, this is an isomorphism onto the subring $H^{\mathrm{even}}(M)$ of cohomology with even degrees (Atiyah-Hirzebruch)!

Problem 8. The goal of this problem is to prove $c(\mathbb{C}P^n) = (1 + x)^n$, where $x \in H^2(\mathbb{C}P^n, \mathbb{Z})$ is Poincaré dual to $\mathbb{C}P^1 \subseteq \mathbb{C}P^n$.

Let $\gamma = \gamma_1(n)$ be the tautological line bundle over $\mathbb{C}P^n$. Fix the standard Hermitian inner product on $\mathbb{C}^{n+1}$, which induces a Hermitian metric on the rank $n+1$ trivial bundle $\underline{\mathbb{C}}^{n+1}$ over $\mathbb{C}P^n$. Let $\gamma^\perp$ be the orthogonal complement in $\underline{\mathbb{C}}^{n+1}$ of γ.

(a) Prove that $\mathrm{Hom}(\gamma, \gamma) = \underline{\mathbb{C}}$ is the trivial bundle, and $\mathrm{Hom}(\gamma, \gamma^\perp) \cong T\mathbb{C}P^n$.

(b) Use (a) to prove that $T\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong (\gamma^*)^{\oplus(n+1)}$. Conclude that $c(T\mathbb{C}P^n) = (1 + x)^{n+1}$ where $x \in H^2(\mathbb{C}P^n)$ is dual to $\mathbb{C}P^1 \subseteq \mathbb{C}P^n$.

Remark 1: In particular, $\overline{T\mathbb{C}P^n}$ (which is just $T^*\mathbb{C}P^n$ (why?)) is not isomorphic to $T\mathbb{C}P^n$. This means that there are two different complex manifolds $\mathbb{C}P^n$ and $\overline{\mathbb{C}P^n}$ with the same underlying real manifold $\mathbb{C}P^n$.

Remark 2: In the lecture we showed that TS^2 is stably trivial. On the other hand, this problem shows that $TS^2 = T\mathbb{C}P^1$ is not stably trivial. How can this be?