Math 550 – Homework 9

For discussion: 3/19 at 5-6 pm in SEO 427.

Reminder: You should solve at least 3 of the following problems. If you can regularly attend discussion session, please do so and volunteer to present the problems you have solved. If you have a recurring conflict with the discussion session, please submit solutions of at least 3 problems.

Problem 1 (Generalized Bianchi identity). Let $\mathcal{H}$ be a connection on a principal bundle $P \to M$ with connection form ω and curvature form Ω. Recall the Bianchi identity $d\Omega = [\Omega, \omega]$. Prove that for any $k \geq 1$, we have $d(\Omega^k) = [\Omega^k, \omega]$.

Problem 2 (Pullback connections). Let $\mathcal{H}$ be a connection on a principal bundle $P \to N$ with connection form ω and let $f : M \to N$ a smooth map. Let $\tilde{f} : f^*P \to P$ be the canonical map. Prove that $\tilde{f}^*\omega$ is a connection on f^*P and that its curvature form is $\tilde{f}^*\Omega$.

Problem 3 (Maurer-Cartan form). Let G be a connected Lie group. Define a $\mathfrak{g}$-valued 1-form θ on G, called the *Maurer-Cartan form*, as follows: Write $\mathfrak{g} = T_eG$, and on T_gG, define θ by $L_{g^{-1}*}$, where $L_{g^{-1}}$ is left-translation by g^{-1}.

(a) Prove that θ satisfies the Maurer-Cartan equation $d\theta + \frac{1}{2}[\theta, \theta] = 0$.

(b) Let M be any manifold and $\mathrm{pr}_2 : M \times G \to G$ denote the projection onto the second factor. Prove that $\mathrm{pr}_2^*\theta$ defines a connection (called the *Maurer-Cartan connection*) on the trivial bundle $\pi : M \times G \to M$. What is it as a distribution? Compute its curvature.

Problem 4. Let $P \to M$ be a principal G-bundle and let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation. A V-valued k-form $\alpha \in \Omega^k(P, V)$ is called

- *G-equivariant* (or *pseudotensorial of type ρ*) if $\alpha \circ g = \rho(g)^{-1} \circ \alpha$ for all $g \in G$, and
- *horizontal* if $\alpha(v_1, \ldots, v_k) = 0$ whenever one of the v_i is vertical.

A form that is both G-equivariant and horizontal is called *tensorial of type ρ*. Let $\Omega_\rho(P, V)$ be the algebra of tensorial forms of type ρ. Prove that the exterior covariant derivative D maps $\Omega_\rho(P, V)$ into itself, and is an antiderivation (of degree 1).

Problem 5. Let $\pi : P \to M$ be a principal G-bundle and $\Phi : P \to P$ be a gauge transformation with associated map $\eta : P \to G$ (see previous HW set, last problem). Let $\mathcal{H}$ be a connection on P with connection form ω and curvature Ω.

(a) Show that $\mathcal{H}^\Phi := \Phi_*\mathcal{H}$ is also a connection.

(b) Let ω^Φ be the connection form of $\mathcal{H}^\Phi$. Prove that $\omega^\Phi = (\Phi^{-1})^*\omega$.

(c) Prove that $\Phi^*\omega = \mathrm{Ad}(\eta^{-1}) \circ \omega + \eta^*\omega_G$, where $\omega_G \in \Omega^1(G; \mathfrak{g})$ is the *Maurer-Cartan form* on G, namely for $v \in T_gG$, set $\omega_G(v) := L_{g^{-1}*}v \in \mathfrak{g}$, where L_g is left-translation by g.

Remark: Part (b) and (c) give a formula for ω^Φ (apply (c) to the gauge transformation Φ^{-1}).