A Riemannian Manifold is a smooth manifold M together with an inner product g on each tangent space which varies smoothly from point to point. The metric is written $g_p(v_p, w_p)$ for vectors in the tangent space at p, i.e., $v_p, w_p \in T_pM$. But, we usually consider all points on the manifold at once and then consider the metric g acting on vector fields, $g(V, W)$. This will be a smooth function of p.

If we have a chart (U, ϕ) where $\phi : U \to \mathbb{R}^n$ then we write $\phi(p) = (x^1(p), \ldots, x^n(p))$ and say $(x^1, \ldots, x^n)$ are local coordinates for the manifold. We define the vector $\frac{\partial}{\partial x^i}$ in $\phi(U) \subset \mathbb{R}^n$ by $\frac{\partial}{\partial x^i}(\phi(p)) = e_i = (0, \ldots, 1, \ldots, 0)$ for all $\phi(p) \in \phi(U)$. We pull these vector fields back to the manifold with the chart and abuse notation by using the same symbol: $\frac{\partial}{\partial x^i} = d\phi^{-1}(\frac{\partial}{\partial x^i})$. With these vector fields define $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and then form the matrix (g_{ij}). This is called the Gram matrix of the metric g in the coordinates $(x^1, \ldots, x^n)$. We write (g^{ij}) for the inverse of the Gram matrix (which is invertible since g is positive definite). Some examples are

$$\mathbb{R}^2 : (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g^{ij}),$$

$$\mathbb{H}^2 : (h_{ij}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}, (h^{ij}) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$$

$$\mathbb{R}^2_{polar} : (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

We can now extend the differential operators we love: The gradient on a manifold ∇ is defined such that for all smooth functions $f \in C^\infty(M)$ we have $g(\nabla f, X) = X(f)$ for all smooth vector fields. This corresponds to what we have in flat space: $\nabla f \cdot v = Df(v)$ Say we have local coordinates $(x^1, \ldots, x^n)$ then we can express the gradient of a function as

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Note that in $\mathbb{R}^n$ we have the usual gradient.

We can also extend the Laplacian to a manifold. This is easiest to express in coordinates $(x^1, \ldots, x^n)$ where we get

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^j} \left(g^{ij} \frac{\partial f}{\partial x^i} \sqrt{\det(g_{ij})} \right).$$

Sometimes geometers use a negative sign in this definition to make the eigenvalues positive instead of negative.

And finally integration: Suppose we have local coordinates $(x^1, \ldots, x^n)$ in a chart (U, ϕ), then for a top dimensional form $f dx^1 \wedge \cdots \wedge dx^n$ with support in U we can integrate to get

$$\int_U f dx^1 \wedge \cdots \wedge dx^n = \int_{\phi(U)} (f \circ \phi^{-1}) \, dx_1 \cdots dx_n$$

The problem is this form and these coordinates might not extend over the whole manifold, so there is no cannonical way of integrating smooth function. Luckily, a Riemannian manifold has enough structure to let us do this. We define the Riemannian volume for dV to be the unique top form that evaluates to $+1$ on every set of n orthonormal vector fields. In coordinates we can write this explicitly as $dV = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$. For example, in polar coordinates we have $\sqrt{\det(g_{ij})} = \sqrt{r^2} = r$ so that $dV = r dr \wedge d\theta$. On the hyperbolic plane we have $dV = \sqrt{\frac{1}{y^4}} dx \wedge dy = \frac{1}{y^2} dx \wedge dy$.

Now, define a Borel measure on M by defining for $A \subset M$ open,

$$\mu(A) = \int_A dV.$$

This measure can be completed with the standard measure theoretic tools to get the Riemannian Volume Measure. Hence, for smooth function $f \in C^\infty(M)$ we can integrate $\int_M f\ dV$. Once again, if we have coordinates we can write this as

$$\int_U f\ dV = \int_U f\sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n.$$

