Math 550 – Homework 8

For discussion: 3/12 at 5-6 pm in SEO 427.

Problem 1. Prove that a principal bundle is trivial if and only if it admits a section.

Problem 2. Prove that the construction of vector bundles from principal bundles actually yields a vector bundle, and compute its cocycle: Show that if $P \to M$ is a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ is a representation, then $E := P \times_\rho V \to P/G \cong M$ is a vector bundle with cocycle $\{\rho \circ g_{UV}\}$, where $\{g_{UV}\}$ is the cocyle data of P.

Problem 3. Prove that the construction of a frame bundle from a vector bundle, and the construction of a vector bundle from a principal $\mathrm{GL}(k, \mathbb{R})$-bundle (using the standard representation) are mutually inverse.

Problem 4. Let $\pi_1 : P_1 \to M$ and $\pi_2 : P_2 \to M$ be principal S^1-bundles over a manifold M. Consider the bundle $\pi_1 \oplus \pi_2 : P_1 \oplus P_2 \to M$ (consisting of the points $(p_1, p_2) \in P_1 \times P_2$ with $\pi_1(p_1) = \pi_2(p_2)$).

(a) Define an $S^1 \times S^1$-action on $P_1 \times P_1$ as follows: For $z := (z_1, z_2) \in S^1 \times S^1$ and $p := (p_1, p_2) \in P_1 \oplus P_2$, set $p \cdot z := (p_1 \cdot z_1, p_2 \cdot z_2)$. Prove that this action is well-defined and is smooth, free, and proper.

(b) Consider the diagonally embedded copy of $\Delta(S^1)$ in $S^1 \times S^1$, and write

$$P_1 + P_2 := (P_1 \oplus P_2)/\Delta(S^1).$$

Show that $P_1 + P_2 \to M$ is a principal S^1-bundle, and that the operation $+$ makes the set of principal S^1-bundles over M into an abelian group.

Hint: For the construction of the (group-theoretic) inverse of a principal bundle, work on the level of cocycles.

Problem 5. Let $\pi : P \to M$ be a principal G-bundle. A *gauge transformation* of P is a principal bundle automorphism $\Phi : P \to P$ over identity on M.

(a) For $p \in P$, write $\eta(p) \in G$ for the unique element with $\Phi(p) = p\,\eta(p)$. Prove that $\eta : P \to G$ is smooth and is equivariant for the conjugation action on G, i.e.

$$\eta(p \cdot g) = g^{-1}\eta(p)g$$

for all $p \in M$.

(b) Show that Φ is uniquely determined by η, and this gives a one-to-one correspondence between gauge transformations and sections of $P \times_G G \to P/G = M$.

Problem 6. Let G be a Lie group and $H \subseteq G$ a closed subgroup. View $G \to G/H$ as a principal H-bundle. Consider the representation $\rho : H \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ via the quotient of the adjoint representation (of H on $\mathfrak{g}$). Prove that the associated vector bundle $G \times_\rho \mathfrak{g}/\mathfrak{h} \to G/H$ is isomorphic to the tangent bundle of G/H.