A Riemannian Manifold is a smooth manifold $M$ together with an inner product $g$ on each tangent space which varies smoothly from point to point. The metric is written $g_p(v_p, w_p)$ for vectors in the tangent space at $p$, i.e., $v_p, w_p \in T_pM$. But, we usually consider all points on the manifold at once and then consider the metric $g$ acting on vector fields, $g(V, W)$. This will be a smooth function of $p$.

If we have a chart $(U, \phi)$ where $\phi : U \to \mathbb{R}^n$ then we write $\phi(p) = (x^1(p), \ldots, x^n(p))$ and say $(x^1, \ldots, x^n)$ are local coordinates for the manifold. We define the vector $\frac{\partial}{\partial x^i}$ in $\phi(U) \subset \mathbb{R}^n$ by $\frac{\partial}{\partial x^i}(\phi(p)) = e_i = (0, \ldots, 1, \ldots, 0)$ for all $\phi(p) \in \phi(U)$. We pull these vector fields back to the manifold with the chart and abuse notation by using the same symbol: $\frac{\partial}{\partial x^i} = d\phi^{-1}(\frac{\partial}{\partial x^i})$. With these vector fields define $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and then form the matrix $(g_{ij})$. This is called the Gram matrix of the metric $g$ in the coordinates $(x^1, \ldots, x^n)$. We write $(g^{ij})$ for the inverse of the Gram matrix (which is invertible since $g$ is positive definite). Some examples are

$$
\mathbb{R}^2 : (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g^{ij}),
$$

$$
\mathbb{H}^2 : (h_{ij}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}, (h^{ij}) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}
$$

$$
\mathbb{R}^2_{polar} : (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}
$$

We can now extend the differential operators we love: The gradient on a manifold $\nabla$ is defined such that for all smooth functions $f \in C^\infty(M)$ we have $g(\nabla f, X) = X(f)$ for all smooth vector fields. This corresponds to what we have in flat space: $\nabla f \cdot v = Df(v)$ Say we have local coordinates $(x^1, \ldots, x^n)$ then we can express the gradient of a function as

$$
\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.
$$

Note that in $\mathbb{R}^n$ we have the usual gradient.

We can also extend the Laplacian to a manifold. This is easiest to express in coordinates $(x^1, \ldots, x^n)$ where we get

$$
\Delta f = \frac{1}{\det(g_{ij})} \frac{\partial}{\partial x^j} \left( g^{ij} \frac{\partial f}{\partial x^i} \sqrt{\det(g_{ij})} \right).
$$

Sometimes geometers use a negative sign in this definition to make the eigenvalues positive instead of negative.

And finally integration: Suppose we have local coordinates $(x^1, \ldots, x^n)$ in a chart $(U, \phi)$, then for a top dimensional form $f dx^1 \wedge \cdots \wedge dx^n$ with support in $U$ we can integrate to get

$$
\int_U f dx^1 \wedge \cdots \wedge dx^n = \int_{\phi(U)} (f \circ \phi^{-1}) dx_1 \cdots dx_n
$$

The problem is this form and these coordinates might not extend over the whole manifold, so there is no canonical way of integrating smooth function. Luckily, a Riemannian manifold has enough structure to let us do this. We define the Riemannian volume for $dV$ to be the unique top form that evaluates to $+1$ on every set of $n$ orthonormal vector fields. In coordinates we can write this explicitly as $dV = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$. For example, in polar coordinates we have $\sqrt{\det(g_{ij})} = \sqrt{r^2} = r$ so that $dV = rdr \wedge d\theta$. On the hyperbolic plane we have $dV = \sqrt{\frac{1}{y^2}} dx \wedge dy = \frac{1}{y^2} dx \wedge dy$.

Now, define a Borel measure on $M$ by defining for $A \subset M$ open,

$$
\mu(A) = \int_A dV.
$$
This measure can be completed with the standard measure theoretic tools to get the Riemannian Volume Measure. Hence, for smooth function $f \in C^\infty(M)$ we can integrate $\int_M f \, dV$. Once again, if we have coordinates we can write this as

$$
\int_U f \, dV = \int_U f \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n.
$$