Consequently, we can define all the lovely spaces from analysis: $L^p(M), W^{p,s}(M)$, etc. all defined in the obvious way. Sobolev inequalities on compact manifolds usually follow from their analog in Euclidean space and a partition of unity argument as well as the following fact: there exists a constant C (depending on p) such that for a function $u : M \to \mathbb{R}$ with compact support in a chart U and its coordinate representation $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ we have

$$\frac{1}{C}\|\tilde{u}\|_p \leq \|u\|_p \leq C\|\tilde{u}\|_p.$$

This follows from compactness: $\sqrt{\det(g_{ij})} : M \to (0, \infty)$ is a smooth function on a compact set, which is nonzero since g is positive definite and so it has a maximum and minimum. So we have a C such that $\frac{1}{C} \leq \sqrt{\det(g_{ij})} \leq C$. Consequently,

$$\begin{aligned}
\frac{1}{C}\|\tilde{u}\|_p &= \frac{1}{C}\int_{\phi(U)} |u \circ \phi^{-1}|^p\ dx_1 \cdots dx_n \\
&= \frac{1}{C}\int_U |u|^p dx^1 \wedge \cdots \wedge dx^n \\
&\leq \int_U |u|^p \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n \\
&= \int_U |u|^p dV \\
&\leq C\int_U |u|^p dx^1 \wedge \cdots \wedge dx^n \\
&= C\int_{\phi(U)} |u \circ \phi^{-1}|^p dx_1 \cdots dx_n = C\|\tilde{u}\|_p.
\end{aligned}$$

Now for our question:

Let k be the Gaussian curvature of a compact Riemannian 2-manifold then the Gauss-Bonnet Theorem tells us

$$\int_M k\ dA = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler Characteristic of the manifold M, which is a constant. We get sign conditions on k from the possible cases: $\chi(M) < 0, = 0, > 0$. This leads to the question "If K obeys the sign conditions stipulated by the Euler characteristic of the manifold, is K the curvature of Riemannian metric on the manifold?"

We make this problem easier to answer by fixing a Riemannian metric g on M and then asking if K is the curvature of some other metric $\tilde{g}$ which is conformally equivalent to g, meaning $\tilde{g} = e^{2u}g$ for some smooth $u \in C^\infty(M)$.

If we let k and δ be the curvature and Laplacian of the metric g then asking that K be the curvature of $\tilde{g} = e^{2u}g$ gives the equation

$$Ke^{2u}\omega_1 \wedge \omega_2 = (k - \Delta u)\omega_1 \wedge \omega_2,$$

where (ω_1, ω_2) is a local orthonormal coframe (orthonormal 1-forms that form a basis for the cotangent space at each point in the chart). Hence we want a solution to the PDE

$$\Delta u = k - Ke^{2u}.$$

If we can show a u exists satisfying this equation then $\tilde{g} = e^{2u}g$ will have curvature K.

Both k and K are most likely non-constant, so we make a change of variables. Let v be a solution of $\Delta v = k - \bar{k}$, where $\bar{k}$ is the average of k on the manifold M. Now let $w = 2(u - v)$. Then w satisfies

$$\begin{aligned}\Delta w &= 2\Delta u - 2\Delta v \\ &= 2(k - Ke^{2u}) - 2(k - \bar{k}) \\ &= 2\bar{k} - 2Ke^{2u} \\ &= 2\bar{k} - (2Ke^{2v})e^{2w}.\end{aligned}$$

Now $\bar{k}$ is constant, and we free the notation from its geometric background by writing

$$\Delta u = c - he^u,$$

where h is some given smooth function. The analysis of this PDE turns out to drastically depend on c more so than h. The following will maybe explain why. The integral of the laplacian of a smooth function on M is 0 by e.g., Greens Theorem. So, if we integrate the PDE we get

$$0 = \int_M c\, dA - \int_M he^u\, dA \quad \Longrightarrow \quad \int_M he^u\, dA = c\, \text{Area}(M).$$

So c is essentially playing the role of the Euler Characteristic. The theory of this PDE depends on the sign of c, and at the time the paper this is from was written, there were only partial answers for the various cases of c and for specific 2-dimensional manifolds. Most specifically, the paper discusses $\mathbb{S}^2$ and $\mathbb{RP}^2$. But I haven't read far enough to comment on what happens.