Math 550 – Homework 7

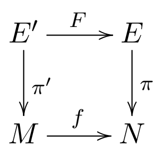
For discussion: 3/5 at 5-6 pm in SEO 427.

From Lee's book, Chapter 10: Problems 6 (note we discussed the equivalence relation in class), 10, 12, 15.

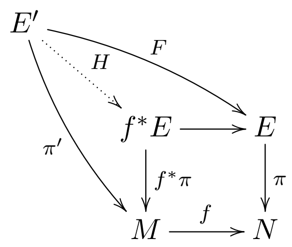
In addition:
Problem 1. Prove that a line bundle is trivial if and only if it admits a nonvanishing section.

Problem 2. Prove that the Möbius bundle is nontrivial. (You probably know that the Möbius band is not diffeomorphic to a cylinder, but try to give the easiest proof of the problem that you can find using tools from this course.)

Problem 3. Prove that the pullback of vector bundles has the following universal property: Suppose that $E \xrightarrow{\pi} N$ is a vector bundle and $f : M \to N$ is a smooth map. Suppose $E' \xrightarrow{\pi'} M$ is a vector bundle and $F : E' \to E$ is a bundle morphism such that the diagram



commutes. Prove that there is a unique bundle morphism $H : E' \to f^*E$ such that the diagram



commutes.

Problem 4. Complete the proof of the universality of tautological bundles: Let $\pi : E \to M$ be a rank k vector bundle, and choose an embedding $\iota : E \hookrightarrow M \times \mathbb{R}^N$ into some trivial bundle. Let $p_2 : M \times \mathbb{R}^N \to \mathbb{R}^N$ be the projection on the second coordinate, and define $f : M \to \mathrm{Gr}_k(\mathbb{R}^N)$ by $f(x) := p_2(E_x)$. Prove that E is isomorphic to $f^*E_k(N)$, where $E_k(N) \to \mathrm{Gr}_k(\mathbb{R}^N)$ is the tautological bundle.