Consequently, we can define all the lovely spaces from analysis: $L^p(M), W^{p,s}(M),$ etc. all defined in the obvious way. Sobolev inequalities on compact manifolds usually follow from their analog in Euclidean space and a partition of unity argument as well as the following fact: there exists a constant $C$ (depending on $p$) such that for a function $u : M \to \mathbb{R}$ with compact support in a chart $U$ and its coordinate representation $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ we have

$$
\frac{1}{C} \|\tilde{u}\|_p \leq \|u\|_p \leq C \|\tilde{u}\|_p.
$$

This follows from compactness: $\sqrt{\det (g_{ij})} : M \to (0, \infty)$ is a smooth function on a compact set, which is nonzero since $g$ is positive definite and so it has a maximum and minimum. So we have a $C$ such that $\frac{1}{C} \leq \sqrt{\det (g_{ij})} \leq C$. Consequently,

$$
\frac{1}{C} \|\tilde{u}\|_p = \frac{1}{C} \int_{\phi(U)} |u \circ \phi^{-1}|^p \, dx_1 \cdots dx_n
$$

$$
= \frac{1}{C} \int_U |u|^p dx^1 \wedge \cdots \wedge dx^n
$$

$$
\leq \int_U |u|^p \sqrt{\det (g_{ij})} dx^1 \wedge \cdots \wedge dx^n
$$

$$
= \int_U |u|^p \, dV
$$

$$
\leq C \int_U |u|^p dx^1 \wedge \cdots \wedge dx^n
$$

$$
= C \int_{\phi(U)} |u \circ \phi^{-1}|^p \, dx_1 \cdots dx_n = C \|\tilde{u}\|_p.
$$

Now for our question:

Let $k$ be the Gaussian curvature of a compact Riemannian 2-manifold then the Gauss-Bonnet Theorem tells us

$$
\int_M k \, dA = 2\pi \chi(M),
$$

where $\chi(M)$ is the Euler Characteristic of the manifold $M$, which is a constant. We get sign conditions on $k$ from the possible cases: $\chi(M) < 0, = 0, > 0$. This leads to the question “If $K$ obeys the sign conditions stipulated by the Euler characteristic of the manifold, is $K$ the curvature of Riemannian metric on the manifold?”

We make this problem easier to answer by fixing a Riemannian metric $g$ on $M$ and then asking if $K$ is the curvature of some other metric $\tilde{g}$ which is conformally equivalent to $g$, meaning $\tilde{g} = e^{2u}g$ for some smooth $u \in C^\infty(M)$.

If we let $k$ and $\delta$ be the curvature and Laplacian of the metric $g$ then asking that $K$ be the curvature of $\tilde{g} = e^{2u}g$ gives the equation

$$
K e^{2u} \omega_1 \wedge \omega_2 = (k - \Delta u) \omega_1 \wedge \omega_2,
$$

where $(\omega_1, \omega_2)$ is a local orthonormal coframe (orthonormal 1-forms that form a basis for the cotangent space at each point in the chart). Hence we want a solution to the PDE

$$
\Delta u = k - K e^{2u}.
$$

If we can show a $u$ exists satisfying this equation then $\tilde{g} = e^{2u}g$ will have curvature $K$. 

2
Both \( k \) and \( K \) are most likely non-constant, so we make a change of variables. Let \( v \) be a solution of \( \Delta v = k - \bar{k} \), where \( \bar{k} \) is the average of \( k \) on the manifold \( M \). Now let \( w = 2(u - v) \). Then \( w \) satisfies

\[
\Delta w = 2\Delta u - 2\Delta v = 2(k - Ke^{2u}) - 2(k - \bar{k}) = 2\bar{k} - 2Ke^{2u} = 2\bar{k} - (2Ke^{2v})e^{2w}.
\]

Now \( \bar{k} \) is constant, and we free the notation from its geometric background by writing

\[
\Delta u = c - he^u,
\]

where \( h \) is some given smooth function. The analysis of this PDE turns out to drastically depend on \( c \) more so than \( h \). The following will maybe explain why. The integral of the laplacian of a smooth function on \( M \) is 0 by e.g., Greens Theorem. So, if we integrate the PDE we get

\[
0 = \int_M c \, dA - \int_M he^u \, dA \quad \Rightarrow \quad \int_M he^u \, dA = c \, \text{Area}(M).
\]

So \( c \) is essentially playing the role of the Euler Characteristic. The theory of this PDE depends on the sign of \( c \), and at the time the paper this is from was written, there were only partial answers for the various cases of \( c \) and for specific 2-dimensional manifolds. Most specifically, the paper discusses \( S^2 \) and \( \mathbb{R}P^2 \). But I haven’t read far enough to comment on what happens.